On the Minimizers of the Möbius Cross Energy of Links

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CONTENTS

We give a geometric interpretation for the Euler-Lagrange equation for the Möbius cross energy of (nontrivially linked) 2-component links in the euclidean 3-space. The minimizer of this energy is conjectured to be a Hopf link of 2 round circles. We prove some elementary properties of the minimizers using the Euler-Lagrange equations. In particular, we give a rigorous proof of the fact that the minimizer is topologically a Hopf link.

1. INTRODUCTION

Let $\gamma_1, \gamma_2 : S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \longrightarrow \mathbb{R}^3$ be a pair of loops. The *Möbius cross energy* of the pair is defined by

$$E(\gamma_1, \gamma_2) = \iint\limits_{S^1 \times S^1} \frac{|\gamma_1'(u)| \cdot |\gamma_2'(v)| du\ dv}{|\gamma_1(u) - \gamma_2(v)|^2}. \tag{1–1}$$

The energy is Möbius invariant [Freedman et al. 94]. In other words, if $T = \widehat{\mathbb{R}}^3 = \mathbb{R}^3 \cup \{\infty\} \longrightarrow \widehat{\mathbb{R}}^3$ is a Möbius transformation (i.e., a composition of inversions on spheres of $\widehat{\mathbb{R}}^3$), then

$$E(T\gamma_1, T\gamma_2) = E(\gamma_1, \gamma_2). \tag{1–2}$$

This follows by the following elementary formula,

$$\frac{|T_x'|\ |T_y'|}{|T(x) - T(y)|^2} = \frac{1}{|x-y|^2}, \tag{1–3}$$

where $|T_x'| = \lim\limits_{t \to 0} \frac{|T(x+th) - T(x)|}{|th|}$ for any $h \in \mathbb{R}^3 \setminus \{0\}$. Because T preserves angles, $|T_x'|$ is independent of the choice of h.

We will also use γ_k to denote the set of points $\{\gamma_k(u); u \in S^1\}$. The pair of loops (γ_1, γ_2) is (nontrivially) *linked* if there is no smoothly embedded 2-sphere Σ in $\mathbb{R}^3$ so that each component of $\mathbb{R}^3 \setminus \Sigma$ contains exactly one loop, γ_1 or γ_2. Thus, for any topological projection $proj : \mathbb{R}^3 \to \mathbb{R}^2$, $proj(\gamma_1) \cap proj(\gamma_2) \neq \phi$.

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It is proved in [Freedman et al. 94] that for any linked pair of loops (γ_1, γ_2),

$$E(\gamma_1, \gamma_2) \geq 4\pi. \tag{1–4}$$

We conjecture the following:

1. The minimum of $E(\gamma_1, \gamma_2)$ over all linked pairs of loops in $\mathbb{R}^3$ equals $2\pi^2$.
2. The minimum is attained by a link formed by (round) circles.
3. The minimizer is unique up to Möbius transformation.

If (γ_1, γ_2) are not required to be linked, then $E(\gamma_1, \gamma_2)$ can be arbitrarily small. For example, γ_1 and γ_2 can be a pair of circles of arbitrarily small radii centered at two points at a distance equal to 1.

It is elementary to prove the existence of the minimizers of $E(\gamma_1, \gamma_2)$ over all linked pairs of loops (see Section 2). In this paper, we will give a geometric interpretation of the Euler-Lagrange equation and prove some elementary properties of the minimizers. In particular, we will show that any minimizer is topologically equivalent to the Hopf link.

We wish to point out that there has been some experimental work done which supports this conjecture [Kim and Kusner 93], [Kusner and Sullivan 97]. Also, note that Abrams et al. settled a related minimization problem for a class of knot energies [Abrams et al. 00]. Their result solves a question suggested in O'Hara's original papers on knot energies ([O'Hara 91], [O'Hara 98]) and considerably generalizes the earlier Freedman et al. theorem [Freedman et al. 94] that circles are the only minimizers of the Möbius energy.

In Section 2, we will discuss some elementary properties of a minimizer using the idea of average crossing number, and give details about the shape of the conjectured minimizers. In Section 3, we will give a geometric interpretation of the Euler-Lagrange equation for any critical pair of E, and derive some consequences. In Section 4, we give the geometric properties of any minimizers. We will try to get as close as possible to the conjectured picture of the minimizers. In particular, we will see that the minimizer is ambiently isotopic to the Hopf link.

2. CONJECTURED MINIMIZERS OF THE CROSS ENERGY

Let $\gamma_k : I_k \to \widehat{\mathbb{R}}^3$ be a curve, $k = 1, 2$, where I_k is an interval in $\mathbb{R}$ or S^1. The cross energy of the pair (γ_1, γ_2) is

$$E(\gamma_1, \gamma_2) = \iint \frac{|\gamma_1'(u)|\ |\gamma_2'(v)| du\ dv}{|\gamma_1(u) - \gamma_2(v)|^2}, \tag{2–1}$$

where the integral is evaluated in $(I_1 \setminus \gamma_1^{-1}(\infty)) \times (I_2 \setminus \gamma_2^{-1}(\infty))$. For any Möbius transformation $T : \widehat{\mathbb{R}}^3 \to \widehat{\mathbb{R}}^3$,

$$E(T\gamma_1, T\gamma_2) = E(\gamma_1, \gamma_2). \tag{2–2}$$

Lemma 2.1. *There is a linked pair of loops $\eta_1, \eta_2 : S^1 \to \widehat{\mathbb{R}}^3$ which minimizes E in the sense that for any linked pair of loops (γ_1, γ_2) in $\widehat{\mathbb{R}}^3$,*

$$E(\eta_1, \eta_2) \leq E(\gamma_1, \gamma_2). \tag{2–3}$$

Proof. The proof resembles the elementary theorem that a shortest length geodesic exists between any pair of points in a complete Riemannian manifold. Let (γ_1^n, γ_2^n) be a sequence of linked pairs of loops so that $E(\gamma_1^n, \gamma_2^n)$ converges to infinity. By means of Möbius transformations, we may assume that $\infty \in \gamma_1^n$, $0 \in \gamma_2^n$ and the (Euclidean) diameter of each γ_2^n is 1. Because $E(\gamma_1^n, \gamma_2^n)$ is uniformly bounded, for any ball B in $\mathbb{R}^3$ centered at 0, all of the $\gamma_1^n \cap B$ have uniformly bounded length. On the other hand, as each γ_2^n is contained in the closed ball of radius 1 centered at 0, γ_1^n must intersect the closed ball; otherwise the pair (γ_1^n, γ_2^n) would be unlinked. It follows that γ_1^n contains an arc connecting the point at infinity to a point in the ball of radius 1 centered at 0. Thus, the uniform boundedness of $E(\gamma_1^n, \gamma_2^n)$ implies the uniform boundedness of the lengths of γ_2^n.

Thus, replacing by subsequences if necessary, we may assume that γ_1^n and γ_2^n converge locally uniformly in $\mathbb{R}^3$ to some curves η_1 and η_2 with $\infty \in \eta_1$ and $0 \in \eta_2$ and diameter $(\eta_2) = 1$. Moreover, as a limit of linked pairs, (η_1, η_2) is also linked. The lemma follows because $E(\eta_1, \eta_2) \leq \lim_{n\to\infty} E(\gamma_1^n, \gamma_2^n) = \text{minimum of } E$.

Following ideas of Gauss, the average crossing number was first introduced in [Freedman and He 91] in the study of lower bounds for the energy of incompressible flows in certain physical problems. Later, it was used in [Freedman et al. 94] in the study of Möbius energy and the cross Möbius energy. For example, it is shown that the average crossing number of a knotted loop is bounded by its Möbius energy. More studies were done by other mathematicians on related problems (see, e.g., [Buck and Simon 99],[Cantarella et al. 00],[Cantarella et al. 98],[Diao 01],[Kusner and Sullivan 98]). Here, we will need to use the average crossing number of a pair (γ_1, γ_2),

$$ac(\gamma_1, \gamma_2) = \frac{1}{4\pi} \iint \frac{|\langle \gamma_1'(u), \gamma_2'(v), \gamma_1(u) - \gamma_2(v)\rangle|\, dudv}{|\gamma_1(u), \gamma_2(v)|^3}, \tag{2–4}$$

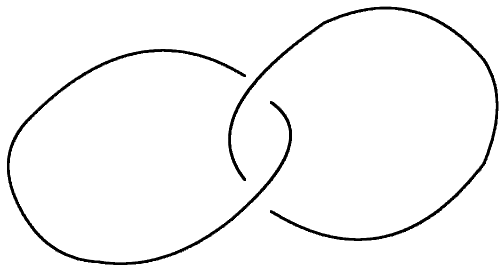

FIGURE 1. The Hopf link.

where $\langle \cdot, \cdot, \cdot \rangle$ denotes the triple scalar product, and the domain of integration is $(S^1 \setminus \gamma_1^{-1}(\infty)) \times (S^1 \setminus \gamma_2^{-1}(\infty))$. This number is actually equal to the expected value of the number of over-crossings of γ_1 on γ_2 in a random orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2$. If γ_1 and γ_2 are disjoint loops with $ac(\gamma_1, \gamma_2) < 2$, then either the pair (γ_1, γ_2) is not linked, or it can be deformed while keeping the pair disjoint from each other (but not necessarily isotopic) to the Hopf link (such deformation is usually referred to as link homotopy). See Figure 1. Note that $ac(\gamma_1, \gamma_2)$ is not Möbius-invariant.

Comparing (2–1) and (2–4), we easily arrive at

$$E(\gamma_1, \gamma_2) \geq 4\pi \ ac(\gamma_1, \gamma_2). \tag{2–5}$$

Let (η_1, η_2) be a mininizer of $E(\cdot, \cdot)$ among all linked pairs of loops in $\widehat{\mathbb{R}^3}$. Then by (2–7),

$$E(\eta_1, \eta_2) \leq 2\pi^2. \tag{2–6}$$

Lemma 2.2. *Let (η_1, η_2) be a minimizer of $E(\cdot, \cdot)$ among all linked pairs of loops in $\widetilde{\mathbb{R}^3}$. Then η_1 and η_2 are mutually disjoint simple loops with linking number $= \pm 1$.*

It does not follow directly from the lemma that the pair (η_1, η_2) is topologically a Hopf link. At this point, we do not yet know whether η_1 (or η_2) is topologically unknotted.

Proof: First, η_1 and η_2 must be disjoint from each other, otherwise $E(\eta_1, \eta_2)$ would be infinite. A simple proof follows. Without loss of generality, assume that η_1 and η_2 are parametrized by arc-length, and assume that $\eta_1(u_0) = \eta_2(v_0)$. Then,

$$\begin{aligned} E(\eta_1, \eta_2) &\geq \int_0^\delta \int_0^\delta \frac{dsdt}{|\eta(u_0+s) - \eta(v_0+t)|^2} \\ &\geq \int_0^\delta \int_0^\delta \frac{dsdt}{(s+t)^2} = \int_0^\delta \left(\frac{1}{s} - \frac{1}{\delta + s} \right) ds = +\infty, \end{aligned}$$

where $\delta > 0$ is any number smaller than the lengths of η_1 and η_2.

By means of a Möbius transformation, we may assume that $\eta_1, \eta_2 \subset \mathbb{R}^3$. By (2–5) and (2–6), we deduce

$$ac(\eta_1, \eta_2) \leq \frac{1}{4\pi} E(\eta_1, \eta_2) \leq \frac{2\pi^2}{4\pi} = \frac{\pi}{2} < 2.$$

It follows that for some orthogonal projection $proj : \mathbb{R}^3 \to \mathbb{R}^2$, the loop $proj(\eta_1)$ crosses $proj(\eta_2)$ less than 4 times. We may also assume that the curves $proj(\eta_1)$ and $proj(\eta_2)$ are transversal with respect to each other. Thus, by topological considerations, there must be either 0 or 2 crossings. Since the pair is linked, there must be at least one over-crossing (i.e., η_1 over η_2) and one under-crossing (i.e., η_1 under η_2). Thus, we must have exactly one over-crossing, and one under-crossing, as shown in Figure 2. The linking number of the pair is clearly ± 1, depending on the orientations of the curves.

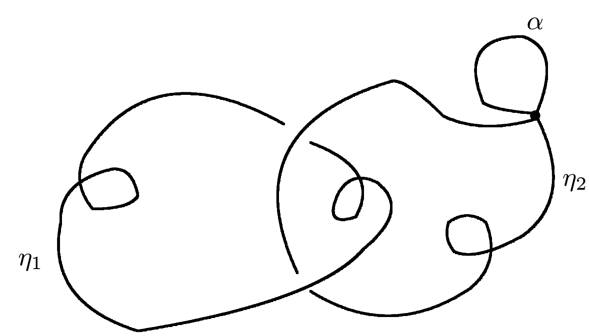

FIGURE 2. The projections of η_1 and η_2 intersect at two points.

If one of η_1 and η_2, say η_2, is not simple, then η_2 contains a subloop α which does not cross η_1 when projected to the plane by $proj$. By removing α from η_2, we obtain a new loop $\widetilde{\eta}_2$ which has linking number ± 1 with η_1 and therefore, $(\eta_1, \widetilde{\eta}_2)$ is linked, but $E(\eta_1, \widetilde{\eta}_2) < E(\eta_1, \eta_2) =$ minimum of E. This is a contradiction, thus proving the lemma. □

Let $\widetilde{\gamma}_1$ be the extended line $x_1 = x_2 = 0$ (including the point at infinity, and let $\widetilde{\gamma}_2$ be the circle $x_1^2 + x_2^2 = 1, x_3 = 0$. Then $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ is topologically the Hopf link in $\widehat{\mathbb{R}^3} \cong S^3$:

$$\begin{aligned} E(\widetilde{\gamma}_1, \widetilde{\gamma}_2) &= \int_{-\infty}^\infty du \int_0^{2\pi} \frac{dv}{|(0,0,u) - (\cos v, \sin v, 0)|^2} \\ &= 2\pi \int_{-\infty}^\infty \frac{du}{1+u^2} = 2\pi^2. \end{aligned} \tag{2–7}$$

The conjecture is that $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ is a minimizer of E among all linked pairs of loops in $\widehat{\mathbb{R}^3}$, and any other minimizer is Möbius-equivalent to $(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$.

3. A GEOMETRIC INTERPRETATION

Let $\gamma : I \to \mathbb{R}^3$ be a curve of finite length. Then the center of mass of γ is the point $q \in \mathbb{R}^3$ such that

$$\int_I (\gamma(u) - q)|\gamma'(u)|du = 0. \tag{3–1}$$

Lemma 3.1. *Let (η_1, η_2) be a critical pair of the functional $E(\cdot,\cdot)$. Let $p \in \eta_1$ and let $T_p = \widehat{\mathbb{R}}^3 \to \widehat{\mathbb{R}}^3$ be a Möbius transformation with $T_p(p) = \infty$. (For example, T_p can be the inversion on a sphere centered at p.). Let L be the line which is asymptotic to $T_p\eta_1$ at $\infty = T_p(p)$. Then the center of mass of $T_p\eta_2$ is on L. The same property holds if η_1 and η_2 are switched.*

In fact, the proof below shows that the converse of the lemma also holds (at least in the case that η_1 and η_2 admit integrable second order derivatives). If for any $j, k = 0, 1$ and for any $p \in \eta_k$, the center of mass of $T_p\eta_j$ is contained on the line which is asymptotic to $T_p\eta_k - \{\infty\}$ at ∞, then (η_1, η_2) is a critical pair for E. This gives a geometric interpretation for the Euler-Lagrange equation for E. See [He 00] for a similar interpretation for the Möbius energy of knots.

Proof: Without loss of generality, we may assume that $p = \eta_1(u_0) = 0 \in \mathbb{R}^3$, $\eta_1'(u_0) = (0,0,1)$ and $(\eta_1'/|\eta_1'|)' = 0$ at p. That is, the osculating circle of η_1 at $p = 0$ is the extended line L: $x_1 = x_2 = 0$. Let $T_p(x) = x/|x|^2$. Then $T_pL = L$ is the line asymptotical to $T_p\eta_1$ at $\infty = T_p(0)$. We need to show that the center of mass of $T_p\eta_2$ is on the x_3-axis.

It is elementary to show that η_1 and η_2 are smooth curves. Let $h_1 = S^1 \to \mathbb{R}^3$ be any smooth function. Since (η_1, η_2) is a critical pair of E, we have by the definition of E:

$$\begin{aligned}
0 &= \nabla_1 E(\eta_1, \eta_2)h_1 \\
&= \lim_{t\to 0} \frac{E(\eta_1 + th_1, \eta_2) - E(\eta_1, \eta_2)}{t} \\
&= \iint \Bigg[\left\langle h_1'(u), \frac{\eta_1'(u)}{|\eta_1'(u)|}\right\rangle \frac{1}{|\eta_1(u) - \eta_2(v)|^2} \\
&\quad - \frac{2\langle \eta_1(u) - \eta_2(v), h_1(u)\rangle}{|\eta_1(u) - \eta_2(v)|^4}\, |\eta_1'(u)|\Bigg] |\eta_2'(v)|du\, dv \\
&= \int \Bigg\langle h_1(u), \int \Bigg[-\left(\frac{\eta_1'(u)}{|\eta_1'(u)|}\right)' \frac{1}{|\eta_1'(u)|}\, \frac{1}{|\eta_1(u) - \eta_2(u)|^2} \\
&\quad + \frac{\eta_1'(u)}{|\eta_1'(u)|} \frac{2\langle \eta_1(u) - \eta_2(v), \eta_1'(u)\rangle}{|\eta_1(u) - \eta_2(v)|^4} \\
&\quad - \frac{2(\eta_1(u) - \eta_2(v))}{|\eta_1(u) - \eta_2(v)|^4}\Bigg] |\eta_2'(v)|dv \Bigg\rangle\, |\eta_1'(u)|du.
\end{aligned}$$

For any nonzero $y \in \mathbb{R}^3$, let $P_{y^\perp} : \mathbb{R}^3 \to \mathbb{R}^3$ denote an orthogonal projection onto the plane through 0 in $\mathbb{R}^3$ orthogonal to y:

$$P_{y^\perp}x = x - \frac{y}{|y|}\left\langle x, \frac{y}{|y|}\right\rangle,$$

where $x \in \mathbb{R}^3$. Then,

$$0 = \nabla_1 E(\eta_1, \eta_2)h_1 = -\int_{S^1} \langle h_1(u), H_1(u)\rangle |\eta_1'(u)|du, \tag{3–2}$$

where

$$\begin{aligned}
H_1(u) = \int_{S^1} \Bigg[&-\left(\frac{\eta_1'(u)}{|\eta_1'(u)|}\right)' \frac{1}{|\eta_1'(u)|} \\
&+ 2P_{\eta_1'(u)^\perp}\left(\frac{\eta_2(v) - \eta_1(u)}{|\eta_2(v) - \eta_1(u)|^2}\right)\Bigg] \frac{|\eta_2'(v)|dv}{|\eta_2(v) - \eta_1(u)|^2}.
\end{aligned} \tag{3–3}$$

Because $h_1 : S^1 \to \mathbb{R}^3$ is arbitrary, we deduce that $H_1(u) = 0$ for all $u \in S^1$.

We may define $H_2(v)$ by computing $\nabla_2 E(\eta_1, \eta_2)h_2$ in a similar way. It is clear from the above that (η_1, η_2) is a critical pair for E if and only if both $H_1(u)$ and $H_2(v)$ vanish for all u and all v.

Let $u = u_0$, so that $\eta_1(u) = 0 \in \mathbb{R}^3$, $\eta_1'(u) = (0,0,1)$, $P_{\eta_1'(u)^\perp}(x_1, x_2, x_3) = (x_1, x_2, 0)$, and $(\eta_1'(u)/|\eta_1'(u)|)' = 0$. Then Equation (3–3) becomes

$$0 = H_0(u_0) = 2\int_{S^1} \left[P_{(0,0,1)^\perp}\left(\frac{\eta_2(v)}{|\eta_2(v)|^2}\right)\right] \frac{|\eta_2'(v)|dv}{|\eta_2(v)|^2}. \tag{3–4}$$

But $T_px = x/|x|^2$. So $T_p\eta_2(v) = \eta_2(v)/|\eta_2(v)|^2$ and

$$|(T_p\eta_2)'(v)| = \frac{|\eta_2'(v)|}{|\eta_2(v)|^2}.$$

Equation (3–4) reduces to

$$\int_{S^1} P_{(0,0,1)^\perp}(T_p\eta_2(v))|(T_p\eta_2)'(v)|dv = 0. \tag{3–5}$$

The above relation means that the center of mass of $T_p\eta_2$ is on the x_3-axis. The lemma is proved. □

Lemma 3.2. *Let (η_1, η_2) be a critical pair of E and let $p \in \gamma_2$. Let S be a round 2-sphere in $\widehat{\mathbb{R}}^3$ which contains the osculating circle of η_2 at p. Then $S \cap \eta_1 \neq \emptyset$. In fact, if η_1 is not contained in S, then each component of $\widehat{\mathbb{R}}^3 \setminus S$ contains points of η_1.*

Proof: Using Möbius transformations, we may assume that $p = \infty$ and thus the osculating circle is an extended line which is asymptotic to η_1 at ∞. By Lemma 3.1, the center of gravity of η_2 is on the line and therefore is on the extended plane S. This implies that η_2 intersects S. The last statement in the lemma also follows. □

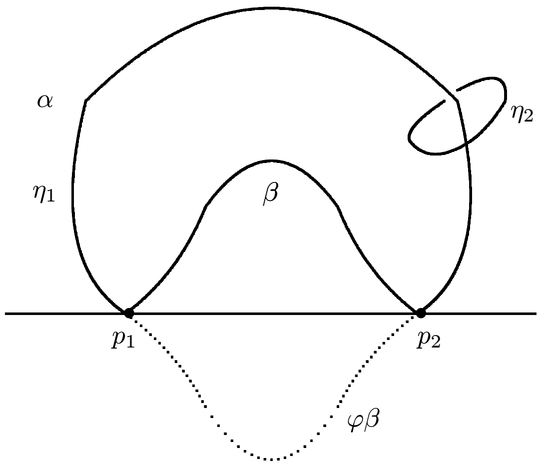

FIGURE 3. Replacing β by $\phi\beta$ decreases Möbius cross energy.

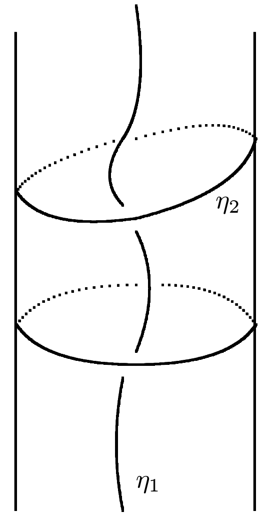

FIGURE 4. $\eta_1 \setminus \{\infty\}$ is strictly inside the cylinder $\Sigma \times \mathbb{R}^1$.

4. THE MINIMIZER IS A HOPF LINK

Throughout this section, (η_1, η_2) will denote a minimizer of E among linked loops in $\widehat{\mathbb{R}}^3$.

Lemma 4.1. *Let B be an open (round) ball in $\widehat{\mathbb{R}}^3$ disjoint from $\eta_1 \cup \eta_2$. Then for $k = 1$ or 2, $\partial B \cap \eta_k$ contains at most one point.*

Proof: Without loss of generality, assume that $k = 1$, $B = \mathbb{R}^2 \times (-\infty, 0)$, and $\infty \notin \eta_1 \cup \eta_2$. Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ denote reflection in the plane $\mathbb{R}^2 \times \{0\}$:

$$\varphi(x_1, x_2, x_3) = (x_1, x_2, -x_3). \tag{4–1}$$

Suppose, to the contrary, that $\partial B \cap \eta_1$ contains at least two points, p_1, p_2. Let α and β be the subarcs of η_1 with ends at p_1 and p_2. (See Figure 3.)

Because (η_1, η_2) is linked and $\eta_1 \subseteq \mathbb{R}^2 \times [0, \infty)$, the loop η_2 cannot be entirely contained in $\mathbb{R}^2 \times \{0\}$ (otherwise η_2 can be deformed to a nearby curve in the lower half-space $\mathbb{R}^2 \times (-\infty, 0)$). So by Lemma 3.1, no osculating circle of η_1 can be contained in $\mathbb{R}^2 \times \{0\}$. Thus, neither of the arcs α and β can entirely be contained in $\mathbb{R}^2 \times \{0\}$.

By Lemma 2.2, the linking number of η_1 and η_2 is ± 1. Therefore, in the orthogonal projection in some direction parallel to the x_1x_2-plane, the loop η_2 over-crosses α or β an odd number of times. Let us assume the former. Let $\widetilde{\eta}_1 = \alpha \cup (\varphi\beta)$. Then the linking number of $\widetilde{\eta}_1$ and η_2 is odd, and by comparing the integrals in the definition of E, we obtain

$$E(\widetilde{\eta}_1, \eta_2) < E(\eta_1, \eta_2),$$

contradicting the minimality of $E(\eta_1, \eta_2)$. The lemma is proved. □

Let $p_1 : \mathbb{R}^3 \to \mathbb{R}^2$ denote the projection:

$$p_1(x_1, x_2, x_3) = (x_1, x_2). \tag{4–2}$$

Theorem 4.2. *Assume that $\infty \in \eta_1$ and the asymptotic line to η_1 at ∞ is $x_1 = x_2 = 0$. Then p_1 maps η_2 homeomorphically onto a strictly convex (C^∞) smooth curve Σ with nonzero curvature in $\mathbb{R}^2$. Moreover, $\eta_1 \setminus \{\infty\}$ is contained in the interior of the cylinder $\Sigma \times \mathbb{R} \subseteq \mathbb{R}^3$.*

As a consequence of Theorem 4.2, $p_1(\eta_1 \setminus \{\infty\})$ is an open curve (from 0 to 0) contained in the convex Jordan domain (in $\mathbb{R}^2$) bounded by Σ. (See Figure 4.)

Proof: Let D be the interior of the convex hull of $p_1(\eta_2) \subseteq \mathbb{R}^2$. By Lemma 3.1, the center of mass of η_2 is on the line $x_1 = x_2 = 0$. Thus, $0 \in D$. The open curve $p_1(\eta_1 \setminus \{\infty\})$ clearly has both endpoints at 0. Thus, $p_1(\eta_1 \setminus \{\infty\}) \subseteq D$. Otherwise, there would be a vertical extended plane Γ disjoint from $D \times \mathbb{R}^1$ which is tangent to $\eta_1 \setminus \{\infty\}$ so that $\eta_1 \setminus \{\infty\}$ is in one closed half-space bounded by Γ. As $0 \in D$ and 0 lies at the ends of $p_1(\eta_1 \setminus \{\infty\})$, we deduce that η_1 and $\overline{D} \times \mathbb{R}^1$ (hence $\eta_2 \subset \overline{D} \times \mathbb{R}^1$) are all in the same half-space bounded by Γ. But $\Gamma \cap \eta_1$ contains ∞ and another point, a contradiction by Lemma 4.1.

We claim that $\partial D \subseteq p_1(\eta_2)$. If not, there would be a straight arc on ∂D. Let $L \subseteq \mathbb{R}^2$ be a straght line containing such an arc. Then $L \cap p_1(\eta_2)$ has at least two points. Let $\Gamma_1 = (L \times \mathbb{R}^1) \cup \{\infty\} \subseteq \widehat{\mathbb{R}}^3$. Then $\eta_1 \cup \eta_2 \subseteq \overline{D} \times \mathbb{R}$ is on one side of Γ_1, but Γ_1 intersects η_2 in two points, again contradicting Lemma 4.1.

Let $f = p_1|_{p_1^{-1}(\partial D)\cap\eta_2} : p_1^{-1}(\partial D) \cap \eta_2 \to \partial D$. Since $\partial D \subseteq p_1(\eta_2)$, f is onto. We now show that f is injective. Let $q \in \partial D$, and let Γ_2 be the vertical extended plane through $q \times \mathbb{R}^1$ which is tangent to $\overline{D} \times \mathbb{R}^1$. Then, again, $\eta_1 \cup \eta_2 \subseteq \overline{D} \times \mathbb{R}^1$ are on the same side of Γ_2. By Lemma 4.1, there is at most one point in $\Gamma_2 \cap \eta_2 \supseteq p_1^{-1}(q) \cap \eta_2 = f^{-1}(q)$. So $f^{-1}(q)$ has at most one point, and therefore f is injective.

Being a closed subset of $\eta_2 \cong S^1$, $p_1^{-1}(\partial D) \cap \eta_2$ is compact. It follows that f is a homeomorphism because it is an injective map from a compact space onto a Hausdorff space. In particular, $p_1^{-1}(\partial D) \cap \eta_2 \subseteq \eta_2 \cong S^1$ is homeomorphic to S^1. As no proper subset of S^1 is homeomorphic to S^1, we deduce $p_1^{-1}(\partial D) \cap \eta_2 = \eta_2$ and thus p_1 maps η_2 homeomorphically onto ∂D. Let $\Sigma = \partial D$. The theorem would be complete if we could show that Σ is a smooth curve with nonzero curvature.

We claim that the tangent vector of η_2 is never vertical (i.e., never parallel to the vector $(0, 0, 1)$). By contradiction, assume that the tangent to η_2 at a point q is vertical. Then by applying the fact that p_1 maps η_2 homeomorphically onto Σ, the osculating circle of η_2 at $q \subseteq \partial D \times \mathbb{R}^1$ must be the extended vertical line through q. It is contained in the extended tangent plane S of $\partial D \times \mathbb{R}^1$ at q. This contradicts Lemma 3.2 because $\eta_1 \setminus \{\infty\}$ lies inside $D \times \mathbb{R}^1$ which is on one side of (but not contained in) S.

It follows that $p_1\eta_2$ is a (C^∞) smooth convex curve in $\mathbb{R}^2$. Its curvature is nowhere vanishing, otherwise the osculating circle at the point of vanishing curvature on η_2 would be a circle contained in an extended plane S tangent to $\overline{D} \times \mathbb{R}^1$, contradicting the second part of Lemma 3.2. This completes the proof of the theorem. □

Theorem 4.3. *Let (η_1, η_2) be a minimizer of E over all linked pairs of loops in $\widehat{\mathbb{R}}^3$. Then the pair is ambiently isotopic to the Hopf link.*

Proof: By means of Möbius transformations, we may assume that $\infty \in \eta_1$ and that the asymptotic line to η_1 at ∞ is $x_1 = x_2 = 0$. It follows by Theorem 4.2 that η_2 is unknotted. By symmetry, η_1 is also unknotted. Therefore, using Theorem 4.2 again, η_1 is contained in the unknotted "solid torus" $(D \times \mathbb{R}^1) \cup \{\infty\}$. It is elementary to show that η_1 is isotopic within the unknotted "solid torus" to an extended line. On the other hand, η_1 is isotopic to a circle on an orthogonal plane with its center on the extended line. The theorem is thus proved. □

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