On Some Inequalities Concerning $\pi(x)$

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CONTENTS

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We investigate the inequalities $\pi(M+N) \leq a\pi(M/a)+\pi(N)$ and $\pi(M+N) \leq a\,(\pi(M/a)+\pi(N/a))$ with $a \geq 1$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\pi(x)$, as usual, denote the number of primes not exceeding x. Further by M, N, K and x, y we mean, respectively, positive integers and positive real numbers.

The conjecture that

$$\pi(M+N) \leq \pi(M)+\pi(N) \qquad (1\text{–}1)$$

for $M, N \geq 2$ takes its origin from Hardy and Littlewood [Hardy and Littlewood 23]. There are many results concerning this conjecture of which we will mention a few. Schinzel and Sierpinski [Schinzel and Sierpinski 58] (see also [Schinzel 61]) proved the inequality (1–1) for $2 \leq \min(M,N) \leq 146$ and from [Gordon and Rodemich 98] it follows that inequality (1–1) is valid in a wider region,

$$2 \leq \min(M,N) \leq 1731. \qquad (1\text{–}2)$$

Dusart [Dusart 98, Theorem 2.6] obtained the result that if $x \leq y \leq \frac{7}{5}x \log x \log\log x$, then

$$\pi(x+y) \leq \pi(x)+\pi(y).$$

However, in general it is believed that (1–1) is not valid, as Hensley and Richards [Hensley and Richards 74] have shown that this inequality is incompatible with another Hardy-Littlewood conjecture, the so called

Prime *k*-tuples conjecture. *Let $b_1 < b_2 < \ldots < b_k$ be a set of integers, such that for each prime p, there is some congruence class (mod p) which contains none of the integers b_i. Then there exist infinitely many integers $n > 0$ for which all of the numbers $n+b_1, \ldots, n+b_k$ are prime.*

More precisely, Hensley and Richards [Hensley and Richards 74], under prime k-tuples conjecture, proved that for $x \geq x_0$,

$$\limsup_{y\to\infty}(\pi(y+x)-\pi(y))-\pi(x) \geq (\log 2-\epsilon)\frac{x}{\log^2 x}.$$

From this it follows easy, that the inequality

$$\pi(M+N) \leq a\pi\left(\frac{M}{a}\right)+\pi(N) \tag{1–3}$$

is not valid for $1 \leq a < 2$. Under the same assumption, Clark and Jarvis [Clark and Jarvis 01] showed that it is also not valid for $a = 2$.

The inequality

$$\pi(M+N) \leq 2\pi(M)+\pi(N) \quad \text{for } M \geq 1, N \geq 2,$$

proved by Montgomery and Vaughan [Montgomery and Vaughan 73], suggests some a for which (1–3) is satisfied.

Theorem 1.1. *Let M and N be integers. If $a \geq \sqrt{M}$, then for $\frac{M}{a} \geq 3$ and $N \geq 1$,*

$$\pi(M+N) \leq a\pi\left(\frac{M}{a}\right)+\pi(N).$$

If $a \geq 2\sqrt{M}$, then this inequality is true for $\frac{M}{a} \geq 2$ and $N \geq 1$.

For $M \geq N$, a much smaller coefficient a can be chosen in the inequality (1–3). Panaitopol [Panaitopol 00] proved that for $M \geq N \geq 2$ and $M \geq 6$,

$$\pi(M+N) \leq 2\pi\left(\frac{M}{2}\right)+\pi(N).$$

Theorem 1.2. *If $M \geq N \geq 7$ are integers, then*

$$\pi(M+N) \leq 1.11\pi\left(\frac{M}{1.11}\right)+\pi(N).$$

The proof of Theorem 1.2 requires some computer calculations; we also make use of Dusart's evaluations [Dusart 98, Dusart 99] for the prime counting function:

$$\pi(x) \geq \frac{x}{\log x - 1}, \qquad x \geq 5393, \tag{1–4}$$

$$\pi(x) \leq \frac{x}{\log x - 1.1}, \qquad x \geq 60184, \tag{1–5}$$

$$\pi(x) \geq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\right), \qquad x \geq 32299, \tag{1–6}$$

$$\pi(x) \leq \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\right), \qquad x \geq 355991. \tag{1–7}$$

It is easy to obtain the symmetric version of Theorem 1.2:

Corollary 1.3. *If $M, N \geq 13$ are integers, then*

$$\pi(M+N) \leq 1.11\pi\left(\frac{M}{1.11}\right)+1.11\pi\left(\frac{N}{1.11}\right).$$

Udrescu [Udrescu 75] has proved that (1–1) is 'ϵ-exact,' i.e., that for any $\epsilon > 0$ and any $x, y \geq 17$ with $x + y \geq 1 + e^{4(1+1/\epsilon)}$,

$$\pi(x+y) \leq (1+\epsilon)(\pi(x)+\pi(y)).$$

Using estimates (1–6), (1–7) we obtain

Theorem 1.4. *For any $0 < \epsilon < 1$ and any $x, y \geq 32299$ with $x + y \geq e^{\frac{3}{4(\epsilon-\epsilon^2/2)}+13}$,*

$$\pi(x+y) \leq (1+\epsilon)\left(\pi\left(\frac{x}{1+\epsilon}\right)+\pi\left(\frac{y}{1+\epsilon}\right)\right).$$

2. PROOFS OF THE THEOREMS

To prove Theorem 1.1, we first obtain several auxiliary inequalities.

Lemma 2.1. *Let x be a real number and $c > b \geq 1$. Then for $\frac{x}{c} > e^{\frac{4}{\log^2 \frac{c}{b}}}$,*

$$b\pi\left(\frac{x}{b}\right) < c\pi\left(\frac{x}{c}\right).$$

Proof: The lemma follows immediately from the following result of Panaitopol [Panaitopol 00]: If $a > 1$ and $x > e^{4(\log a)^{-2}}$ then $\pi(ax) < a\pi(x)$. □

Lemma 2.2. *Let M be an integer. If $1 \leq a \leq 12$, $\frac{\sqrt{M}}{a} \geq 3$ and $M \leq 1731$, then*

$$\pi(M) \leq a\sqrt{M}\pi\left(\frac{\sqrt{M}}{a}\right). \tag{2–1}$$

The latter inequality is also true for $2 \leq a \leq 12$, $\frac{\sqrt{M}}{a} \geq 2$ and $M \leq 1731$.

Proof: Let $b \geq 1$, $c \geq 0$ and $[x]$ denotes the greatest integer not exceeding x. If

$$\pi(M) \leq b\sqrt{M}\pi\left(\frac{\sqrt{M}}{b+c}\right),$$

then the inequality (2–1) is valid for $a \in [b, b+c]$. Hence in order to prove the lemma, we check the following inequalities with a computer:

$$\pi(i) \le (1+0.21j)\sqrt{i}\,\pi\left(\max\left(\frac{\sqrt{i}}{1+0.21j+0.21}, 3\right)\right)$$

for $j = 0, 1, \dots, 5$; $i = \left[3^2(1+0.21j)^2\right] + 1, \dots, 1731$ and

$$\pi(i) \le (2+0.091j)\sqrt{i}\,\pi\left(\max\left(\frac{\sqrt{i}}{2+0.091j+0.091}, 2\right)\right)$$

for $j = 0, 1, \dots, 121$; $i = \left[2^2(1+0.091j)^2\right] + 1, \dots, 1731$. This proves the lemma. □

Lemma 2.3. *Let M be an integer. If $2.44 \le a \le 4$ and $\frac{\sqrt{1720}}{a} \le \frac{\sqrt{M}}{a} \le \min\left(17, e^{\frac{4}{\log^2 a}}\right)$, then*

$$\frac{2M}{\log M} \le a\sqrt{M}\pi\left(\frac{\sqrt{M}}{a}\right). \tag{2–2}$$

Proof: The proof is analogous to the proof of Lemma 2.2. Here we check the inequalities

$$\frac{2i}{\log i} \le (2.44+j)\sqrt{i}\,\pi\left(\frac{\sqrt{i}}{2.44+j+1}\right)$$

where $j = 0, 1$;
$i = 1720, \dots, \min\left((2.44+j)^2 17^2, \left[(2.44+j)^2 e^{\frac{8}{\log^2(2.44+j)}}\right]\right)$. □

Proof of Theorem 1.1. Montgomery and Vaughan [Montgomery and Vaughan 73] have shown that

$$\pi(M+N) - \pi(N) \le \frac{2M}{\log M} \quad \text{for } M \ge 2,\ N \ge 1. \tag{2–3}$$

Then, in view of the inequality ([Rosser and Shoenfeld 62])

$$\pi(x) > \frac{x}{\log x} \quad \text{for } x \ge 17, \tag{2–4}$$

if $d \ge 1$, then for $M \ge 17^2 d^2$, $N \ge 1$,

$$\pi(M+N) - \pi(N) \le \frac{M}{\log\frac{\sqrt{M}}{d}} < d\sqrt{M}\,\pi\left(\frac{\sqrt{M}}{d}\right) \tag{2–5}$$

By (1–2) and Lemma 2.2, for $1 \le d \le 12$, $\frac{\sqrt{M}}{d} \ge 3$, $M \le 1731$ and for $2 \le d \le 12$, $\frac{\sqrt{M}}{d} \ge 2$, $M \le 1731$,

$$\pi(M+N) - \pi(N) \le \pi(M) < d\sqrt{M}\,\pi\left(\frac{\sqrt{M}}{d}\right). \tag{2–6}$$

From (2–5) and (2–6), since $17^2 d^2$ is less than 1731 if $1 \le d \le 2.44$, we prove the theorem for $\sqrt{M} \le a \le 2.44\sqrt{M}$.

By Lemma 2.1, we obtain

$$\sqrt{M}\pi(\sqrt{M}) < d\sqrt{M}\,\pi\left(\frac{\sqrt{M}}{d}\right) \quad \text{for } \frac{\sqrt{M}}{d} > e^{\frac{4}{\log^2 d}}.$$

We have already proved that

$$\pi(M+N) - \pi(N) \le \sqrt{M}\,\pi\left(\sqrt{M}\right) \text{ for } \sqrt{M} \ge 3,\ N \ge 1.$$

From this and (2–5), (2–6), (2–3) and Lemma 2.3, since $e^{\frac{4}{\log^2 d}} \le \frac{\sqrt{1731}}{d}$ if $d \ge 4$, we obtain the theorem for the remaining case $a > 2.44\sqrt{M}$. □

The next two lemmas will be useful in the proof of Theorem 1.2.

Lemma 2.4. *If $x \ge y \ge 5393$ and $x + y \ge 60184$, then*

$$\pi(x+y) < 1.11\pi\left(\frac{x}{1.11}\right) + \pi(y).$$

Proof: From (1–4) and (1–5) we have

$$\begin{aligned}
&(1+a)\pi\left(\frac{x}{1+a}\right) + \pi(y) - \pi(x+y) \\
&\ge x\frac{\log\left(1+\frac{y}{x}\right) + \log(1+a) - 0.1}{\left(\log\frac{x}{1+a} - 1\right)\left(\log(x+y) - 1.1\right)} \\
&\quad + y\frac{\log\left(1+\frac{x}{y}\right) - 0.1}{\left(\log y - 1\right)\left(\log(x+y) - 1.1\right)} > 0
\end{aligned}$$

when $a \ge 0.106$. □

Lemma 2.5. *If $M \ge 619\,901$, then*

$$1.11\pi\left(\frac{M}{1.11}\right) > \pi(M + 5393).$$

Proof: Most of the calculations below were made using a computer. For $619\,901 \le M < 1\,040\,000$, we check the lemma directly. For the remaining range we will use P. Dusart's inequalities for the prime counting function. Let us define

$$f(x) := \frac{x}{\log\frac{x}{1.11}}\left(1 + \frac{1}{\log\frac{x}{1.11}} + \frac{1.8}{\log^2\frac{x}{1.11}}\right)$$

and

$$\begin{aligned}
g(x) &:= \frac{x+5393}{\log(x+5393)} \\
&\quad \times \left(1 + \frac{1}{\log(x+5393)} + \frac{2.51}{\log^2(x+5393)}\right).
\end{aligned}$$

Then by (1–6) and (1–7), the lemma for $M \geq 1\,040\,000$ will follow from the inequality

$$f(x) > g(x) \text{ if } x \geq 1\,040\,000. \tag{2–7}$$

As $f(1\,040\,000) > g(1\,040\,000)$, it is enough to prove that, for $x \geq 1\,040\,000$,

$$(f(x) - g(x))' > 0. \tag{2–8}$$

After removing the denominator, we see that, for $x > 5393$, inequality (2–8) becomes equivalent to the inequality

$$\begin{aligned}\Delta(x) \quad := \quad & 100\log^4(5393+x)\log^3\frac{x}{1.11} \\ & -100\log^4\frac{x}{1.11}\log^3(5393+x) \\ & -20\log^4(5393+x)\log\frac{x}{1.11} \\ & -51\log^4\frac{x}{1.11}\log(5393+x) \\ & -540\log^4(5393+x) + 753\log^4\frac{x}{1.11} > 0.\end{aligned} \tag{2–9}$$

Now using

$$\log\frac{x}{1.11} = \log x - \log 1.11,$$
$$\log(5393+x) =: \log x + \frac{5393a}{x}$$

where $a = a(x)$, and $|a| \leq 1$, we rewrite $\Delta(x)$ as

$$\Delta(x) = M(\log x) + R\left(\log x, \frac{a}{x}\right), \tag{2–10}$$

where

$$\begin{aligned} M(y) &= 753\log^4 1.11 - (3012\log^3 1.11 + 51\log^4 1.11)y \\ &+ (4518\log^2 1.11 + 204\log^3 1.11)y^2 \\ &- (3012\log 1.11 + 306\log^2 1.11 + 100\log^4 1.11)y^3 \\ &+ (213 + 224\log 1.11 + 300\log^3 1.11)y^4 \\ &- (71 + 300\log^2 1.11)y^5 + 100\log(1.11)y^6,\end{aligned}$$

and $R\left(\log x, \frac{a}{x}\right)$ is the remaining, 'small' part of $\Delta(x)$. If $x \geq 1\,040\,000$, then it is easy to compute, where b_{ijk} are appropriate coefficients, that

$$\begin{aligned}\left|R\left(\log x, \frac{a}{x}\right)\right| &= \left|\sum_{\substack{0\leq i\leq 4\\ 0\leq j\leq 6\\ 1\leq k\leq 4}} b_{ijk}\log^i 1.11\,\log^j x\left(\frac{a}{x}\right)^k\right| \\ &\leq \sum |b_{ijk}|\log^i 1.11\,\log^j x\left(\frac{1}{x}\right)^k \\ &< 4\times 10^6,\end{aligned} \tag{2–11}$$

Considering the main part, we have $M'(y) > 0$ for $y > 2$ and $M(\log 1\,040\,000) > 4 \times 10^7$. Then

$$M(\log x) > 4\times 10^7 \text{ for } x \geq 1\,040\,000.$$

From this and (2–7)–(2–11), we obtain the lemma for $x \geq 1\,040\,000$. This finishes the proof. □

Proof of Theorem 1.2. From Lemma 2.4, it follows that the inequality of the theorem holds if $M \geq N \geq 5393$ and $M + N \geq 60184$. By Lemma 2.5, it also holds if $M \geq 619901$ and $7 \leq N \leq 5393$. A computer check for the remaining cases completes the proof of the theorem. □

Proof of Corollary 1.3. For $13 \leq M \leq N \leq 1644$, we check the inequality of the corollary with a computer. By (1–6) and (1–7) we know that $1.11\pi(N/1.11) \geq \pi(N)$ for $N \geq 355991$ and a computer check shows that this inequality is true for $N \geq 1644$. Now Corollary 1.3 follows from Theorem 1.2. □

We will use the following lemma in the proof of Theorem 1.4.

Lemma 2.6. *Let $f''(x) \leq 0$ for $x \geq x_0 \geq 0$ and let $f'(x_0)x_0 \leq f(x_0)$. Then, if $x_1, x_2 \geq x_0$,*

$$f(x_1 + x_2) \leq f(x_1) + f(x_2).$$

Proof: Let the line $l : y = kx + c$ cut the curve $y = f(x)$ at points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Then the point $(x_1 + x_2, f(x_1) + f(x_2) - c)$ lies on l and, because of the concavity of $f(x)$, this point is above the curve $y = f(x)$. Thus

$$f(x_1) + f(x_2) - c \geq f(x_1 + x_2).$$

Now we will prove that $c \geq 0$. Let $x_1 \leq x_2$ (the case $x_1 \geq x_2$ is analogous). By Lagrange's theorem, there exists $x_1 \leq \xi \leq x_2$, such that $k = f'(\xi)$. Then

$$c = f(x_1) - f'(\xi)x_1.$$

Let the line $y = k_0x + c_0$ be a tangent to the curve $y = f(x)$ at (x_0, y_0). Since $f'(x)$ is not increasing,

$$c_0 = f(x_0) - f'(x_0)x_0 \leq f(x_0) - f'(\xi)x_0.$$

Once again, by Lagrange's theorem, there exist $x_0 \leq \xi_0 \leq x_1$ and $\xi_0 \leq \xi_1 \leq \xi$, such that

$$c - c_0 \geq (f'(\xi_0) - f'(\xi))(x_1 - x_0) = f''(\xi_1)(\xi_0 - \xi)(x_1 - x_0).$$

Thus $c - c_0 \geq 0$. Since $c_0 \geq 0$, the lemma is proved. □

Proof of Theorem 1.4. Let's define

$$f(x) \quad := \quad \frac{x}{\log \frac{x}{1+\epsilon}} \left(1 + \frac{1}{\log \frac{x}{1+\epsilon}} + \frac{1.8}{\log^2 \frac{x}{1+\epsilon}} \right)$$

and

$$g(x) \quad := \quad \frac{x}{\log x} \left(1 + \frac{1}{\log x} + \frac{2.51}{\log^2 x} \right).$$

Then, if $x \geq 32299$,

$$(f(x) - g(x)) \frac{100}{x} \log^3 x \log^3 \frac{x}{1+\epsilon} \geq 100 \log(1+\epsilon) \log^4 x - 71 \log^3 x.$$

Thus, $f(x+y) \geq g(x+y)$, if the conditions of the theorem are satisfied. Since $f''(x) \leq 0$ and

$$f(x) - f'(x)x = \frac{27x}{5 \log^4 \frac{x}{1+\epsilon}} + \frac{2x}{\log^3 \frac{x}{1+\epsilon}} + \frac{x}{\log^2 \frac{x}{1+\epsilon}} \geq 0,$$

by Lemma 2.6, we see that $f(x) + f(y) \geq f(x+y) \geq g(x+y)$. From this, (1–6), and (1–7), the theorem follows. □

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