Sampling Theory for Functions with Fractal Spectrum

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CONTENTS

We investigate in greater detail a sampling formula given by the first author for functions whose spectrum lies in a Cantor set K of a special type introduced by Jorgensen and Pedersen, where the sampling set is extremely thin, and the sampling function is quite different from the usual sinc function. We obtain new properties of the sampling function, and we give approximate descriptions of both local and global behavior of functions with spectrum in K. Some experimental results are described, and more can be found at http://mathlab.cit.cornell.edu/~tillman.

1. INTRODUCTION

Functions on $\mathbb{R}$ are said to be *bandlimited* if their Fourier transform has compact support. Such functions have many remarkable properties. In particular, they are determined by their values on certain discrete subsets of $\mathbb{R}$, such as a multiple of the integer lattice, and sampling theory provides concrete formulas for recovering the function from its "sampled" values. Functions whose Fourier transforms have support in a fractal set, such as a Cantor set, are even more special, and should possess a sampling theory that allows for a much thinner sampling set. However, not much is known in general. In [Strichartz 2000a], the first author presented some results of this type for a special class of Cantor sets first studied by Jorgensen and Pedersen [1998]. Here we will continue the investigation, and to be specific we will deal with just a single family of Cantor sets. It is likely that the results presented here could be extended to the class of sets discussed in [Strichartz 2000a].

Let R denote a positive even integer, $R \geq 4$. In our examples we take either $R = 4$ or $R = 6$. Let K

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denote the Cantor set defined by the iterated function system consisting of the mappings

$$F_{\pm}x = R^{-1}x \pm \tfrac{1}{4}.$$

We may also construct K from the interval

$$\left[\frac{-R}{4(R-1)}, \frac{R}{4(R-1)}\right]$$

by successively deleting the middle $(R-2)/R$ portion of each interval. Let μ denote the associated Cantor measure that assigns the value 2^{-n} to each of 2^n intervals in the nth stage of the construction. Let Λ denote the discrete set of nonnegative integers expressible base R using only the digits 0 and 1. It was shown in [Jorgensen and Pedersen 1998] and [Strichartz 1998] that Λ forms a spectrum for $L^2(\mu)$, in the sense that the functions $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ form an orthonormal basis for $L^2(\mu)$. In [Strichartz 2000a] this result was used to show that Λ may be used as a sampling set for functions whose Fourier transform is supported in K. We say that a function F on $\mathbb{R}$ is *K-bandlimited* if there exists a finite measure $\hat{F}$ supported in K such that

$$F(x) = \int e^{2\pi ixy}\, d\hat{F}(y).$$

A special class of such functions are the *strongly K-bandlimited* functions of the form

$$F(x) = \int_K e^{2\pi ixy}\varphi(y)\, d\mu(y) \tag{1–1}$$

for $\varphi \in L^2(\mu)$. In other words, $\hat{F} = \varphi d\mu$. But there are many other K-bandlimited functions, such as $\cos ax$ or $\sin ax$ for $a \in 2\pi K$. The sampling formula for K-bandlimited functions in [Strichartz 2000a] says that

$$F(x) = \sum_{\lambda\in\Lambda} F(\lambda)\hat{\mu}(x-\lambda) \tag{1–2}$$

with uniform convergence on compact sets, where $\hat{\mu}$, the Fourier transform of μ (given by (1–1) with $\varphi = 1$), is called the *sampling function.* Here we will be interested in a more precise description of (1–2). What is the nature of the functions $\hat{\mu}(x-\lambda)$? How well does (1–2) converge? If you are only going to use a finite number of terms, can you do better than taking a partial sum of (1–2)? How can you recognize whether or not a function is K-bandlimited? We will give both theoretical and experimental results. Although some of our conclusions are expressed in precise theorems, our best answers to the above questions are of an informal nature, with certain errors that are small but not negligible.

In Section 2 we study the asymptotic behavior of the sampling function. It is a special case of a class of Fourier transforms of Cantor measures that are called *multiperiodic functions*, since it satisfies a type of multiplicative periodicity identity

$$\hat{\mu}(Rx) = \Big(\cos\frac{\pi}{2}Rx\Big)\hat{\mu}(x).$$

Such functions have been studied extensively [Fan and Lau 1998; Janardhan et al. 1992; Strichartz 1990; 1993a; 1993b], but the property we describe here does not seem to have been previously noticed: viewed at larger and larger scales, the graph of the sampling function appears to converge to a limiting "picture". We express this by showing that the graphs of $\hat{\mu}(R^n x)$, as subsets of $\mathbb{R}\times\mathbb{R}$, converge to a limit set Γ, which is not itself the graph of a function. We do identify a limit function

$$g(x) = \lim_{n\to\infty} \hat{\mu}(R^n x), \tag{1–3}$$

but this function is zero almost everywhere, and (1–3) does not in fact describe what one sees. The limit set Γ consists of the x-axis together with a countable number of vertical line segments, and has the appearance of a chaotic comb.

In Section 3 we study the behavior of the translates of the sampling function $\hat{\mu}(x-\lambda)$ that appear in the sampling formula (1–2), but this time viewed on a fixed finite interval. We find that if $\lambda \in \Lambda$ is large, we can find λ' from a fixed finite segment of Λ so that $\hat{\mu}(x-\lambda)$ is approximately a constant times $(x-\lambda')\hat{\mu}(x-\lambda')$ on the interval. The precise relationship between the interval, the segment of Λ and the approximation error is a bit complicated. But informally, we reach the conclusion that the space of K-bandlimited functions restricted a fixed interval is close to being a finite dimensional space, with basis $\{\hat{\mu}(x-\lambda')\}\cup\{(x-\lambda')\hat{\mu}(x-\lambda')\}$, where λ' varies over the initial segment of Λ. This is consistent with an information theoretic description based on the thinness of the set K [Gulisashvili 1993; $\geq$ 2001].

In Section 4 we study the sampling formula via examples, and prove one important consequence:

the asymptotic description of the sampling function in Section 2 extends to all strongly K-bandlimited functions. The limit set Γ in the general case has the same chaotic comb appearance, and is determined from the function by just two parameters (the sup and inf of the function). The two types of examples of K-bandlimited functions we examine are the translates $\hat{\mu}(x-y)$ of the sampling function, and the exponentials (actually cosines) with frequencies from K. The first type are strongly K-bandlimited, and so the sampling formula converges uniformly. But it does not converge rapidly, since the coefficients $F(\lambda) = \hat{\mu}(\lambda - y)$ are not quite $O(\lambda^{-1})$. Our experimental evidence shows that error is relatively smaller on the sampling interval than overall. The second type of example is not strongly K-bandlimited, and the approximation error is considerably worse. If we double the number of sampling points we can extend the interval of approximation by a factor of R with the same error, but there are unacceptable errors if we try to go much beyond the sampling interval. If we look at the graphs of the approximation error, with the x-axis rescaled, we seem to see a convergence to a limit set analogous to the asymptotic convergence of the graph of the scaling function, but to a completely different type of limit set. At present we have no explanation for this phenomenon. However, in [Strichartz 2000b] it is shown that this type of "shape of the error" behavior is typical of other, more standard approximation processes, but with the y-axis rather than the x-axis being rescaled. Figure 8 was the direct inspiration for the results of [Strichartz 2000b].

In Section 5 we consider a different type of sampling formula (actually two variants), based on the approximations of Section 3. The goal is to obtain better accuracy on a fixed interval (at the expense of much worse global accuracy). Whether or not these modified sampling formulas actually work remains unproved, but we represent some experimental evidence of their effectiveness.

The results presented here give us a basis for asserting that K-bandlimited functions are indeed easy to recognize. In the short term, all K-bandlimited functions are approximately linear combinations of a finite list of explicit functions. In the long view, the graph of any strongly K-bandlimited function has a characteristic chaotic comb appearance chosen from a certain two parameter space of such sets. Note that this does not give us a definitive test for deciding whether or not a given function is K-bandlimited, and it will not help us correct small errors in data for such functions. It merely gives us a "rule of thumb". In contrast, if we depart from the special class of Cantor sets described in [Jorgensen and Pedersen 1998] or [Strichartz 1998], absolutely nothing is known about K-bandlimited functions (for example, the original $1/3$ Cantor set). This is an enticing problem which demands entirely different methods than those presented here.

This paper mixes theoretical and experimental results. Many more experimental results are available online (see Electronic Availability at the end).

It must be emphasized that the experimental results in all cases preceded the theoretical results. The theorems were concocted to explain the pictures. On the other hand, the pictures were concocted to illustrate the theorems from [Strichartz 2000a]. The chicken and the egg.

2. ASYMPTOTIC BEHAVIOR OF THE SAMPLING FUNCTION

The Fourier transform $\hat{\mu}$ of the Cantor measure μ, which we call the sampling function, is given by the infinite product

$$\hat{\mu}(x) = \prod_{k=0}^{\infty} \cos\left(\frac{\pi}{2}\frac{x}{R^k}\right). \tag{2–1}$$

For moderate values of x it is not necessary to take very many terms in the product to get an accurate approximation, and the number of terms need only grow on the order of $\log|x|$. Since the function is even we only deal with $x \geq 0$. In Figure 1 we show the graph for $R = 4$ on four different intervals, each multiplied by a factor of 4 from the previous one. In Figure 2 we do the same for $R = 6$, now using a factor of 6 in changing the x scale. The results are visually striking, but how can they be explained?

Consider first the pointwise convergence of $\hat{\mu}(R^n x)$ as $n \to \infty$. The limit function g is given simply by

$$g(x) = \prod_{k=-\infty}^{\infty} \cos\left(\frac{\pi}{2}\frac{x}{R^k}\right) = \hat{\mu}(x) \prod_{m=1}^{\infty} \cos\left(\frac{\pi}{2}R^m x\right); \tag{2–2}$$

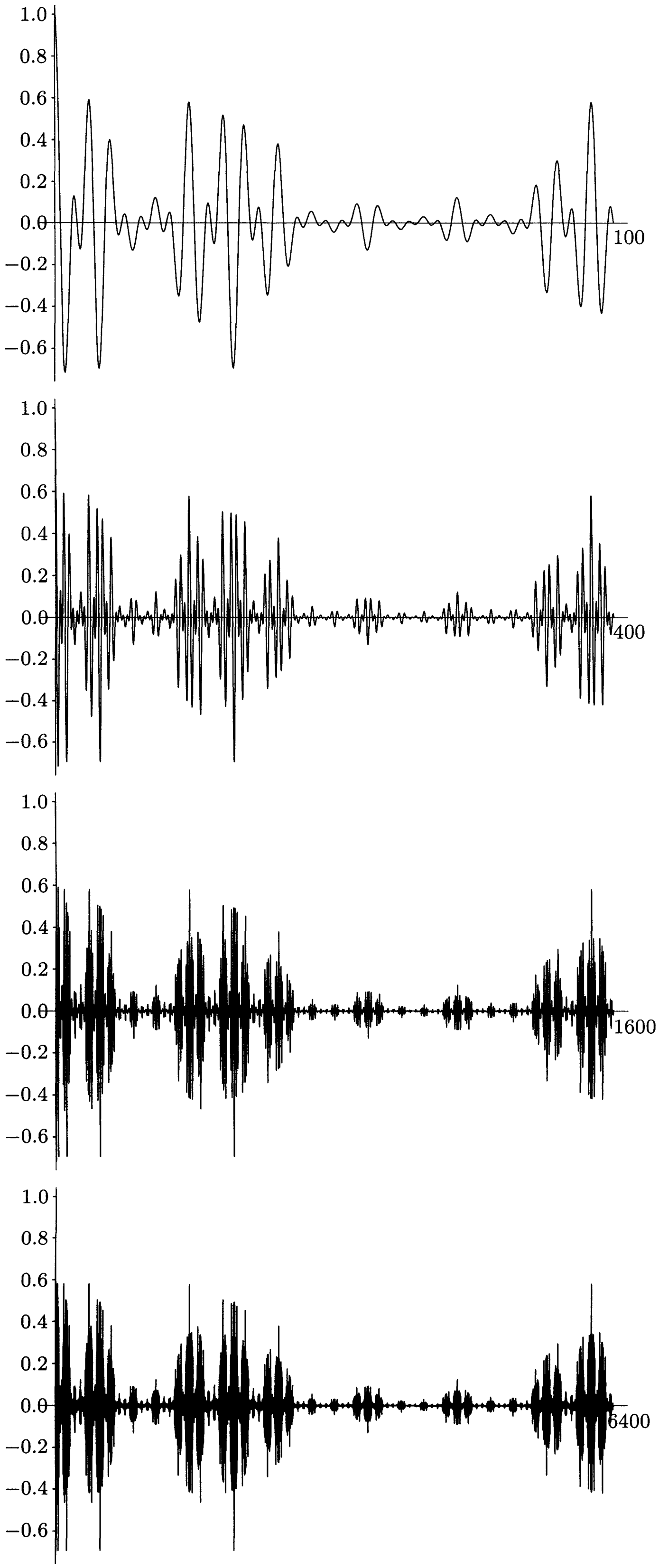

FIGURE 1. The graph of the sampling function $\hat{\mu}(x)$ for $R = 4$ on four intervals: $[0, 100]$, $[0, 400]$, $[0, 1600]$ and $[0, 6400]$.

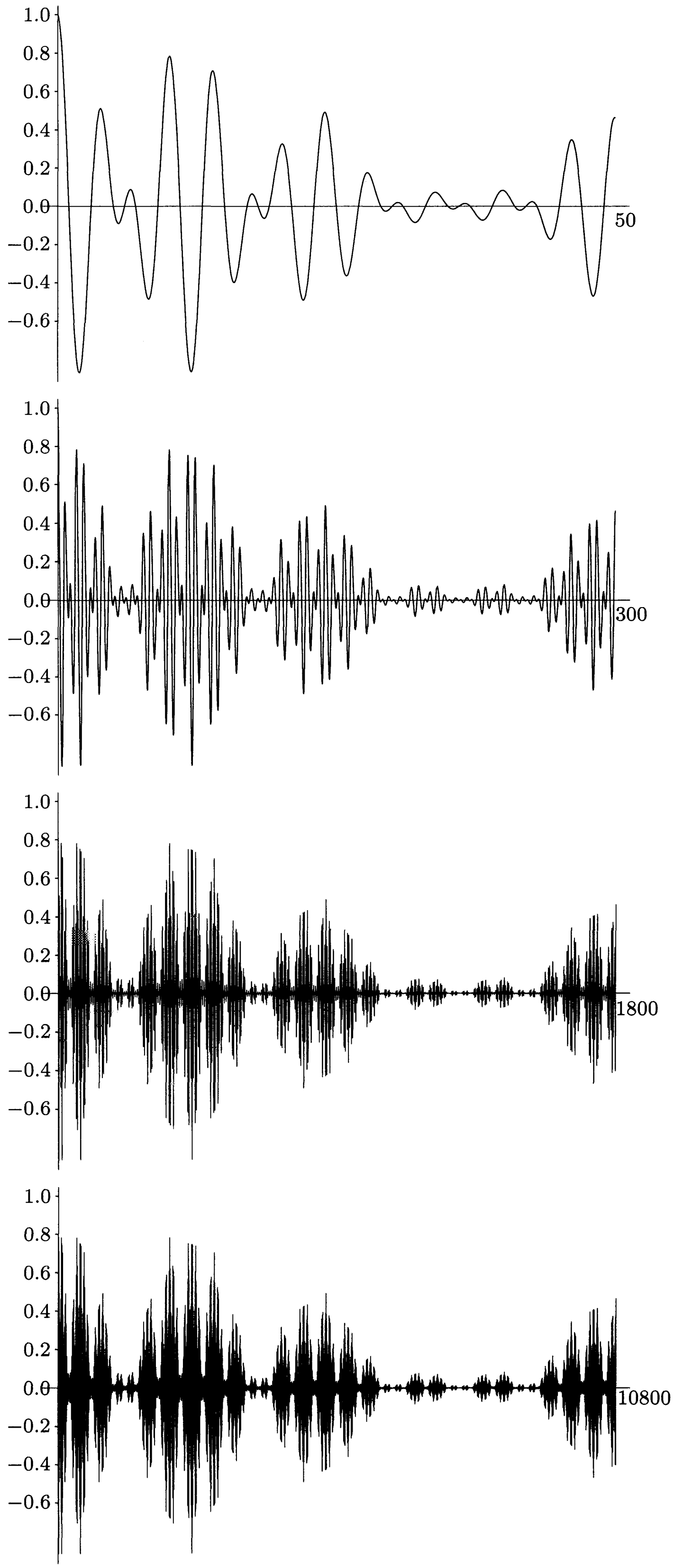

FIGURE 2. The graph of the sampling function $\hat{\mu}(x)$ for $R = 6$ on four intervals: $[0, 50]$, $[0, 300]$, $[0, 1800]$ and $[0, 10800]$.

here we specifically do not follow the convention of calling an infinite product divergent if it converges to 0.

Lemma 2.1. *For R an even integer, $R \geq 4$, the infinite product in (2–2) converges for every x. It is different from 0 when $x = 0$ or $x = p/R^m$ for integers $m \geq 0$ and even integers p not divisible by R, and 0 otherwise. Thus g is everywhere discontinuous. Also*

$$g(x) = \lim_{n\to\infty} \hat{\mu}(R^n x) \quad \textit{for every } x. \tag{2–3}$$

Proof. For $x = p/R^m$, note that every factor in (2–2) corresponding to $k < -m$ is equal to one, since it is the cosine of an even multiple of π. Thus the product converges to $\hat{\mu}(p)$, and $\hat{\mu}(p) \neq 0$ if p is an even integer not divisible by R, while $\hat{\mu}(p) = 0$ if p has the form $(2q+1)R^m$.

Next suppose x is not expressible as p/R^m. Then the base R expansion of x is infinite and does not end with an infinite string of all 0 (or all $R-1$) digits:

$$x = \sum_{j=-N}^{\infty} x_j R^{-j}.$$

We then have

$$\left|\cos\frac{\pi}{2}R^k x\right| = \left|\cos\frac{\pi}{2}\sum_{j=0}^{\infty} x_{j+k}R^{-j}\right|.$$

If (x_{k+1}, x_{k+2}) is not equal to $(0,0)$ or $(R-1, R-1)$, we can bound $\sum_{j=0}^{\infty} x_{j+k}R^{-j}$ away from the set of integers, hence we will have a bound

$$\left|\cos\frac{\pi}{2}R^k x\right| \leq c$$

for a suitable constant $c < 1$ depending only on R. Since this holds for an infinite number of factors in the product (2–2), we conclude that $g(x) = 0$.

It is clear that g is everywhere discontinuous, and the arguments above also prove (2–3). □

Rather than the pointwise limit (2–3), which is not uniform, we will be interested in the convergence of the graphs Γ_n of the function $\hat{\mu}(R^n x)$ to a certain limit set Γ, which itself is not the graph of a function. First we will describe the set Γ, and then explain the type of convergence that we have.

Note that $\hat{\mu}(0) = 1$ is the maximum value of $\hat{\mu}$, and $\hat{\mu}$ attains its minimum value, which we denote by b, at a point x_0 near $x = 2$. Indeed, the identity $\hat{\mu}(Rx) = \left(\cos\frac{\pi}{2}Rx\right)\hat{\mu}(x)$ shows that the maximum value is attained at $x = 0$, and also

$$\sup_{x\geq R} |\hat{\mu}(x)| \leq \max_{1\leq x\leq R} |\hat{\mu}(x)|.$$

However, routine estimates show that on the interval $1 \leq x \leq R$ the maximum value of $|\hat{\mu}(x)|$ occurs where $\hat{\mu}(x)$ attains its minimum (this occurs near $x = 2$, but not exactly at $x = 2$ because $\hat{\mu}'(2) \neq 0$). Thus this local maximum is in fact a global minimum. We define the set Γ to be the region between the graphs of $g(x)$ and $bg(x)$. In other words, Γ consists of the x-axis together with the countable number of vertical line segments joining $(x, g(x))$ and $(x, bg(x))$ for x of the form p/R^k (p an even integer not divisible by R or $p = 0$). It is not obvious that Γ is closed, but we will prove this below.

We will show $\Gamma_n \to \Gamma$ in the following sense. A simple compactness argument [Kuratowski 1968, p. 49] shows that this implies convergence in the Hausdorff metric on any bounded region. We do not know whether $\Gamma_n \to \Gamma$ in the Hausdorff metric globally.

Definition 2.2. Let $\{A_n\}$ be a sequence of nonempty sets in a metric space M. Let

$$\limsup_{n\to\infty} A_n \subset M$$

be the set of all limits of convergent sequences $\{x_n\}$ with $x_n \in A_n$, and let

$$\liminf_{n\to\infty} A_n \subset M$$

be the set of all accumulation points of sequences $\{x_n\}$ with $x_n \in A_n$. Clearly

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

and $\limsup_{n\to\infty} A_n$ is a closed set. We say the sequence $\{A_n\}$ *converges weakly* to a set A (necessarily closed) if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = A$.

Theorem 2.3. *Γ_n converges weakly to Γ.*

The key to proving Theorem 2.3 is the following Lemma.

Lemma 2.4. *For any real t and p an even integer,*

$$\hat{\mu}(R^n p + t) = \frac{\hat{\mu}(t)}{\hat{\mu}(R^{-n}t)}\hat{\mu}(p + R^{-n}t). \tag{2–4}$$

Proof. Split the product defining $\hat{\mu}(R^n p+t)$ at $k=n$. For $k<n$ we have

$$\cos\frac{\pi}{2}(R^{n-k}p+R^{-k}t)=\cos\frac{\pi}{2}R^{-k}t$$

since $R^{n-k}p/2$ is an even integer. Thus

$$\prod_{k=0}^{n-1}\cos\frac{\pi}{2}R^{-k}(R^n p+t)=\hat{\mu}(t)/\hat{\mu}(R^{-n}t).$$

But

$$\prod_{k=n}^{\infty}\cos\frac{\pi}{2}R^{-k}(R^n p+t)=\prod_{k=0}^{\infty}\cos\frac{\pi}{2}R^{-k}(p+R^{-n}t),$$

proving (2–4). □

When p is an odd integer the same identity holds, except with a minus sign if $R/2$ is odd.

Lemma 2.5. *If $(x,y)\notin\Gamma$, there exists $\varepsilon>0$ such that the ball of radius ε in the plane about (x,y) is disjoint from Γ_n for all sufficiently large n.*

Proof. Assume first that x is not of the form p/R^m. We have $y\neq 0$, and we may assume without loss of generality that $y>0$. The argument in the proof of Lemma 2.1 to show $g(x)=0$ actually shows that for a certain finite N we have

$$\left|\prod_{j=0}^{N}\cos\frac{\pi}{2}R^j x\right|\leq y/3.$$

By continuity there exists $\delta>0$ such that

$$\left|\prod_{j=0}^{N}\cos\frac{\pi}{2}R^j t\right|\leq y/2$$

provided $|t-x|<\delta$. Since $\hat{\mu}(R^n t)$ contains the factor

$$\prod_{j=0}^{N}\cos\frac{\pi}{2}R^j t \quad \text{for all} \quad n\geq N,$$

we have $|\hat{\mu}(R^n t)|\leq y/2$ for $|t-x|<\delta$, which gives the desired result. The argument when $x=p/R^m$ for p odd is similar.

The last case is when $x=p/R^m$ and p is an even integer not divisible by R (or $p=0$). We may assume without loss of generality that $g(x)=\hat{\mu}(p)>0$. A point near x will have the form $x+t=p/R^m+t$, where t is small. We take $n\geq m$. Then (2–4) yields

$$\hat{\mu}(R^n(x+t))=\hat{\mu}(R^{n-m}p+R^n t)=\frac{\hat{\mu}(R^n t)\hat{\mu}(p+R^m t)}{\hat{\mu}(R^m t)} \tag{2–5}$$

By taking t small enough we can make $\hat{\mu}(p+R^m t)$ arbitrarily close to $\hat{\mu}(p)$, and $\hat{\mu}(R^m t)$ arbitrarily close to 1. The term $\hat{\mu}(R^n t)$ is bounded above by 1 and below by b. Thus the portion of the graph Γ_n corresponding to values $x+t$ lies between the values $g(x)$ and $bg(x)$ with an arbitrarily small error. Thus if $y>g(x)$ or $y<bg(x)$ we can find a ball about (x,y) disjoint from all Γ_n for $n\geq m$. □

Lemma 2.6. *For any $(x,y)\in\Gamma$ and any $\varepsilon>0$, there exists N such that for any $n\geq N$ there exists $(x_n,y_n)\in\Gamma_n$ within a distance of ε of (x,y).*

Proof. If $y=0$ the results is obvious, since Γ_n intersects the x-axis at points p/R^n for p odd. Thus we may assume $x=p/R^m$ for p even and not divisible by R (or $p=0$), and $b\hat{\mu}(p)\leq y\leq\hat{\mu}(p)$ (again assuming $\hat{\mu}(p)>0$). We choose $(x_n,y_n)=(x+t_n,\hat{\mu}(R^n(x+t_n)))$, where t_n is required to satisfy $|t_n|<\varepsilon/2$. By (2–5) we will have $\hat{\mu}(R^n(x+t_n))\approx\hat{\mu}(R^n t_n)\hat{\mu}(p)$ if $n\geq m$ and ε is small enough. But then we can certainly arrange that $\hat{\mu}(R^n t_n)$ attains any value between b and 1, hence $|y_n-y|<\varepsilon/2$. □

Proof of Theorem 2.3. By Lemma 2.6,

$$\Gamma\subseteq\liminf_{n\to\infty}\Gamma_n.$$

By Lemma 2.5, the complement of Γ is disjoint from $\limsup_{n\to\infty}\Gamma_n$, hence $\limsup_{n\to\infty}\Gamma_n\subseteq\Gamma$. It follows that $\liminf_{n\to\infty}\Gamma_n=\limsup_{n\to\infty}\Gamma_n=\Gamma$, hence $\Gamma_n\to\Gamma$ weakly and Γ is closed. □

Part of the content of Theorem 2.3 is the fact that Γ is a closed set. This is by no means apparent from the description, since the vertical line segments in Γ can have other vertical line segments as limits in rather complicated ways. The Theorem implies that these vertical line segments are also contained in Γ. The graphs of $\hat{\mu}$ over the larger scales shown in Figures 1 and 2 begin to give an impression of the appearance of Γ.

3. TRANSLATES OF THE SAMPLING FUNCTION

The translates $\hat{\mu}(x-\lambda)$ of the sampling function for $\lambda\in\Lambda$ are the basis elements in the sampling formula. In this section we study the behavior of the restriction of these translates to a finite interval. The essence of our observations is that although the space spanned by these restrictions is infinite dimensional, if we are willing to tolerate a small error then it becomes a finite dimensional space. This is in agreement with the general philosophy that K-bandlimited functions carry very little information [Gulisashvili 1993; $\geq$ 2001].

Choose a positive integer N, and let Λ_N denote the elements of Λ expressible as $\lambda = \sum R^{n_j}$ with all $n_j < N$. Then we may write the general element of Λ as $\lambda + \lambda'$ where $\lambda \in \Lambda_N$ and $\lambda' = \sum R^{n_j}$ with all $n_j \geq N$. Consider the interval $I_N = \{x : -aR^N \leq x \leq bR^N\}$ for certain fixed parameters a and b of moderate size, so that $\Lambda \cap I_N$ is essentially Λ_N. We will show that $\hat{\mu}(x-\lambda-\lambda')$ restricted to I_N is approximately equal to a multiple of $(x-\lambda)\hat{\mu}(x-\lambda)$ for $\lambda' \neq 0$. It is somewhat difficult to specify the error precisely, and we are not able to make the error go to zero by manipulating the parameters, but the estimates are independent of N. Informally, we may say that to within a reasonable error, the space of K-bandlimited functions restricted to I_N is spanned by the 2^{N+1} functions $\hat{\mu}(x-\lambda)$ and $(x-\lambda)\hat{\mu}(x-\lambda)$ for $\lambda \in \Lambda_N$. This observation will lead to the modified sampling formulas in Section 5. Note that $(x-\lambda)\hat{\mu}(x-\lambda)$ is not K-bandlimited in the sense defined in [Strichartz 2000a], since its Fourier transform is a distribution, rather than a measure, supported on K (one might say it is K-bandlimited in the broad sense). In particular, it is unbounded, so it does not approximate $\hat{\mu}(x-\lambda-\lambda')$ outside an interval of the order of size I_N.

Theorem 3.1. *There exist positive constants a_0, b_0 such that if $0 \leq a \leq a_0$ and $0 \leq b \leq b_0$ then there exists ε (depending on a, b and R) satisfying $0 < \varepsilon < 1$ such that for every N and $\lambda + \lambda' \in \Lambda$ with $\lambda \in \Lambda_N$ and $\lambda' \neq 0$ there exists a constant $c_{\lambda'}$ such that*

$$\hat{\mu}(x-\lambda-\lambda') = c_{\lambda'}(x-\lambda)\hat{\mu}(x-\lambda)(1+\varepsilon(x)) \tag{3–1}$$

with

$$|\varepsilon(x)| \leq \varepsilon \quad \text{for } x \in I_N.$$

Proof. Let N_0 denote the first nonzero digit of λ', so $N_0 \geq N$. For $k < N_0$ we see that $R^{-k}\lambda'$ is an even integer (divisible by 4 unless $R/2$ is odd and $k = N_0 - 1$). Thus

$$\prod_{k=0}^{N_0-1} \cos\frac{\pi}{2}\Big(\frac{x-\lambda-\lambda'}{R^k}\Big) = \pm \prod_{k=0}^{N_0-1} \cos\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big). \tag{3–2}$$

On the other hand $R^{-N_0}\lambda'$ is an odd integer, so

$$\begin{aligned}\cos\frac{\pi}{2}\Big(\frac{x-\lambda-\lambda'}{R^{N_0}}\Big) &= \pm\sin\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)\\ &= \pm\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)\cos\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big).\end{aligned} \tag{3–3}$$

For $k > N_0$ we simply use standard trigonometric identities to write

$$\begin{aligned}\cos\frac{\pi}{2}\Big(\frac{x-\lambda-\lambda'}{R^k}\Big) &= \cos\frac{\pi}{2}\frac{\lambda'}{R^k}\cos\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big)\\ &\quad\times\Big(1+\tan\Big(\frac{\pi}{2}\frac{\lambda'}{R^k}\Big)\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big)\Big).\end{aligned} \tag{3–4}$$

We therefore choose

$$c_{\lambda'} = \pm\frac{\pi}{2R^{N_0}}\prod_{k=N_0+1}^{\infty}\cos\Big(\frac{\pi}{2}\frac{\lambda'}{R^k}\Big).$$

Combining (3–2), (3–3) and (3–4) in the definition of $\hat{\mu}$ yields (3–1) with

$$\begin{aligned}1+\varepsilon(x) &= \frac{\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)}{\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)}\\ &\quad\times\prod_{k=N_0+1}^{\infty}\Big(1+\tan\Big(\frac{\pi}{2}\frac{\lambda'}{R^k}\Big)\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big)\Big).\end{aligned} \tag{3–5}$$

It remains to show that $\varepsilon(x)$ is not too large.

For $\lambda \in \Lambda_N$ we have $0 \leq \lambda \leq (R^N-1)/(R-1)$, so

$$|x-\lambda| \leq \max\Big(a+\frac{1}{R-1},\, b\Big)R^N.$$

We may choose a and b so that $a + 1/(R-1) \leq \frac{1}{2}$ and $b \leq \frac{1}{2}$. In that case

$$1 \leq \frac{\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)}{\frac{\pi}{2}\Big(\frac{x-\lambda}{R^{N_0}}\Big)} \leq \frac{4}{\pi},$$

with the worst case occuring when $N_0 = N$. To estimate the terms in the product on the right side of (3–5) we observe that $\tan\frac{\pi}{2}\big(\frac{x-\lambda}{R^k}\big)$ converges rapidly to zero, so the real difficulty arises when λ'/R^k is close to an odd integer. If we write $\lambda' = R^{N_0} + R^{N_0+k_1} + \cdots + R^{N_0+k_m}$ this happens exactly when $k = N_0 + k_j$, with

$$\begin{aligned}\lambda'/R^k = 1 &+ R^{k_{j-1}-k_j} + R^{k_{j-2}-k_j}\\ &+ \cdots + R^{k_1-k_j} + R^{-k_j} \pmod 2.\end{aligned}$$

This leads to the estimate

$$\left|\tan\frac{\pi}{2}\frac{\lambda'}{R^k}\right| \le \Big(\frac{\pi}{2}\big(R^{k_{j-1}-k_j}+R^{k_{j-2}-k_j}+\cdots+R^{k_1-k_j}+R^{-k_j}\big)\Big)^{-1}.$$

But $\left|\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big)\right| \le \frac{\pi}{4}R^{N-N_0-k_j}$, so

$$\left|\tan\Big(\frac{\pi}{2}\frac{\lambda'}{R^k}\Big)\tan\frac{\pi}{2}\Big(\frac{x-\lambda}{R^k}\Big)\right| \le \tfrac{1}{2}\big(R^{k_{j-1}}+R^{k_{j-2}}+\cdots+R^{k_1}+1\big)^{-1}.$$

The result follows by routine estimates. □

It is clear from the argument that the estimates on $\varepsilon(x)$ improve as R increases. We have not attempted to find the best estimate when $R = 4$, but it does not appear to be very good. It seems plausible that (3–1) is just the first in a sequence of improving approximations, but we will not attempt to pursue the matter here.

Figure 3 shows the sequence of graphs of $\hat{\mu}(x-\lambda)$ for $R = 4$, λ ranging over the first 16 elements of Λ, and x ranging over the interval $[-8, 8]$. The y-scale in the subunits of Figure 3 is adjusted so as to highlight the approximate periodicity of period 4 over λ (starting with $\lambda = 16$). The periodicity holds

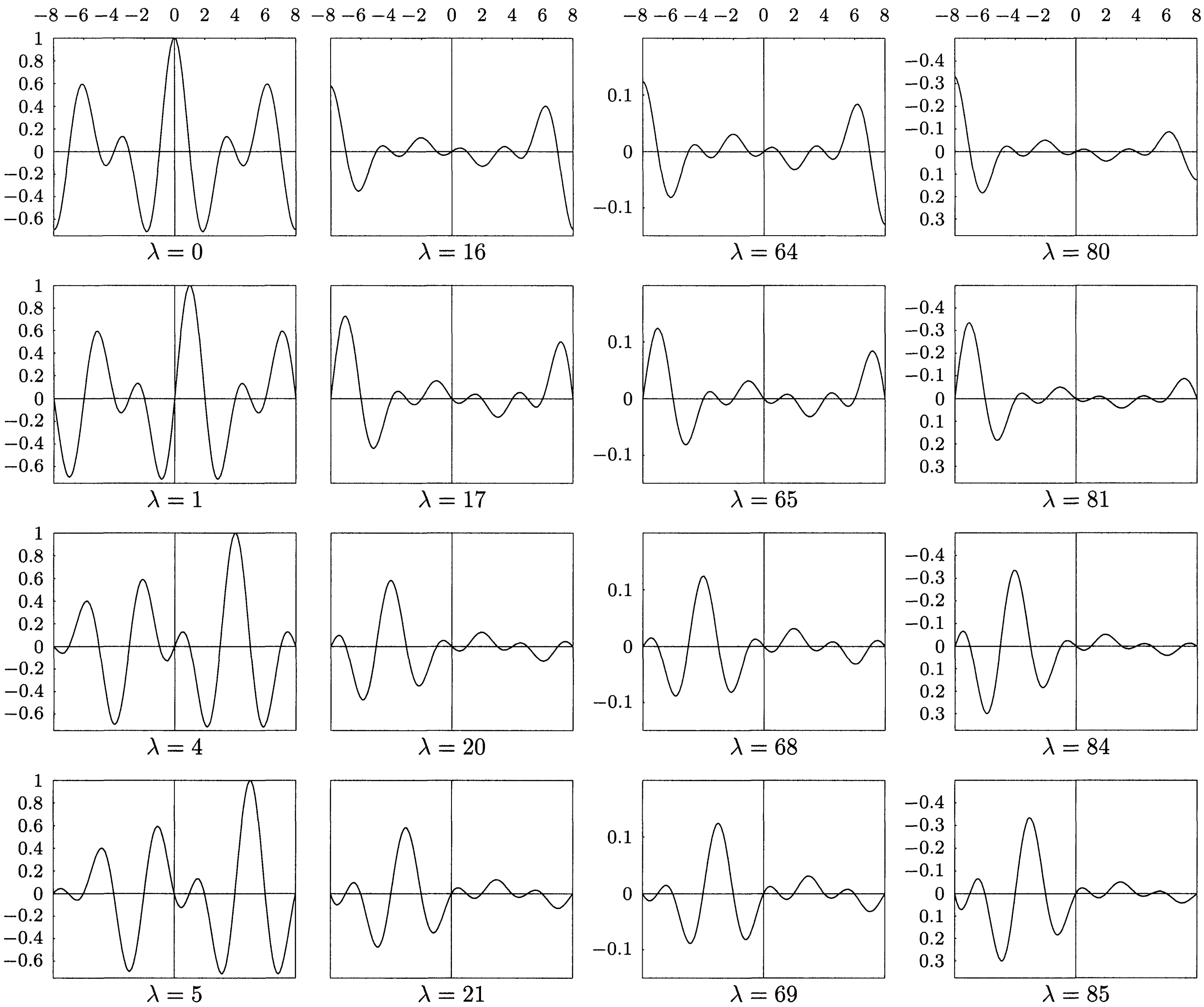

FIGURE 3. Graphs of the translates $\hat{\mu}(x-\lambda)$ of the sampling function for $R = 4$ and $\lambda = 0$, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85 (the first 16 elements of Λ), plotted on the interval $[-8, 8]$. The vertical scale of the graphs on the third and fourth columns is respectively 5 and -2 times the scale of the graphs on the first two.

only up to scaling, due to the different values of the coefficients $c_{\lambda'}$ in (3–1).

This approximate periodicity breaks down on the larger interval, as shown by Figure 4, which extends the plots to the interval $[-32, 32]$. Nonetheless, an approximate periodicity of period 8 can be recovered if one looks at further translates (see the section on Electronic Availability). Conversely, by concentrating on the even smaller interval $[-3, 3]$ one can see an approximate periodicity of period 2.

Figures 5 and 6 show the analogous graphs for $R = 6$.

4. THE SAMPLING FORMULA

As indicated in [Strichartz 2000a], the convergence of the sampling formula

$$F(x) = \sum_{\lambda \in \Lambda} F(\lambda)\hat{\mu}(x - \lambda) \tag{4–1}$$

is uniform if F is strongly K-bandlimited, meaning that

$$F(x) = \int e^{2\pi i x t} f(t)\, d\mu(t) \tag{4–2}$$

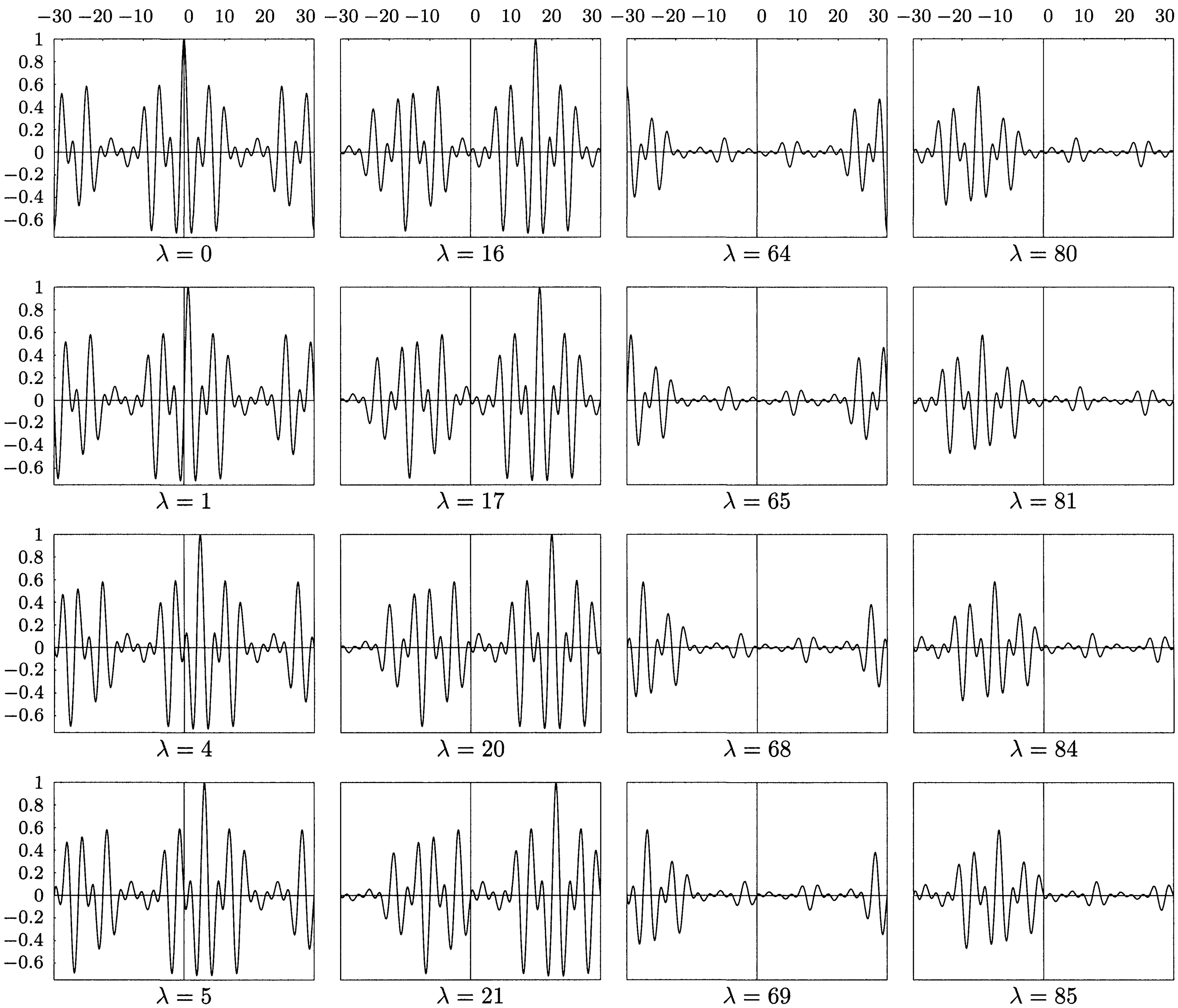

FIGURE 4. Graphs of the same translates $\hat{\mu}(x - \lambda)$ as in the preceding figure, plotted over the interval $[-32, 32]$. The vertical scale is the same for all plots.

for some $f \in L^2(\mu)$. This is equivalent to

$$\sum_{\lambda \in \Lambda} |F(\lambda)|^2 < \infty. \tag{4–3}$$

If we write

$$F_N(x) = \sum_{\lambda \in \Lambda_N} F(\lambda)\hat{\mu}(x-\lambda) \tag{4–4}$$

for the partial sum, then

$$F(x) - F_N(x) = \sum_{\lambda \in \Lambda \setminus \Lambda_N} F(\lambda)\hat{\mu}(x-\lambda),$$

which yields the estimate

$$|F(x) - F_N(x)| \le \Big(\sum_{\lambda \in \Lambda \setminus \Lambda_N} |F(\lambda)|^2 \Big)^{1/2}$$

because

$$\sum_{\lambda \in \Lambda} |\hat{\mu}(x-\lambda)|^2 = 1. \tag{4–5}$$

Thus the rate of convergence of (4–1) is a consequence of the decay rate of $\{F(\lambda)\}$. Unfortunately, we do not know any useful criteria for getting rapid decay of $\{F(\lambda)\}$.

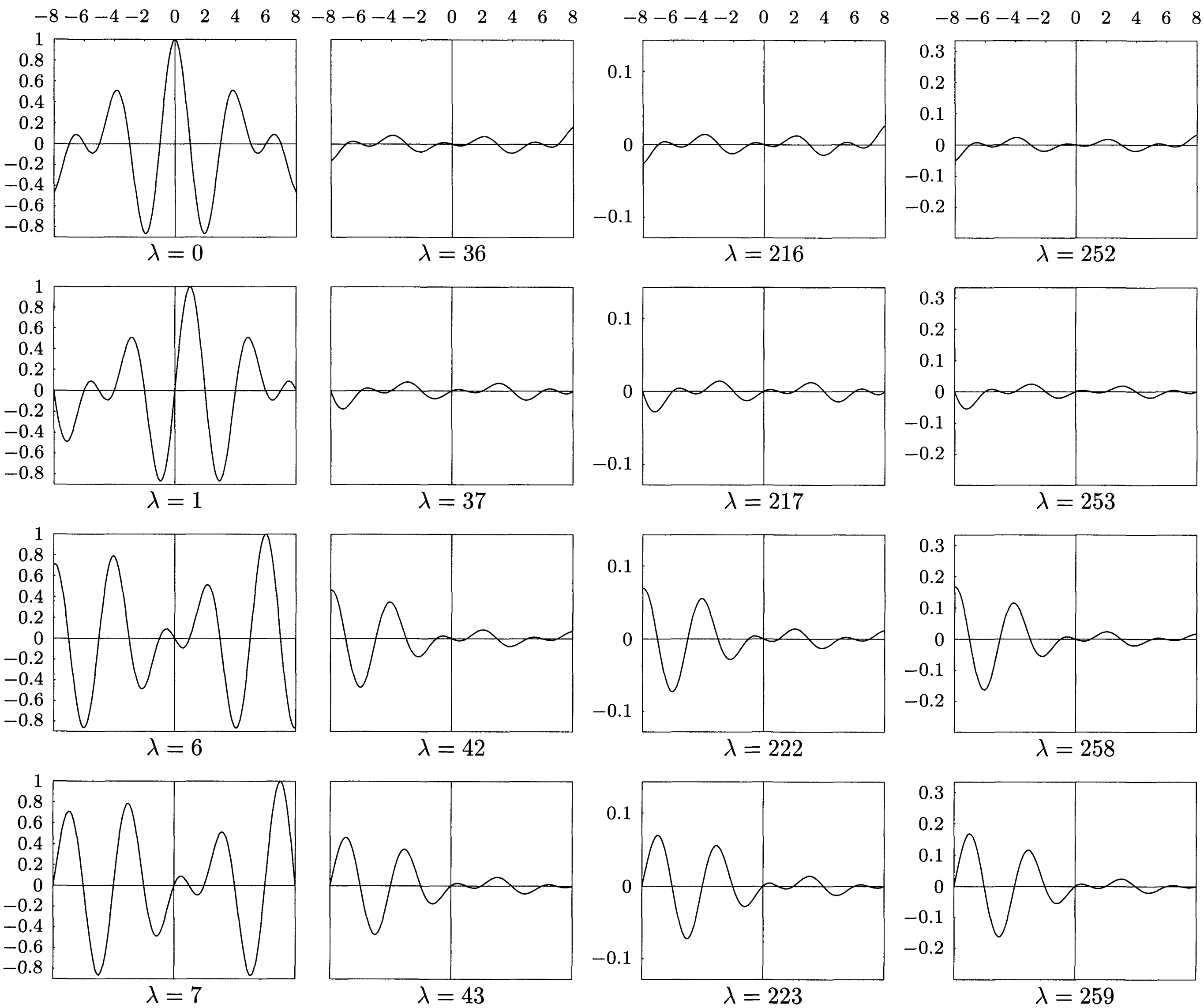

FIGURE 5. Graphs of the translates $\hat{\mu}(x-\lambda)$ of the sampling function for $R = 6$ and $\lambda = 0$, 1, 6, 7, 36, 37, 42, 43, 216, 217, 222, 223, 252, 253, 258, 259. on the interval $[-8, 8]$. The vertical scale of the graphs on the second and third columns is respectively 7 and 3 times the scale of the graphs on the first two.

We now show how the uniform convergence implies a generalization of Theorem 2.3 to all strongly K-bandlimited functions. Let a_F and b_F denote the sup and inf of such a function (it is not clear whether or not these values are actually attained in general). Let Λ_F denote the subset of the plane consisting of the x-axis together with the countable number of vertical line segments joining

$$(x, a_F g(x)) \quad \text{and} \quad (x, b_F g(x))$$

for x of the form p/R^k.

Theorem 4.1. *If F is any nonzero strongly K-bandlimited function, then the graphs of $F(R^n x)$ converge weakly to Γ_F as $n \to \infty$.*

Proof. It is easy to see that $b_F < 0 < a_F$ because K is disjoint from a neighborhood of zero. For any given $\varepsilon > 0$ we can choose N large enough that $|F(x) - F_N(x)| \leq \varepsilon$ uniformly, so by routine limiting arguments it suffices to prove the result for F_N. In other words, without loss of generality we may assume that the sum in (4–1) is finite.

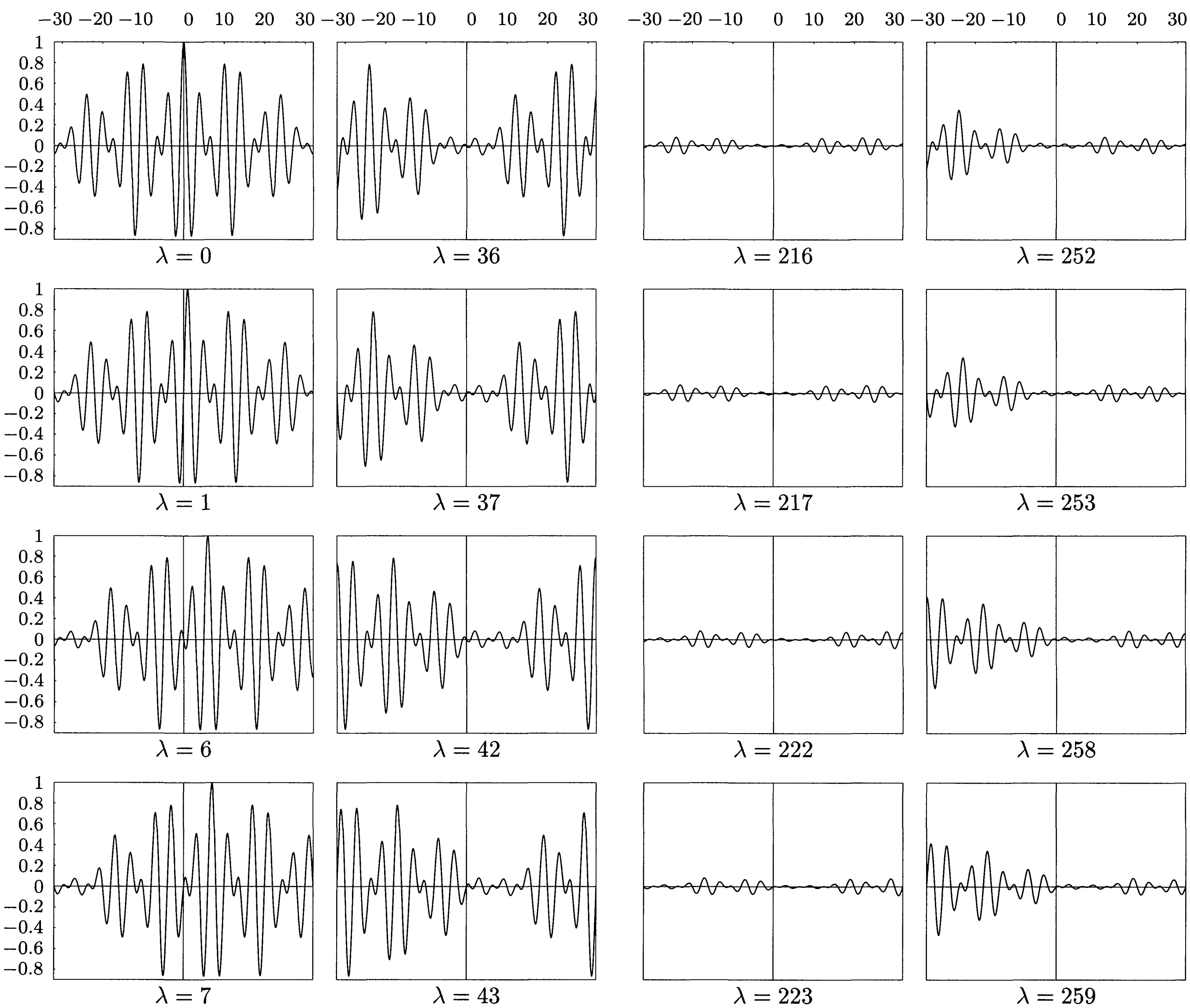

FIGURE 6. Graphs of the same translates $\hat{\mu}(x - \lambda)$ as in the preceding figure, plotted over the interval $[-32, 32]$. The vertical scale is the same for all plots.

Now if x is not of the form p/R^m, or x has this form with p odd,

$$F(R^n x) = \sum F(\lambda)\hat{\mu}(R^n(x-\lambda R^{-n})).$$

For n large enough, all the terms λR^{-n} are very small, and so we can show that $|F(R^n t)|$ can be made arbitrarily small for t close to x. On the other hand, for $x = p/R^n$ with p an even integer not divisible by R (or $p = 0$), we have by (2–5)

$$\begin{aligned} F(R^n(x+t)) &\\ &= \sum F(\lambda)\hat{\mu}(R^{n-m}p + R^n t - \lambda)\\ &= \frac{\sum F(\lambda)\hat{\mu}(R^n t-\lambda)\hat{\mu}(p+R^m t - R^{m-n}\lambda)}{\hat{\mu}(R^m t - R^{m-n}\lambda)}. \end{aligned}$$

When n is large enough $R^{m-n}\lambda$ is small for all λ in the sum. Similarly when t is small enough $R^m t$ is small. Thus $\hat{\mu}(p+R^m t-R^{m-n}\lambda) \approx \hat{\mu}(p)$ and$\hat{\mu}(R^m t - R^{m-n}\lambda) \approx 1$, so

$$\begin{aligned} F(R^n(x+t)) &\approx \sum F(\lambda)\hat{\mu}(R^n t-\lambda)\hat{\mu}(p)\\ &= F(R^n t) g(x) \end{aligned}$$

and $F(R^n t)$ can take on all values between a_F and b_F. The rest of the proof is the same as before. □

In other words, when viewed on a large scale, all strongly K-bandlimited functions look pretty much the same, except for the two vertical stretching factors a_F and b_F.

The simplest way to create interesting strongly K-bandlimited functions is to take translates of $\hat{\mu}$, as $\hat{\mu}(x-y)$ is given by (4–2) with $f(t) = e^{-2\pi i y t}$. The sampling formula (4–1) in this case is just

$$\hat{\mu}(x-y) = \sum_{\lambda\in\Lambda} \hat{\mu}(\lambda-y)\hat{\mu}(x-\lambda) \tag{4–6}$$

or equivalently,

$$\hat{\mu}(x+y) = \sum_{\lambda\in\Lambda} \hat{\mu}(x-\lambda)\hat{\mu}(y+\lambda). \tag{4–7}$$

This may be regarded as an addition formula for the sampling function. Note that (4–5) is the special case $x = y$ of (4–6).

Figure 7 shows the approximations F_N and the error for the function $F(x) = \hat{\mu}(x-10.3)$ when $R = 4$. The values $N = 3, 4, 5$ are shown, all on the interval $[-160, 400]$ (second, fourth, and last rows), and also on the interval $[-10\cdot 4^{N-3}, 25\cdot 4^{N-3}]$ (first and third rows). On the large interval the maximum error decreases from around 15% for $N = 3$ to around 6% for $N = 4$ to around 1.5% for $N = 5$. The smaller intervals show the behavior of the approximation on the sampling interval and its vicinity. The sampling interval increases roughly by a factor of R for each increase of N. For $N = 3$ it is $[0, 21]$, for $N = 4$ it is $[0, 85]$, and for $N = 5$ it is $[0, 341]$. The maximum error on the sampling interval decreases from around 4% for $N = 3$ to around 1% for $N = 4$ to around .5% for $N = 5$. The error increases considerably once we move away from the sampling interval. For example, when $N = 4$ the error jumps to around 3% near $x = -20$, whose distance to the sampling interval is comparable to the maximum distance to a sampling point within the sampling interval (near the midpoint $x = 42.5$). This seems to illustrate the intuition that the sampling formula should do a better job interpolating than extrapolating. In [Strichartz 2000a] it is shown that in this case we have slightly stronger decay of the sample values than the square summability in (4–3), namely the summability condition

$$\sum_{\lambda\in\Lambda} |F(\lambda)| < \infty. \tag{4–8}$$

Here we will give a more precise estimate that implies (4–8).

Lemma 4.2. *For $F(x) = \hat{\mu}(x-y)$, for any $\varepsilon > 0$ there exists c (depending on ε any y) such that*

$$|F(\lambda)| \le c\Big(\frac{\pi}{2}\Big(\frac{R}{R-1}+\varepsilon\Big)\Big)^m R^{-k_m} \tag{4–9}$$

where

$$\lambda = R^{k_1}+R^{k_2}+\cdots+R^{k_m}, \quad 0 \le k_1 < k_2 < \cdots < k_m.$$

Proof. To estimate $|F(\lambda)| = |\hat{\mu}(\lambda-y)|$ we only use the terms in (2–1) with $k = k_j$. Note that

$$\frac{\lambda-y}{R^{k_j}} \equiv \frac{R^{k_1}+\cdots+R^{k_{j-1}}-y}{R^{k_j}} + 1 \bmod 2,$$

so

$$\begin{aligned} \left|\cos\frac{\pi}{2}\Big(\frac{\lambda-y}{R^{k_j}}\Big)\right| &= \left|\sin\frac{\pi}{2}\Big(\frac{R^{k_1}+\cdots+R^{k_{j-1}}-y}{R^{k_j}}\Big)\right|\\ &\le \frac{\pi}{2}\left|\frac{R^{k_1}+\cdots+R^{k_{j-1}}-y}{R^{k_j}}\right|. \end{aligned} \tag{4–10}$$

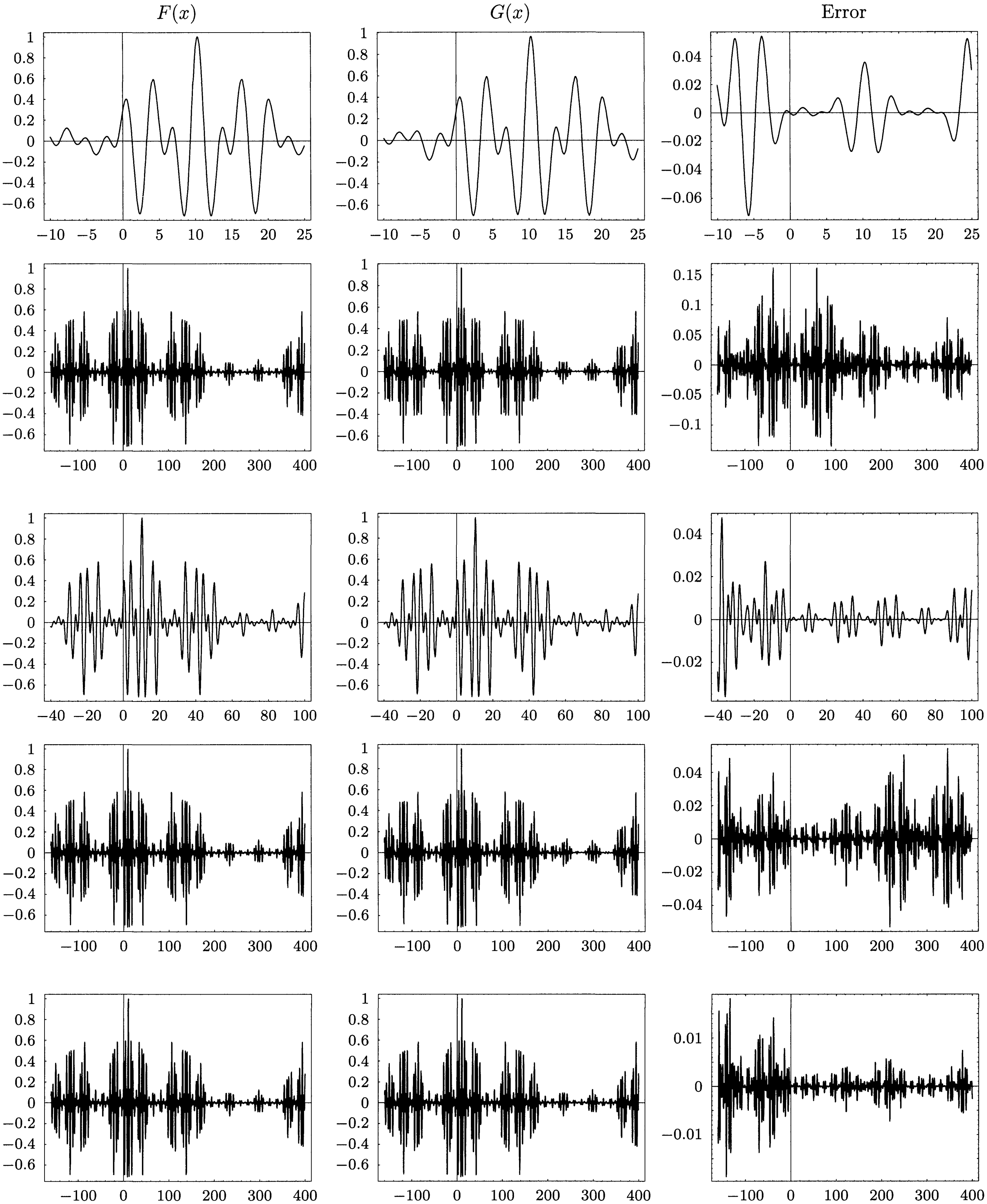

FIGURE 7. The sampling formula approximation and error for the function $F(x) = \hat{\mu}(x - 10.3)$ for $R = 4$. Top two rows: $G = F_3$, on the intervals $[-10, 25]$ and $[-160, 400]$. Next two rows: $G = F_4$ on the intervals $[-40, 100]$ and $[-160, 400]$. Last row: $G = F_5$ on the interval $[-160, 400]$.

Now for fixed y we can find N large enough so that if $k_{j-1} \geq N$ then

$$\left|R^{k_1} + \cdots + R^{k_{j-1}} - y\right| \leq \Big(\frac{R}{R-1} + \varepsilon\Big) R^{k_{j-1}} \tag{4–11}$$

(for $k_{j-1} < N$ we can also have (4–11) at the cost of a multiplicative constant, and there are at most N such). Using (4–10) and (4–11) we obtain

$$\begin{aligned} |F(\lambda)| &\leq \prod_{j=1}^{m} \left|\cos\frac{\pi}{2}\Big(\frac{\lambda - y}{R^{k_j}}\Big)\right| \\ &\leq c\Big(\frac{\pi}{2}\Big(\frac{R}{R-1} + \varepsilon\Big)\Big)^m \frac{1}{R^{k_1}} \frac{R^{k_1}}{R^{k_2}} \cdots \frac{R^{k_{m-1}}}{R^{k_m}}, \end{aligned}$$

which establishes (4–9). □

To see that (4–9) implies (4–8) observe that if we fix $k_m = k$, then there are exactly $\binom{k}{m}$ choices of λ. Thus the sum of these $|F(\lambda)|$ is estimated by

$$c\binom{k}{m}\Big(\frac{\pi}{2}\Big(\frac{R}{R-1} + \varepsilon\Big)\Big)^m R^{-k}.$$

When we sum over $m \leq k$ (with k still fixed) the estimate is

$$c\Big(1 + \frac{\pi}{2}\Big(\frac{R}{R-1} + \varepsilon\Big)\Big)^k R^{-k}.$$

Finally, the sum over k is estimated by a convergent geometric series because

$$1 + \frac{\pi}{2}\Big(\frac{R}{R-1} + \varepsilon\Big) < R$$

for ε small enough.

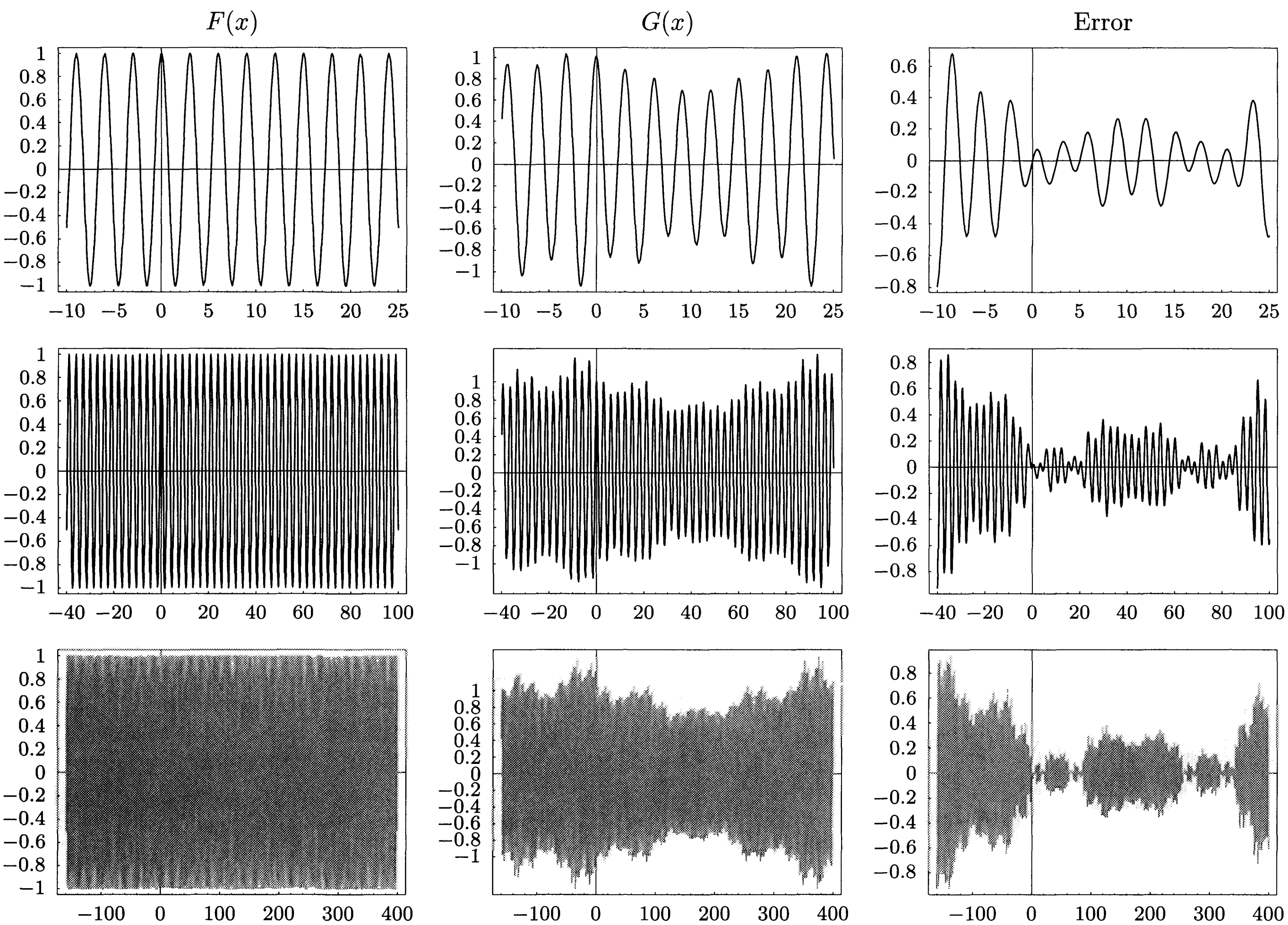

FIGURE 8. The sampling formula approximation and error for the function $F(x) = \cos(2\pi x/3)$ with $R = 4$. Top: $G = F_3$ on the interval $[-10, 25]$. Middle: $G = F_4$ on the interval $[-40, 100]$. Bottom: $G = F_5$ on the interval $[-160, 400]$.

Note that (4–9) is not a very rapid decay rate. It is slightly worse than $F(\lambda) = O(\lambda^{-1})$, so the convergence in (4–8) is due largely to the fact that the set Λ of samples is quite thin.

At the other extreme of K-bandlimited functions that are not strongly K-bandlimited, we have the exponentials $e^{2\pi ixy}$ for a frequency y in K. Since K is chosen to be symmetric about the origin, we can replace the exponential by sines and cosines. In this case we have the analog of (4–2) with f being replaced by a discrete measure supported on $\{\pm y\}$. We have no decay for the sample values $F(\lambda)$, and the convergence of F_N to F is not uniform over $\mathbb{R}$. However, as shown in [Strichartz 2000a], it is uniform on any compact set.

In Figure 8 we show the approximations and errors for $F(x) = \cos(2\pi x/3)$, $R = 4$, and the same in Figure 9 for $F(x) = \cos(3\pi x/5)$, $R = 6$ (see the web site for more examples). We observe that the error is quite substantial, even in the sampling interval. The most striking feature of this data is that the shape of the error appears to converge. That is, the graphs of $F(R^N x) - F_N(R^N x)$ appear to be converging to a limit set. The limit set appears to have a nonempty interior and a fractal boundary, and is of an entirely different nature than the limit sets seen for strongly K-bandlimited functions. We do not have any explanation for this apparent behavior.

5. A MODIFIED SAMPLING FORMULA

Suppose we only have access to values of the function F on a fixed interval I, containing Λ_N. Then we could use the sampling formula (4–1) only to the extent of finding the approximation F_N given by (4–4). If the error of this approximation is deemed too large, there seems to be nothing to be done, since we cannot increase N to improve accuracy. However, the results of Section 3 suggest that all the remaining terms in (4–1) may be approximated on I by multiplies of the functions $(x-\lambda)\hat{\mu}(x-\lambda)$ for $\lambda \in \Lambda_N$. This suggests that we look for an approximation of the form

$$\sum_{\lambda\in\Lambda_N} \big(F(\lambda)\hat{\mu}(x-\lambda) + a_\lambda(x-\lambda)\hat{\mu}(x-\lambda)\big).$$

We now propose two variants of this idea, called the *symmetric* and *nonsymmetric modified sampling formula*, in which the coefficients a_λ are obtained by sampling F at some points in the vicinity of I. The price to be paid for trying to get more accuracy locally is that we wreak havoc globally: the modified sampling formulas must break down entirely once we move outside a neighborhood of I.

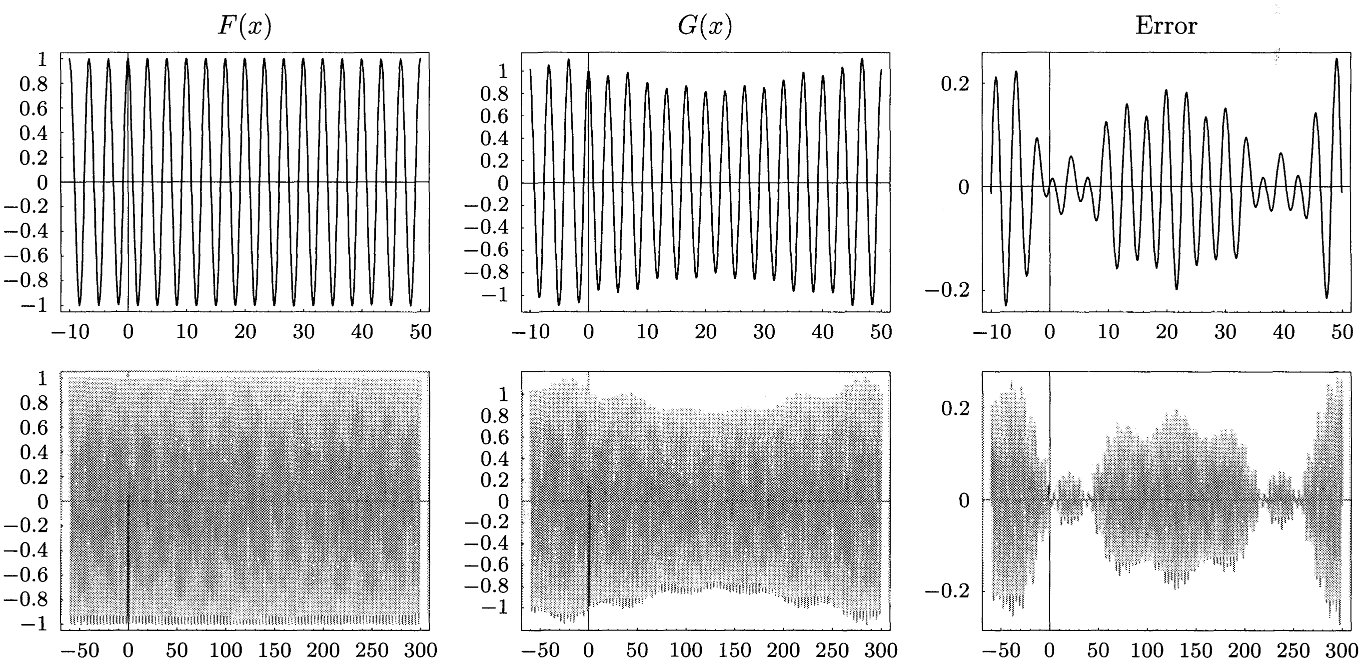

FIGURE 9. The sampling formula approximation and error for the function $F(x) = \cos(3\pi x/5)$ with $R = 6$. Top: $G = F_3$ on the interval $[-10, 50]$. Bottom: $G = F_4$ on the interval $[-60, 300]$.

For the first variant, the nonsymmetric one, we will sample on Λ_N and $\Lambda_N + 2R^{N-1}$. Note that

$$\hat{\mu}(\lambda - \lambda' - 2R^{N-1}) = 0$$

if $\lambda \neq \lambda'$ are both in Λ_N, since if λ and λ' first differ at R^k for $k < N-1$ then

$$\cos\frac{\pi}{2}\Big(\frac{\lambda - \lambda' - 2R^{N-1}}{R^k}\Big) = 0$$

as before, while if they first differ at R^{N-1} then

$$\cos\frac{\pi}{2}\Big(\frac{\lambda - \lambda' - 2R^{N-1}}{R^{N-1}}\Big) = \cos\frac{\pi}{2} \text{ or } \cos\frac{3\pi}{2}.$$

Thus

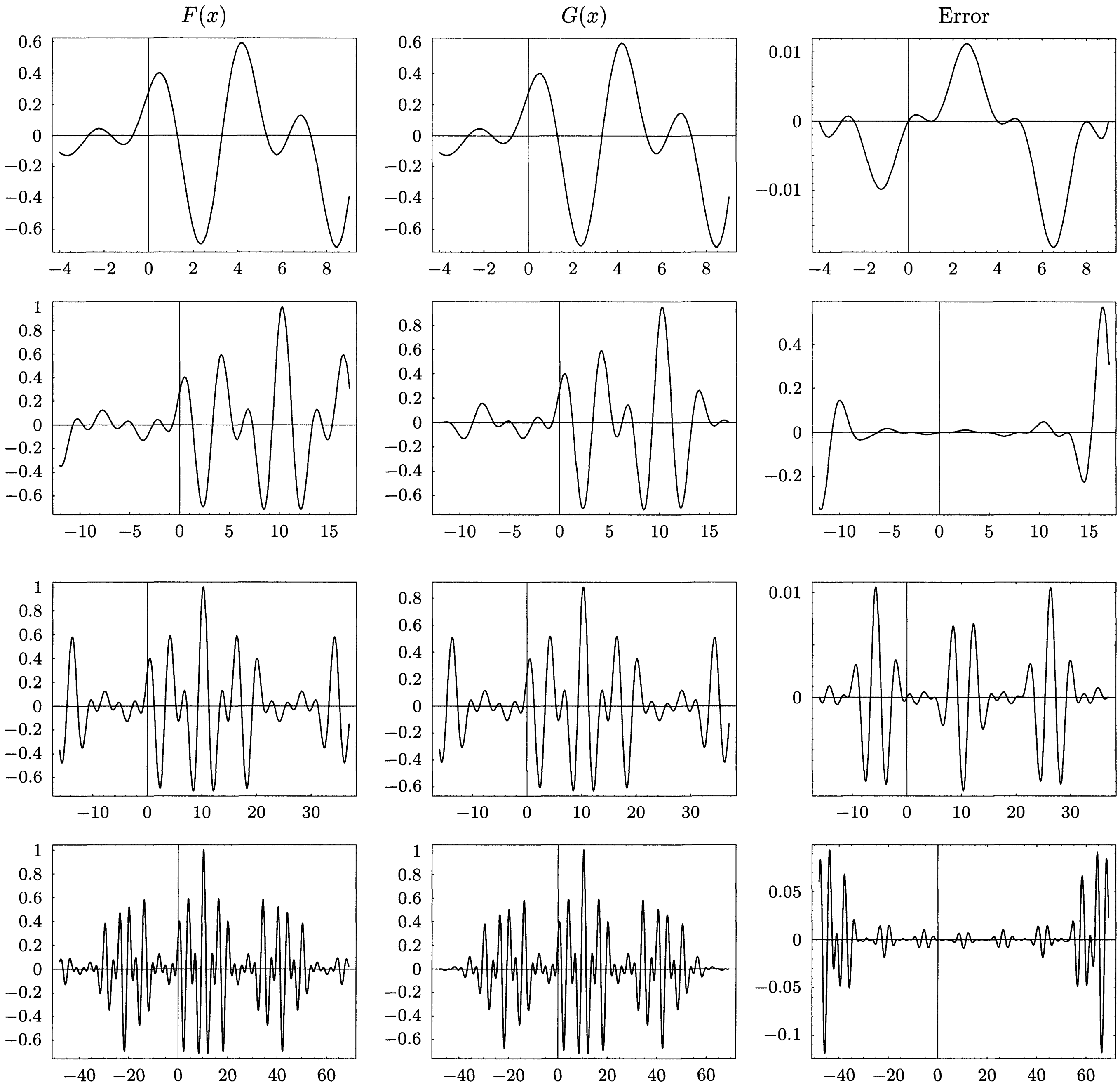

FIGURE 10. The symmetric modified sampling formula for the function $F(x) = \hat{\mu}(x - 10.3)$ with $R = 4$. Top two rows: $G = G_2''$ on the sampling interval $[-4, 9]$ and the slightly larger interval $[-13, 18]$. Bottom rows: $G = G_3''$ on the sampling interval $[-16, 37]$ and the slightly larger interval $[-52, 73]$.

$$G'_N(x) = \sum_{\lambda\in\Lambda_N} F(\lambda)\hat{\mu}(x-\lambda) + \sum_{\lambda\in\Lambda_N}\Big(\frac{F(\lambda+2R^{N-1})}{2R^{N_1}\hat{\mu}(2)} - \frac{F(\lambda)}{2R^{N-1}}\Big)(x-\lambda)\hat{\mu}(x-\lambda)$$

is a strongly K-bandlimited function that agrees with F exactly on $\Lambda_N \cup (\Lambda_N + 2R^{N-1}) = \Lambda'_N$. We call this the nonsymmetric modified sampling function because the original sampling set Λ_N forms the left half of Λ'_N.

To get the symmetric version we translate Λ'_N by $-R^{N-1}$ to form $\Lambda''_N = (\Lambda_N - R^{N-1}) \cup (\Lambda_N + R^{N-1})$. Since $\Lambda_N = \Lambda_{N-1} \cup (\Lambda_{N-1} + R^{N-1})$, we have

$$\Lambda''_N = (\Lambda_{N-1} - R^{N-1}) \cup \Lambda_N \cup (\Lambda_{N-1} + 2R^{N-1}),$$

so Λ_N is situated symmetrically in the center of Λ''_N. Again it is easy to verify that

$$G''_N(x) = \sum_{\lambda\in\Lambda_N} F(\lambda)\hat{\mu}(x-\lambda) + \sum_{\lambda\in\Lambda_{N-1}}\Big(\frac{F(\lambda+2R^{N-1})}{2R^{N-1}\hat{\mu}(2)} - \frac{F(\lambda)}{2R^{N-1}}\Big)(x-\lambda)\hat{\mu}(x-\lambda) + \sum_{\lambda\in\Lambda_{N-1}}\Big(\frac{F(\lambda+R^{N-1})}{2R^{N-1}} - \frac{F(\lambda-R^{N-1})}{2R^{N-1}\hat{\mu}(2)}\Big) \times (x-\lambda-R^{N-1})\hat{\mu}(x-\lambda-R^{N-1})$$

agrees with $F(\lambda)$ on Λ''_N.

We have been unable to prove estimates for the accuracy of the modified sampling formulas, but we have some experimental evidence. Note that G'_N and G''_N have the same number of terms as F_{N+1}. However, the width of Λ'_N and Λ''_N is greater than Λ_N but smaller than Λ_{N+1}. In Figures 10 and 11 we show the performance of the symmetric and nonsymmetric modified sampling formulas on the function $F(x) = \hat{\mu}(x-10.3)$ with $R=4$ that may be compared with the standard sampling formula in Figure 7. As expected, the error blows up rapidly as we move outside the sampling interval. Inside the sampling interval, the error is about 2% for $N=2$ and 1% for $N=3$ for the symmetric version, and about 8% for $N=2$ for the nonsymmetric verison. The nonsymmetric version does not perform as well as the standard sampling formula, and the improvement in the symmetric version is not dramatic.

In Figure 12 we show the performance of the symmetric modified sampling formula for the function $F(x) = \cos(2\pi x/3)$ with $R=4$, to be compared with Figure 8. The error is around 2% in the sampling interval for both $N=2$ and 3 — a dramatic improvement over the 20% error in Figure 8. Moreover, the visual appearance of G''_2 and G''_3 on the

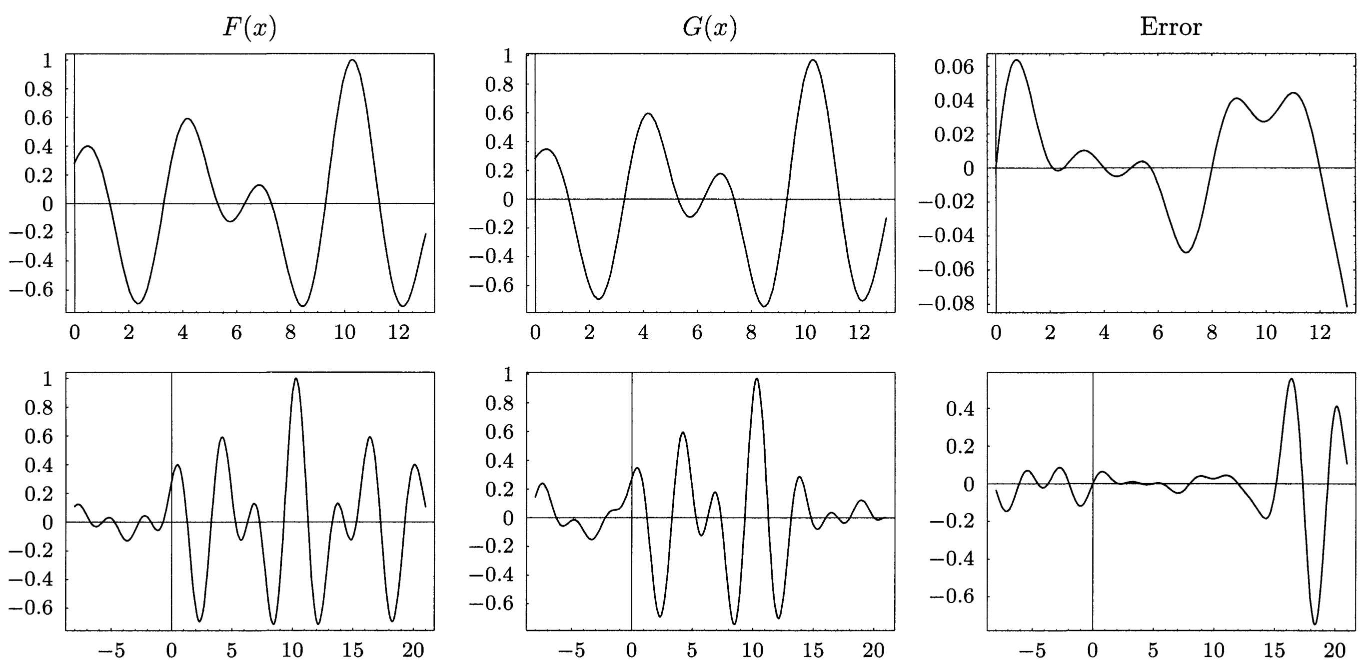

FIGURE 11. The nonsymmetric modified sampling formula for the function $F(x) = \hat{\mu}(x-10.3)$ with $R=4$. Here $G = G'_2$ on the sampling intervals $[0, 13]$ and $[-9, 21]$.

sampling interval is acceptably close to that of F. Figures 13 and 14 show the performance of both the symmetric and nonsymmetric modified sampling formula for $F(x) = \cos(3\pi x/5)$ with $R = 6$, for $N = 2$. In both cases we again have dramatic improvement over Figure 9. The error is about 0.5% for the symmetric version and 3% for the nonsymmetric version on the sampling interval.

This limited evidence suggests that the symmetric version is better than the nonsymmetric version. For strongly K-bandlimited functions there does not seem to be much point to using this approach because of its poor behavior outside the sampling interval. But for K-bandlimited functions with more singular Fourier transform, the symmetric modified sampling formula provides an attractive alternative.

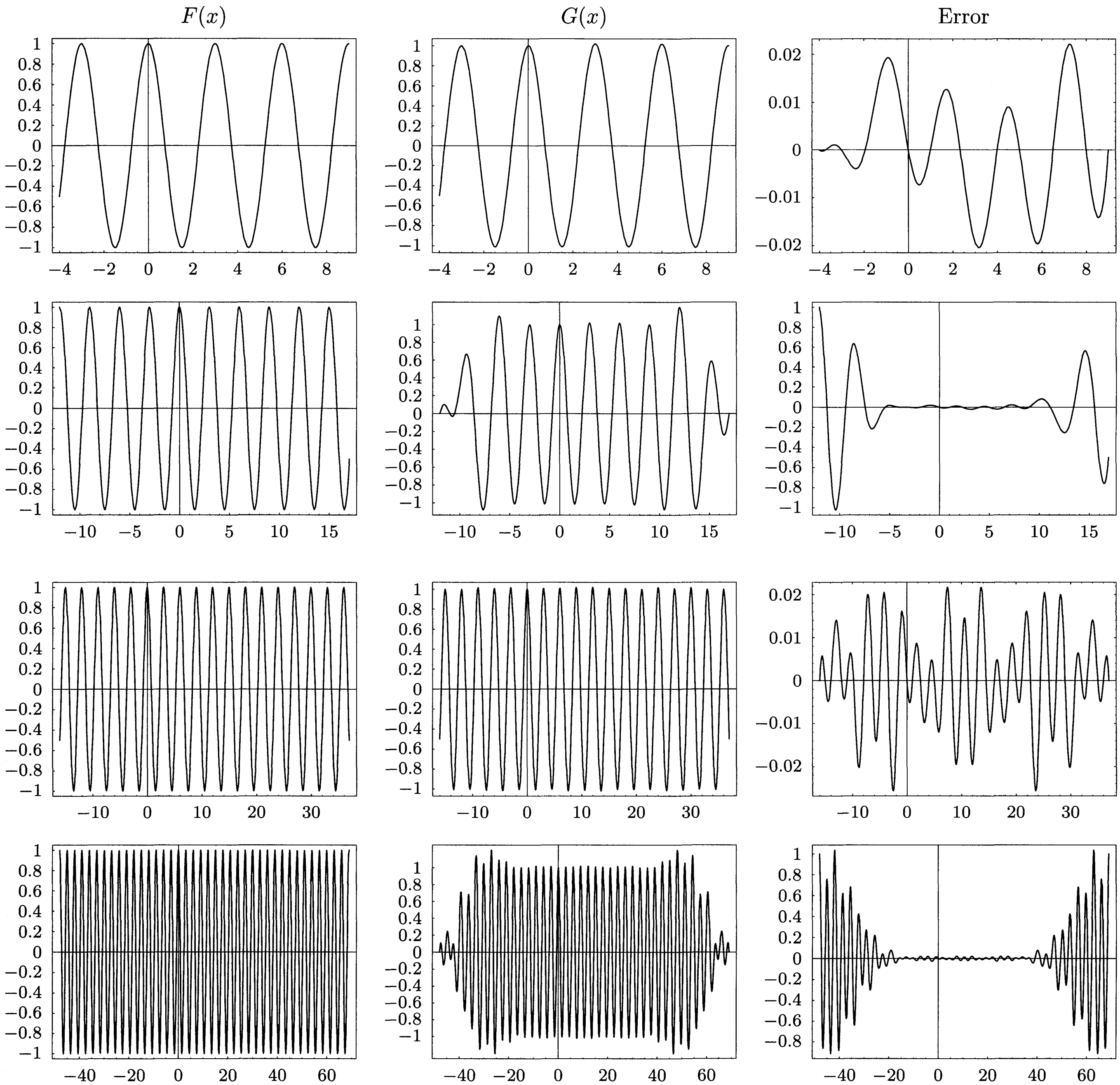

FIGURE 12. The symmetric modified sampling formula for the function $F(x) = \cos(2\pi x/3)$ with $R = 4$. Top two rows: $G = G_2''$ on the sampling interval $[-4, 9]$ and the slightly larger interval $[-13, 18]$. Bottom rows: $G = G_3''$ on the sampling interval $[-16, 37]$ and the slightly larger interval $[-52, 73]$.

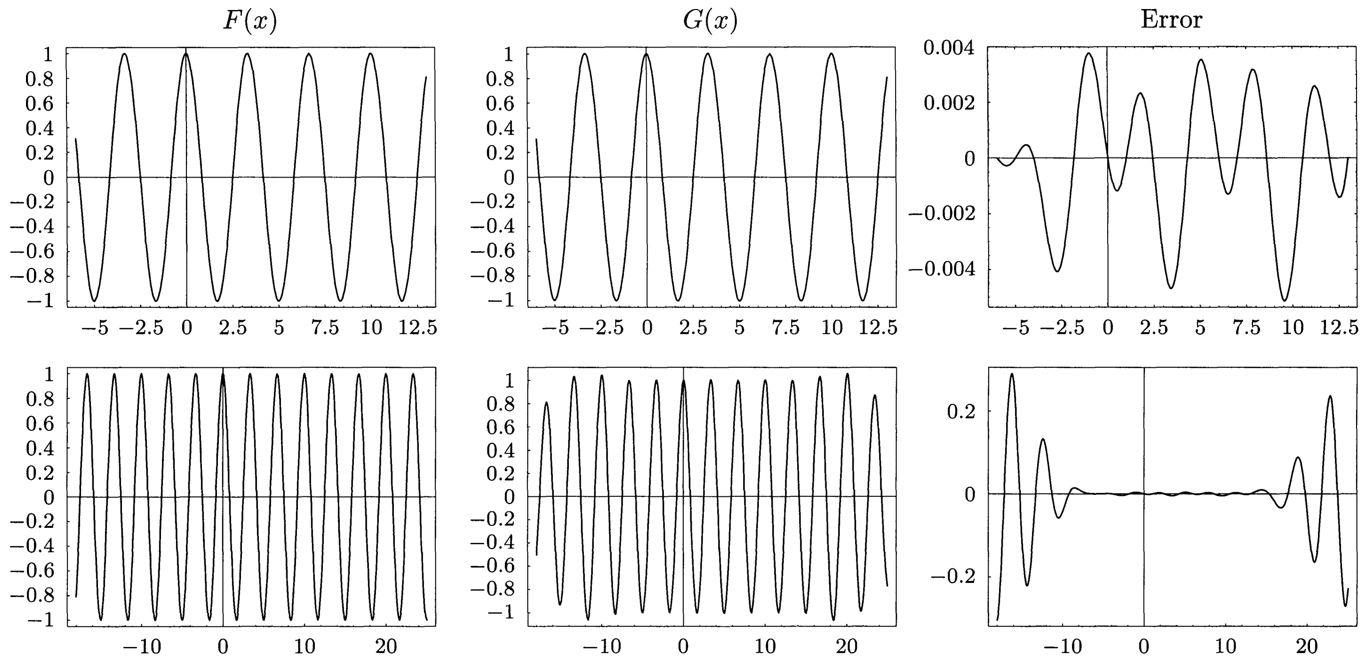

FIGURE 13. The symmetric modified sampling formula for the function $F(x) = \cos(3\pi x/5)$ with $R = 6$. Here $G = G_2''$ on the sampling intervals $[-6, 13]$ and $[-18, 25]$.

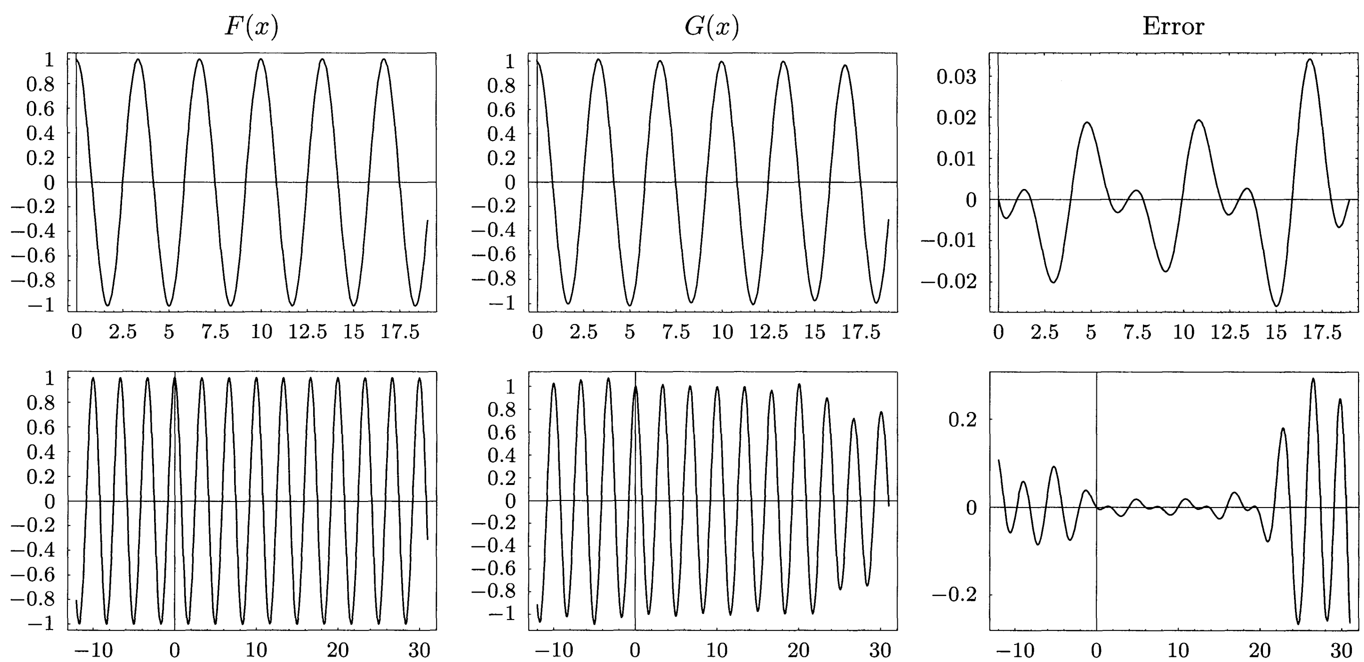

FIGURE 14. The nonsymmetric modified sampling formula for the function $F(x) = \cos(3\pi x/5)$ with $R = 6$. Here $G = G_2''$ on the sampling intervals $[0, 19]$ and $[-12, 31]$.

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ELECTRONIC AVAILABILITY

The web site http://mathlab.cit.cornell.edu/~tillman contains a wealth of experimental data related to the investigations in this article, including many more examples.

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