Polynomial Invariants and Harmonic Functions Related to Exceptional Regular Polytopes

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CONTENTS

We compute certain polynomial invariants for the finite reflection groups of the types H_3, H_4 and F_4. Using this result, we explicitly determine the solution space of functions satisfying a mean value property related to the exceptional regular polytopes, namely, the icosahedron and dodecahedron in three dimensions and the 24-cell, 600-cell, and 120-cell in four dimensions.

1. INTRODUCTION

A classical theorem of Gauss and Koebe states that a function is harmonic if and only if it satisfies the mean value property with respect to a sphere. In this paper, we study the following variant of this property for polytopes.

Given an n-dimensional polytope P and an integer $k \in \{0, 1, \ldots, n\}$, let $P(k)$ be the k-dimensional skeleton of P. An $\mathbb{R}$-valued continuous function $f \in C(\mathbb{R}^n)$ is said to be $P(k)$-*harmonic* if it satisfies the mean value property:

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry)\, d\mu_k(y) \qquad (1\text{–}1)$$

for any $x \in \mathbb{R}^n$ and $r > 0$, where μ_k is the k-dimensional volume element on $P(k)$ and $|P(k)| = \mu_k(P(k))$ is the k-dimensional total mass of $P(k)$. Let $\mathcal{H}_{P(k)}$ denote the set of $P(k)$-harmonic functions. We are interested in the problem of characterizing the function space $\mathcal{H}_{P(k)}$.

From our previous work [Iwasaki 97a], the following facts are known: The space $\mathcal{H}_{P(k)}$ is a *finite-dimensional* linear space of *polynomials*. The space $\mathcal{H}_{P(k)}$ is invariant under partial differentiations, namely, it carries a structure of $\mathbb{R}[\partial]$-module, where $\mathbb{R}[\partial]$ is the ring of partial differential operators with constant coefficients. If the symmetry group $G \subset O(n)$ of P is *irreducible*, then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of *harmonic* polynomials.

Our problem is of particular interest when the polytope is a regular convex polytope. Now we recall the

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type	polytope	dimension
A_n	regular simplex, self dual	$n \geq 3$
B_n	cross polytope and measure polytope	$n \geq 3$
F_4	24-cell, self dual	$n = 4$
H_3	icosahedron and dodecahedron	$n = 3$
H_4	600-cell and 120-cell	$n = 4$
$I_2(m)$	regular convex m-gon, self dual	$n = 2$

TABLE 1. Classification of regular convex polytopes.

classification of regular convex polytopes in terms of their symmetry groups (see [Coxeter 73]). The symmetry groups of the regular convex polytopes are the irreducible finite reflection groups of types A_n, B_n, F_4, H_3, H_4, and $I_2(m)$ (see [Humphreys 90]). The correspondence between the polytopes and the types is given in Table 1. Observe that certain types, e.g., H_3, correspond to two polytopes, which are duals of each other. Polytopes of the types H_3, F_4, and H_4 are called the *exceptional* regular polytopes, as they appear sporadically in the classification.

For any n-dimensional regular convex polytope P, one has $\mathcal{H}_{P(n-1)} = \mathcal{H}_{P(n)}$ by [Iwasaki 97a, Theorem 2.2]. Hence it is sufficient to consider the k-skeleton problem for $k \in \{0, 1, \dots, n-1\}$. For each regular convex polytope, the 0-skeleton problem, or the vertex problem, was thoroughly discussed by many authors [Kakutani and Nagumo 35, Walsh 36, Beckenbach and Reade 43, Beckenbach and Reade 45, Friedman 1957, Flatto 63, Flatto and Wiener 70, Haeuslein 70]. Our main concern is the much more involved higher-skeleton problems. In this direction, Flatto [Flatto 63] solved the $(n-1)$-skeleton problem for a regular n-simplex and an n-dimensional cross polytope. However, attempts to deal with every skeleton have begun only recently. Iwasaki [Iwasaki 97b] settled the problem for all skeletons of a regular n-simplex. This paper focuses on the same problem for the exceptional regular polytopes and gives a complete solution to it. The remaining polytopes will be discussed elsewhere.

To obtain our result, we employ a criterion established in an earlier paper [Iwasaki 99a] by one of the authors. See (3–4) of Theorem 3.2. The new material of the present paper consists of elaborate computations needed to verify the criterion. For this purpose, we used the computer algebra system, Maple.

2. MAIN THEOREM

We take the icosahedron, dodecahedron, 24-cell, 600-cell, and 120-cell in such a manner that their vertices are as

polytope	coordinates of vertices
Icosahedron $\{3,5\}$	$(0,\pm\tau,\pm1), (\pm1,0,\pm\tau), (\pm\tau,\pm1,0)$
Dodecahedron $\{5,3\}$	$(0,\pm\tau^{-1},\pm\tau), (\pm\tau,0,\pm\tau^{-1}), (\pm\tau^{-1},\pm\tau,0)$ $(\pm1,\pm1,\pm1)$
24-cell $\{3,4,3\}$	the permutations of $(\pm1,\pm1,0,0)$
600-cell $\{3,3,5\}$	$(\pm1,\pm1,\pm1,\pm1)$ the permutations of $(\pm2,0,0,0)$ the even permutations of $(\pm\tau,\pm1,\pm\tau^{-1},0)$
120-cell $\{5,3,3\}$	the permutations of $(\pm2,\pm2,0,0)$ the permutations of $(\pm\sqrt{5},\pm1,\pm1,\pm1)$ the permutations of $(\pm\tau,\pm\tau,\pm\tau,\pm\tau^{-2})$ the permutations of $(\pm\tau^2,\pm\tau^{-1},\pm\tau^{-1},\pm\tau^{-1})$ the even permutations of $(\pm\tau^2,\pm\tau^{-2},\pm1,0)$ the even permutations of $(\pm\sqrt{5},\pm\tau^{-1},\pm\tau,0)$ the even permutations of $(\pm2,\pm1,\pm\tau,\pm\tau^{-1})$

TABLE 2. Vertices of the exceptional regular polytopes.

in Table 2, where τ stands for the golden ratio:

$$\tau = \frac{1+\sqrt{5}}{2}.$$

These polytopes are expressed as $\{3,5\}$, $\{5,3\}$, $\{3,4,3\}$, $\{3,3,5\}$, $\{5,3,3\}$, respectively, in Schläfli's symbols. Here $\{p,q\}$ represents a regular polyhedron whose faces are regular p-gons and whose vertex figures are regular q-gons, while $\{p,q,r\}$ represents a 4-dimensional regular polytope whose cells are $\{p,q\}$s and whose vertex figures are $\{q,r\}$s (see [Coxeter 73]). Note that $\{3,5\}$ and $\{5,3\}$ (respectively $\{3,3,5\}$ and $\{5,3,3\}$) are duals of each other.

For each $\sharp = H_3, F_4, H_5$, let $G_\sharp$ be a finite reflection group of type $\sharp$ realized as the symmetry group of $\{3,5\}$, $\{3,4,3\}$, $\{3,3,5\}$, respectively. The groups G_{H_3} and G_{H_4} are also realized as the symmetry groups of $\{5,3\}$ and $\{5,3,3\}$, respectively. Then the fundamental alternating polynomial $\mathit{\Delta}_\sharp = \mathit{\Delta}_{G_\sharp}$ of the group $G_\sharp$ is given in Table 3, where the notation $\prod$ is used in the following sense:

$$\begin{aligned}\prod&(a_0 \pm a_1 \pm \cdots \pm a_m)\\ &= \prod_{\varepsilon_1=\pm1} \cdots \prod_{\varepsilon_m=\pm1} (a_0 + \varepsilon_1 a_1 + \cdots + \varepsilon_m a_m).\end{aligned}$$

The reflecting hyperplanes of the reflection group $G_\sharp$ are given by the locus of the equation $\mathit{\Delta}_\sharp = 0$. Moreover, the order of the group $G_\sharp$ is given by

$$|G_\sharp| = \begin{cases} 120 & (\sharp = H_3), \\ 1152 & (\sharp = F_4), \\ 14400 & (\sharp = H_4). \end{cases}$$

The main theorem of the present paper is now stated as follows:

$$\begin{array}{rrl}
\Delta_{H_3} = & xyz & \prod(\tau x \pm \tau^{-1} y \pm z) \prod(\tau y \pm \tau^{-1} z \pm x) \prod(\tau z \pm \tau^{-1} x \pm y) \\
\Delta_{F_4} = & xyzw & \prod(x \pm y \pm z \pm w) \\
 & & \prod(x \pm y) \prod(x \pm z) \prod(x \pm w) \\
 & & \prod(y \pm z) \prod(y \pm w) \prod(z \pm w) \\
\Delta_{H_4} = & xyzw & \prod(x \pm y \pm z \pm w) \\
 & & \prod(\tau x \pm \tau^{-1} y \pm z) \prod(\tau y \pm \tau^{-1} z \pm x) \prod(\tau z \pm \tau^{-1} x \pm y) \\
 & & \prod(\tau x \pm \tau^{-1} z \pm w) \prod(\tau z \pm \tau^{-1} w \pm x) \prod(\tau w \pm \tau^{-1} x \pm z) \\
 & & \prod(\tau x \pm \tau^{-1} w \pm y) \prod(\tau w \pm \tau^{-1} y \pm x) \prod(\tau y \pm \tau^{-1} x \pm w) \\
 & & \prod(\tau y \pm \tau^{-1} w \pm z) \prod(\tau w \pm \tau^{-1} z \pm y) \prod(\tau z \pm \tau^{-1} y \pm w)
\end{array}$$

TABLE 3. Fundamental alternating polynomials.

Theorem 2.1. *Let P be an n-dimensional exceptional regular convex polytope ($n = 3$ or 4) centered at the origin and let G be its symmetry group. Then for each $k \in \{0, 1, \dots, n\}$, the fundamental alternating polynomial Δ_G of the reflection group G generates the function space $\mathcal{H}_{P(k)}$ as an $\mathbb{R}[\partial]$-module, namely,*

$$\mathcal{H}_{P(k)} = \mathbb{R}[\partial]\, \Delta_G.$$

In particular, the space $\mathcal{H}_{P(k)}$ is independent of the skeletons of P. The dimension of $\mathcal{H}_{P(k)}$ is the order $|G|$ of the group G, that is,

$$\dim \mathcal{H}_{P(k)} = |G|.$$

3. INVARIANT THEORY

The proof of Theorem 2.1 is based on some results in invariant theory established by Iwasaki [Iwasaki 97c, Iwasaki 99a]. Let G be a finite reflection group acting on $\mathbb{R}^n$. The ring of G-invariant polynomials in $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ is generated by an n-tuple of algebraically independent homogeneous G-invariant polynomials. Such an n-tuple $(\phi_1, \dots, \phi_n)$ is called an invariant basis for G, where $\phi_1, \dots, \phi_n$ are arranged so that the degrees $d_i = \deg \phi_i$ $(i = 1, \dots, n)$ satisfy $d_1 \le \cdots \le d_n$. The degrees $(d_1, \dots, d_n)$ depend only on G, that is, independent of a particularly chosen invariant basis. An invariant basis $(\phi_1, \dots, \phi_n)$ is said to be *canonical* if it satisfies the system of nonlinear partial differential equations:

$$\phi_i(\partial)\phi_j = \langle \phi_i, \phi_j \rangle\, \delta_{ij} \qquad (i, j = 1, \dots, n),$$

where $\langle f, g \rangle$ is an inner product on $\mathbb{R}[x]$ defined by

$$\langle f, g \rangle = f(\partial) g|_{x=0} \qquad (f, g \in \mathbb{R}[x]), \tag{3–1}$$

and δ_{ij} is Kronecker's symbol. From a result of [Iwasaki 97c], any finite reflection group admits a canonical invariant basis, which is unique in the following sense: if $(\phi_1, \dots, \phi_n)$ and $(\psi_1, \dots, \psi_n)$ are two canonical invariant bases, then $\phi_1, \dots, \phi_n$ are linear combinations of $\psi_1, \dots, \psi_n$ and vice versa. In particular, if the degrees $(d_1, \dots, d_n)$ satisfy $d_1 < \cdots < d_n$, then for each $i \in \{1, \dots, n\}$ the i-th canonical invariant polynomial ϕ_i is unique up to a nonzero constant multiple. The canonical invariant bases of the types A_n, B_n, D_n, and $I_2(m)$ were explicitly calculated in [Iwasaki 97c]. Other results from our previous work that will be used in Sections 4 and 5 include:

Theorem 3.1. ([Iwasaki 97c]) *Let $(\psi_1, \dots, \psi_n)$ be an orthogonal invariant basis for G relative to the inner product (3–1). Then the system of partial differential equations*

$$\begin{cases} \psi_i(\partial)\phi_j = \langle \psi_i, \phi_j \rangle \delta_{ij} & (i, j = 1, \dots, n), \\ \langle \psi_i, \phi_i \rangle \neq 0 & (i = 1, \dots, n), \end{cases} \tag{3–2}$$

admits a solution $(\phi_1, \dots, \phi_n)$ such that each ϕ_i is a G-invariant smooth function on $\mathbb{R}^n$ with $\phi_i(0) = 0$. Moreover, any such solution $(\phi_1, \dots, \phi_n)$ of (3–2) is a canonical invariant basis for G.

Theorem 3.2. [Iwasaki 99a] *Let P be an n-dimensional polytope having a finite reflection group G as its symmetry group. Assume that the degrees $(d_1, \dots, d_n)$ of G satisfy the condition*

$$d_1 < d_2 < \cdots < d_n, \tag{3–3}$$

and let $(\phi_1, \dots, \phi_n)$ be the canonical invariant basis for G. Then for each $k \in \{0, 1, \dots, n\}$, the fundamental alternating polynomial Δ_G of the group G generates the *function space $\mathcal{H}_{P(k)}$ as an $\mathbb{R}[\partial]$-module and the dimension of $\mathcal{H}_{P(k)}$ is the order $|G|$ of G, if and only if $P(k)$ satisfies*

$$\int_{P(k)} \phi_i(x)\, d\mu_k(x) \neq 0 \qquad (i = 1, \ldots, n). \tag{3–4}$$

In Section 4, we will apply Theorem 3.1 to compute the canonical invariant bases for the groups $G_\sharp$ with $\sharp = H_3, F_4, H_4$. In Section 5, we will verify the criterion (3–4) of Theorem 3.2 to establish Theorem 2.1.

4. CANONICAL INVARIANT BASES

For each $\sharp = H_3, F_4, H_4$, we shall explicitly compute the canonical invariant basis for the group $G_\sharp$. Recall that the degrees of $G_\sharp$ are given by

$$\begin{array}{rl} (2,6,10) & (\sharp = H_3), \\ (2,6,8,12) & (\sharp = F_4), \\ (2,12,20,30) & (\sharp = H_4). \end{array}$$

To state the result, we establish some notation. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, let M_λ denote the associated monomial symmetric polynomial of the variables $(x_1^2, \ldots, x_n^2)$, namely,

$$M_\lambda = \sum x_1^{2\mu_1} \cdots x_n^{2\mu_n},$$

where the sum is taken over all permutations $(\mu_1, \ldots, \mu_n)$ of $(\lambda_1, \ldots, \lambda_n)$. If λ consists of mutually distinct numbers $p_1 > \cdots > p_m$ with p_j appearing k_j times in λ, then we put

$$M_\lambda = [p_1^{k_1}|\cdots|p_m^{k_m}].$$

For example, if $\lambda = (1,0,0)$, $(2,1,0)$, $(1,1,1)$, then

$$\begin{aligned} [1|0^2] &= x_1^2 + x_2^2 + x_3^2, \\ [2|1|0] &= x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 + x_1^2x_2^4 + x_2^2x_3^4 + x_3^2x_1^4, \\ [1^3] &= x_1^2x_2^2x_3^2. \end{aligned}$$

Moreover, let Δ_n be the fundamental alternating polynomial of $(x_1^2, \ldots, x_n^2)$:

$$\Delta_n = \prod_{1 \leq i < j \leq n} \left(x_i^2 - x_j^2\right).$$

Theorem 4.1. *For each $\sharp = H_3$, F_4, H_4, the canonical invariant basis for the group $G_\sharp$ is given as in Tables 4, 5, and 6, respectively.*

$$\begin{aligned} \phi_1 &= [1|0^2] \\ \phi_2 &= 2\,[3|0^2] - 15\,[2|1|0] + 180\,[1^3] + 21\sqrt{5}\Delta_3 \\ \phi_3 &= 5\{2\,[5|0^2] - 45\,[4|1|0] + 42\,[3|2|0] + 1008\,[3|1^2] - 1260\,[2^2|1]\} \\ &\quad -33\sqrt{5}\Delta_3\{3\,[2|0^2] - 11\,[1^2|0]\} \end{aligned}$$

TABLE 4. Canonical invariant basis for G_{H_3}.

We explain how to determine these canonical invariant bases. We pick out the case $\sharp = F_4$ as an example. The remaining cases $\sharp = H_3$, H_4 can be treated in a similar manner. We begin with an invariant basis constructed by Mehta [Mehta 88]. As an invariant basis for the group G_{F_4}, he has given $(\psi_1, \psi_2, \psi_3, \psi_4) = (I_2, I_6, I_8, I_{12})$, where

$$I_{2k} = (8 - 2^{2k-1})S_{2k} + \sum_{i=1}^{k-1} \binom{2k}{2i} S_{2i}S_{2(k-i)} \qquad (k = 1, 3, 4, 6)$$

with $S_m = x_1^m + x_2^m + x_3^m + x_4^m$. Since the degrees $(d_1, d_2, d_3, d_4) = (2, 6, 8, 12)$ are mutually distinct, the invariant basis $(\psi_1, \psi_2, \psi_3, \psi_4)$ is an orthogonal system relative to the inner product (3–1). To determine the canonical invariant basis $(\phi_1, \phi_2, \phi_3, \phi_4)$, we try to solve the system of partial differential equations (3–2). Taking the degrees into account, we can find its solution in the form

$$\left\{ \begin{array}{rcl} \phi_1 &=& \psi_1, \\ \phi_2 &=& \psi_2 + a_1\psi_1^3, \\ \phi_3 &=& \psi_3 + a_2\psi_2\psi_1 + a_3\psi_1^4, \\ \phi_4 &=& \psi_4 + a_4\psi_3\psi_1^2 + a_5\psi_2^2 + a_6\psi_2\psi_1^3 + a_7\psi_1^6, \end{array} \right. \tag{4–1}$$

with some constants $a_1, a_2, \ldots, a_7$. The existence of such constants is guaranteed theoretically. But we must determine them explicitly. Substituting (4–1) into (3–2), we obtain a system of linear equations for $a_1, a_2, \ldots, a_7$. By solving it, $(\phi_1, \phi_2, \phi_3, \phi_4)$ can be determined explicitly. This procedure is quite elaborate and requires computer-assisted calculations (we use Maple for this purpose). Determining $a_1, a_2, \ldots, a_7$ in this manner, we are able to

$$\begin{aligned} \phi_1 &= [1|0^3] \\ \phi_2 &= [3|0^3] - 5\,[2|1|0^2] + 30\,[1^3|0] \\ \phi_3 &= 3\,[4|0^3] - 28\,[3|1|0^2] + 98\,[2^2|0^2] - 84\,[2|1^2|0] + 1512\,[1^4|0] \\ \phi_4 &= [6|0^3] - 22\,[5|1|0^2] + 143\,[4|2|0^2] + 66\,[4|1^2|0] - 308\,[3^2|0^2] \\ &\quad +308\,[3|2|1|0] - 5544\,[3|1^3] - 2310\,[2^3|0] + 4620\,[2^2|1^2] \end{aligned}$$

TABLE 5. Canonical invariant basis for G_{F_4}.

$$
\begin{aligned}
\phi_1 = \;& [1|0^3] \\
\phi_2 = \;& [6|0^3] - 22\,[5|1|0^2] + 99\,[4|2|0^2] + 198\,[4|1^2|0] - 176\,[3^2|0^2] \\
& -66\,[3|2|1|0] - 4752\,[3|1^3] - 330\,[2^3|0] + 3960[2^2|1^2] - 462\sqrt{5}\Delta_4 \\
\phi_3 = \;& 3\,[10|0^3] - 190\,[9|1|0^2] + 2907\,[8|2|0^2] + 5814\,[8|1^2|0] - 14820\,[7|3|0^2] \\
& -63270\,[7|2|1|0] + 61560\,[7|1^3] + 31122\,[6|4|0^2] + 238602\,[6|3|1|0] \\
& +414960\,[6|2^2|0] - 311220[6|2|1^2] - 36556[5^2|0^2] - 204516\,[5|4|1|0] \\
& -1110018\,[5|3|2|0] - 3361176\,[5|3|1^2] + 5913180\,[5|2^2|1] \\
& +1934010\,[4^2|2|0] + 6802380\,[4^2|1^2] - 62244\,[4|3^2|0] \\
& -3267810\,[4|3|2|1] - 28009800\,[4|2^3|0] + 5228496\,[3^3|1] \\
& +17428320\,[3^2|2^2] + 1254\sqrt{5}\Delta_4\{15\,[4|0^3] \\
& -76\,[3|1|0^2] + 158\,[2^2|0^2] - 24\,[2|1^2|0] + 4032\,[1^4]\} \\
\phi_4 = \;& 2\,[15|0^3] - 290\,[14|1|0^2] + 10962\,[13|2|0^2] \\
& +21924\,[13|1^2|0] - 160080\,[12|3|0^2] - 580725\,[12|2|1|0] \\
& -156600\,[12|1^3] + 1124562\,[11|4|0^2] + 6347172\,[11|3|1|0] \\
& +10445220\,[11|2^2|0] + 2401200\,[11|2|1^2] - 4011338\,[10|5|0^2] \\
& -39631806\,[10|4|1|0] - 74993478\,[10|3|2|0] + 93454704\,[10|3|1^2] \\
& -163041480\,[10|2^2|1] + 6894112\,[9|6|0^2] + 153571414\,[9|5|1|0] \\
& +234767325\,[9|4|2|0] - 789274440\,[9|4|1^2] + 487555656\,[9|3^2|0] \\
& +361940880\,[9|3|2|1] + 3353275800\,[9|2^3] - 3877938\,[8|7|0^2] \\
& -350953389\,[8|6|1|0] - 414939366\,[8|5|2|0] + 2156133528\,[8|5|1^2] \\
& -1281658509\,[8|4|3|0] + 1977748380\,[8|4|2|1] - 2823138864\,[8|3^2|1] \\
& -11401137720\,[8|3|2^2] + 465352560\,[7^2|1|0] + 219749820\,[7|6|2|0] \\
& -1551175200\,[7|6|1^2] + 3221273832\,[7|5|3|0] - 13029871680\,[7|5|2|1] \\
& +116338140\,[7|4^2|0] + 5584230720\,[7|4|3|1] + 44208493200\,[7|4|2^2] \\
& +4343290560\,[7|3^2|2] - 4234708296\,[6^2|3|0] + 23526157200\,[6^2|2|1] \\
& -218457174\,[6|5|4|0] - 7528370304\,[6|5|3|1] - 42347082960\,[6|5|2^2] \\
& +9074374920\,[6|4^2|1] - 70578471600\,[6|4|3|2] + 237143664576\,[6|3^3] \\
& +739393512\,[5^3|0] - 4436361072\,[5^2|4|1] + 186327165024\,[5^2|3|2] \\
& -33272708040\,[5|4^2|2] - 186327165024\,[5|4|3^2] + 299454372360\,[4^3|3] \\
& +957\sqrt{5}\Delta_4\{105\,[9|0^3] - 1827\,[8|1|0^2] + 8661\,[7|2|0^2] + 32167\,[7|1^2|0] \\
& -14937\,[6|3|0^2] - 136078\,[6|2|1|0] - 721221\,[6|1^3] + 7350\,[5|4|0^2] \\
& +219203\,[5|3|1|0] + 408861\,[5|2^2|0] + 2676017\,[5|2|1^2] - 204250\,[4^2|1|0] \\
& -335350\,[4|3|2|0] - 1700975\,[4|3|1^2] - 9676225\,[4|2^2|1] \\
& +695970\,[3^3|0] + 8701810\,[3^2|2|1] + 7709820\,[3|2^3]\}
\end{aligned}
$$

TABLE 6. Canonical invariant basis for G_{H_4}.

obtain the result in Table 5 after suitable renormalizations; recall that the canonical invariant polynomials are unique only up to nonzero constant multiples.

Also in the cases $\sharp = H_3, H_4$, the same procedures as explained above with the invariant bases constructed by Mehta [Mehta 88] lead to the results in Tables 4 and 6.

5. MEAN VALUE PROBLEM

The proof of Theorem 2.1 consists of verifying the criterion (3–4) by using the explicit formulas for the canonical invariant bases obtained in Section 4. If $F(k)$ is a fundamental region for the action of G on $P(k)$, then the criterion (3–4) is equivalent to the nonvanishing of the multiple integrals:

$$I_i(k) = \int_{F(k)} \phi_i(x)\,dx \qquad (i = 1, \ldots, n).$$

If P is a regular convex polytope in $\mathbb{R}^n$, then one can take a fundamental region $F(k)$ in the following manner: Take a sequence $P_0, P_1, \ldots, P_n$ of regular polytopes with $P_n = P$ such that P_i is a face of P_{i+1} for each $i \in \{0, 1, \ldots, n-1\}$. Denote by p_i the center of P_i (note that $p_n = 0$). Let $F(k)$ be the k-simplex having $p_0, p_1, \ldots, p_k$ as its vertices.

$\{3,5\}$	p_0	$=$	$(\tau,1,0)$
	p_1	$=$	$(\tau,0,0)$
	p_2	$=$	$\frac{\tau^2}{3}(\tau,0,\tau^{-1})$
$\{5,3\}$	p_0	$=$	$(\tau,0,\tau^{-1})$
	p_1	$=$	$(\tau,0,0)$
	p_2	$=$	$\frac{\tau}{\sqrt{5}}(\tau,1,0)$
$\{3,4,3\}$	p_0	$=$	$(1,0,0,1)$
	p_1	$=$	$(\frac{1}{2},\frac{1}{2},0,1)$
	p_2	$=$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3},1)$
	p_3	$=$	$(0,0,0,1)$
$\{3,3,5\}$	p_0	$=$	$(\tau,1,\tau^{-1},0)$
	p_1	$=$	$(\tau,1,0,0)$
	p_2	$=$	$\frac{2\tau}{3}(\tau,\tau^{-1},0,0)$
	p_3	$=$	$\frac{\tau^2}{4}(\tau^2,1,0,\tau^{-2})$
$\{5,3,3\}$	p_0	$=$	$(\tau^2,1,0,\tau^{-2})$
	p_1	$=$	$\tau(\tau,\tau^{-1},0,0)$
	p_2	$=$	$\frac{2\tau}{\sqrt{5}}(\tau,1,0,0)$
	p_3	$=$	$\frac{\tau^2}{2}(\tau,1,\tau^{-1},0)$

TABLE 7. Vertices of characteristic simplices.

It is easy to see that $F(k)$ becomes a fundamental region for the action of G on $P(k)$. The simplex $F(n-1)$ is called the *characteristic simplex* of P in [Coxeter 73].

For each exceptional regular convex polytope, the vertices $p_0, p_1, \ldots, p_n$ of its characteristic simplex are given as in Table 7. With these data, using Maple, we can evaluate the multiple integrals $I_i(k)$ and check that they do not vanish (see [Iwasaki et al. 01] for full details). This implies that the criterion (3–4) of Theorem 3.2 is verified and therefore Theorem 2.1 is established. Without computer assistance, the proof presented here would not have been possible. We wonder whether there exists a more conceptual proof of the theorem.

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