Kashaev's Conjecture and the Chern-Simons Invariants of Knots and Links

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CONTENTS

R. M. Kashaev conjectured that the asymptotic behavior of the link invariant he introduced [Kashaev 95], which equals the colored Jones polynomial evaluated at a root of unity, determines the hyperbolic volume of any hyperbolic link complement. We observe numerically that for knots 6_3, 8_9 and 8_{20} and for the Whitehead link, the colored Jones polynomials are related to the hyperbolic volumes and the Chern–Simons invariants and propose a complexification of Kashaev's conjecture.

1. INTRODUCTION

In [Kashaev 95], R.M. Kashaev defined a link invariant associated with the quantum dilogarithm, depending on a positive integer N, which is denoted by $\langle L \rangle_N$ for a link L. Moreover, in [Kashaev 97], he conjectured that for any hyperbolic link L, the asymptotics at $N \to \infty$ of $|\langle L \rangle_N|$ gives its volume, that is

$$\mathrm{vol}(L) = 2\pi \lim_{N \to \infty} \frac{\log |\langle L \rangle_N|}{N}$$

with $\mathrm{vol}(L)$ the hyperbolic volume of the complement of L. He showed that this conjecture is true for three doubled knots 4_1, 5_2, and 6_1. Unfortunately, his proof is not mathematically rigorous.

Afterwards, in [Murakami and Murakami 01], the first two authors proved that for any link L, Kashaev's invariant $\langle L \rangle_N$ is equal to the colored Jones polynomial evaluated at $\exp\left(2\pi\sqrt{-1}/N\right)$, which is written by $J_N(L)$, and extended Kashaev's conjecture as follows.

Conjecture 1.1. (Volume conjecture.)

$$\|L\| = \frac{2\pi}{v_3} \lim_{N \to \infty} \frac{\log |J_N(L)|}{N},$$

where $\|L\|$ is the simplicial volume of the complement of L and v_3 is the volume of the ideal regular tetrahedron.

Note that the hyperbolic volume $\mathrm{vol}(L)$ of a hyperbolic link L is equal to $\|L\|$ multiplied by v_3. This conjecture is not true for links in general, as $J_N(L)$ vanishes for a split link L. It is shown by Kashaev and O. Tirkkonen in [Kashaev and Tirkkonen 00] that the volume conjecture holds for torus knots. See [Thurston 99] and [Yokota 00, Yokota 02] for discussions about Kashaev's conjecture for hyperbolic knots from the viewpoint of tetrahedron decomposition.

In this paper, following Kashaev's way to analyze the asymptotic behavior of the invariant, we observe numerically, by using MAPLE V (a product of Waterloo Maple Inc.) and SnapPea [Weeks 02], that for the hyperbolic knots 6_3, 8_9, 8_{20}, and for the Whitehead link, the colored Jones polynomials are related to the hyperbolic volumes and the Chern–Simons invariants. Note that the knots 6_3 and 8_9 are not doubles of the unknot.

We also discuss a relation between the asymptotic behavior of $J_N(L)$ and the Chern–Simons invariant of the complement of the above-mentioned links L, and propose the following conjecture.

Conjecture 1.2. (Complexification of Kashaev's conjecture.) *Let L be a hyperbolic link. Then the following formula holds.*

$$J_N(L) \sim \exp \frac{N}{2\pi}(\mathrm{vol}(L) + \sqrt{-1}\,\mathrm{CS}(L)) \quad (N \to \infty)$$

where $\mathrm{CS}(L)$ is the Chern–Simons invariant of L [Chern and Simons 74, Meyerhoff 86]. Note that the complement of L is a hyperbolic manifold with cusps.

The statement of this conjecture will be given more properly in the last section.

2.   PRELIMINARIES

First, we will briefly review the colored Jones polynomials of links following [Kirby and Melvin 91]. It is obtained from the quantum group $U_q(sl(2,\mathbb{C}))$ and its N-dimensional irreducible representation.

Let L be an oriented link. We consider a $(1,1)$-tangle presentation of L, obtained by cutting a component of the link. We assume that all crossing and local extreme points are as in Figure 1. We can calculate the N-colored Jones polynomial $J_L(N)$ evaluated at the N-th root of unity for L in the following way. We start with a labeling of the edges of the $(1,1)$-tangle presentation with labels $\{0,1,\ldots,N-1\}$. Here we label the two edges containing the end points of the tangle by 0. Following the labeling,
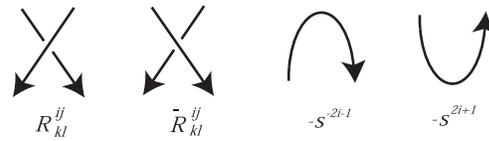

FIGURE 1. Crossings and local extrema in a link diagram.

we associate a positive (respectively, negative) crossing with the element R_{kl}^{ij} (respectively, $\bar{R}_{kl}^{ij}$), a maximal point $\cap$ labeled by i with the element $-s^{-2i-1}$, and a minimal point $\cup$ labeled by i with the element $-s^{2i+1}$ with $s = \exp\left(\frac{\pi\sqrt{-1}}{N}\right)$ as in Figure 1.

Here R_{kl}^{ij} and $\bar{R}_{kl}^{ij}$ are given by

$$R_{kl}^{ij} =$$

$$\sum_{n=0}^{\min(N-1-i,j)} \delta_{l,i+n}\delta_{k,j-n} \frac{(i+n)!(N-1+n-j)!}{(i)!(N-1-j)!(n)!}$$

$$\times\, s^{2(i-\frac{N-1}{2})(j-\frac{N-1}{2})-n(i-j)-\frac{n(n+1)}{2}},$$

$$\bar{R}_{kl}^{ij} =$$

$$\sum_{n=0}^{\min(N-1-j,i)} \delta_{l,i-n}\delta_{k,j+n} \frac{(j+n)!(N-1+n-i)!}{(j)!(N-1-i)!(n)!}(-1)^n$$

$$\times\, s^{-2(i-\frac{N-1}{2})(j-\frac{N-1}{2})-n(i-j)+\frac{n(n+1)}{2}}$$

with $(n)! = (s-s^{-1})(s^2-s^{-2})\cdots(s^n-s^{-n})$.

After multiplying all elements associated with the critical points, we sum over all indices, ignoring framings of links.

We calculate the colored Jones polynomial of the Whitehead link as an example. We can label each edge in the following way, noting Kronecker's deltas in R_{kl}^{ij} and $\bar{R}_{kl}^{ij}$.

We have to rotate a crossing where edges go up. In that case, we use $\cup$ and/or $\cap$ to calculate the invariant.
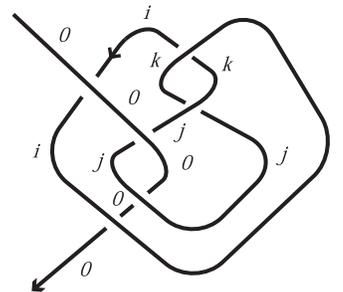

FIGURE 2. Labeling of the Whitehead link.

$$J_N(6_3) = \sum_{\substack{0 \le k,l,m \\ k+l+m \le N-1}} (-1)^{k+l} s^{\frac{(l+k)(l+k+1)}{2} - \frac{(m+k)(m+k+1)}{2} + \frac{k(k+1)}{2} + 2(m-l)(k+1) + N(m-l+k)}$$

$$\times \frac{(N-1-l)!(N-1-m)!(l+m+k)!(N-1)!(1-s^{-2N-2})\cdots(1-s^{-2N-2k})}{(N-1-l-m-k)!(N-1-l-k)!(N-1-m-k)!(l)!(m)!(k)!}.$$

FIGURE 3. Calculation of $J_N(6_3)$.

Then we calculate the formula

$$J_N(L) =$$
$$\sum_{\substack{0 \le i,j,k \le N-1 \\ i,j \ge k}} \frac{(q)_i (q)_j \{(q)_{N-1-k}\}^2}{\{(q)_k\}^2 (q)_{N-1-i}(q)_{N-1-j}(q)_{i-k}(q)_{j-k}} q^{-k(i+j+1)},$$

$$(2\text{-}1)$$

where $q = s^2 = \exp\left(\frac{2\pi\sqrt{-1}}{N}\right)$. Here $(x)_k = (1-x)(1-x^2)\cdots(1-x^k)$.

Next, the Chern–Simons invariant of a link is defined. Let $\mathcal{A}$ be the set of all $SO(3)$-connections of the trivial $SO(3)$-bundle of a closed three-manifold M and cs: $\mathcal{A} \to \mathbb{R}$ the Chern–Simons functional defined by

$$\text{cs}(A) = \frac{1}{8\pi^2} \text{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

The Chern–Simons invariant of the connection A is then defined to be the integral

$$\text{cs}_M(A) = \int_{s(M)} \text{cs}(A) \in \mathbb{R}/\mathbb{Z},$$

where the integral is over a section s of the $SO(3)$-bundle (i.e., an orthonormal frame field on M) [Chern and Simons 74]. If M is hyperbolic, we define $\text{cs}(M)$ to be the Chern–Simons invariant of the connection defined by the hyperbolic metric.

The definition of the Chern–Simons invariant for hyperbolic three-manifolds with cusps is due to R. Meyerhoff [Meyerhoff 86]. It is defined modulo $1/2$ by using a special singular frame field which is linear near the cusps. See [Meyerhoff 86] for details. See [Coulson et al. 00] to examine how it is computed by SnapPea [Weeks 02]. Throughout this paper, we use another normalization $\text{CS}(M) = -2\pi^2 \text{cs}(M)$ so that $\text{vol}(M) + \sqrt{-1}\,\text{CS}(M)$ is a natural complexification of the hyperbolic volume $\text{vol}(M)$ (see [Neumann and Zagier 85, Yoshida 85]).

3.   KNOT 6_3

Let us calculate the colored Jones polynomial of the knot 6_3 using the labeling as in Figure 4.

Putting $k = n_1 + n_2$ and using the formula in [Murakami and Murakami 01]

$$\sum_{i=0}^{N-1} (-1)^i s^{\beta i} \begin{bmatrix} \alpha \\ i \end{bmatrix} = \prod_{j=1}^{\alpha} (1 - s^{\beta+\alpha+1-2j}) \qquad (3\text{-}1)$$

with $\alpha = k$, $i = n_1$, and $\beta = -k - 1 - 2N$, we calculate $J_N(6_3)$ as shown in Figure 3.

The colored Jones polynomial of the knot 6_3 is given by

$$J_N(6_3) = \sum_{\substack{k,l,m \ge 0 \\ k+l+m \le N-1}} \left| \frac{(q)_{k+l+m}}{(q)_l (q)_m} \right|^2 (q)_{k+l} (\bar{q})_{m+k} \, q^{(m-l)(k+1)}.$$

$$(3\text{-}2)$$

We review the technique in [Kashaev 97]. For a complex number p and a positive real number γ with $|\operatorname{Re} p| < \pi + \gamma$, we define

$$S_\gamma(p) = \exp \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{px}}{\sinh(\pi x)\sinh(\gamma x)} \frac{dx}{x}.$$

Here Re denotes the real part. This function has two properties:

(a) $\left(1 + \exp(\sqrt{-1}p)\right) S_\gamma(p + \gamma) = S_\gamma(p - \gamma)$,

(b) $S_\gamma(p) \sim \exp\left(\frac{1}{2\gamma\sqrt{-1}} \text{Li}_2\left(-\exp(\sqrt{-1}p)\right)\right)$ $(\gamma \to 0)$,
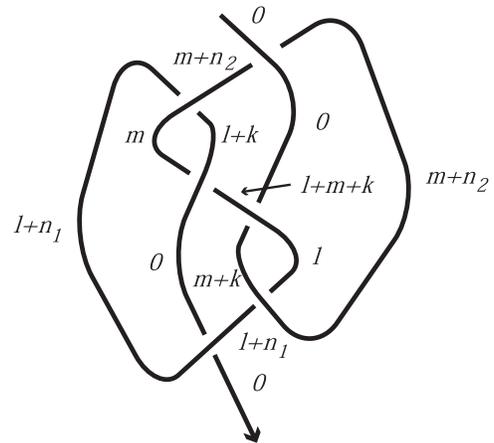

FIGURE 4. Labeling of 6_3.

where

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\,du.$$

We put

$$f_\gamma(p) = \frac{S_\gamma(\gamma - \pi)}{S_\gamma(p)}, \quad \bar{f}_\gamma(p) = \frac{S_\gamma(-p)}{S_\gamma(\pi - \gamma)},$$

so that

$$(q)_k = f_\gamma(-\pi + (2k+1)\gamma), \quad (\bar{q})_k = \bar{f}_\gamma(-\pi + (2k+1)\gamma).$$

Following Kashaev's analysis, we rewrite the formula (3–2) as a multiple integral with appropriately chosen contours. (Note that there is considerable doubt as to the contours.) By using the property (b), it can be asymptotically approximated by

$$\iiint \exp\frac{\sqrt{-1}}{2\gamma} V_{6_3}(z, u, v)\,dz\,du\,dv$$

with $\gamma = \pi/N$. Here z, u, and v correspond to q^k, q^m, and q^l, respectively, and

$$V_{6_3}(z, u, v) = \text{Li}_2(zuv) - \text{Li}_2\left(\frac{1}{zuv}\right) + \text{Li}_2(zv)$$

$$- \text{Li}_2\left(\frac{1}{zu}\right) - \text{Li}_2(u) + \text{Li}_2\left(\frac{1}{u}\right)$$

$$- \text{Li}_2(v) + \text{Li}_2\left(\frac{1}{v}\right) - \log z \log \frac{u}{v}.$$

Then there exists a stationary point
$(z_0, u_0, v_0) = (0.204323 - 0.978904\sqrt{-1}, 1.60838 + 0.558752\sqrt{-1}, 0.554788 + 0.192734\sqrt{-1})$ of V_{6_3} with

$$\text{Im}\, V_{6_3}(z_0, u_0, v_0) < 0, \quad \arg z_0 + \arg u_0 + \arg v_0 \le 2\pi,$$

and we have

$$- \text{Im}\, V_{6_3}(z_0, u_0, v_0) = 5.693021\ldots,$$

$$\text{Re}\, V_{6_3}(z_0, u_0, v_0) = 0.$$

From values of $\text{vol}(6_3)$ and $\text{CS}(6_3)$ given by SnapPea, we see that the equation

$$\exp\frac{\sqrt{-1}}{2\gamma} V_{6_3}(z_0, u_0, v_0) = \exp\frac{\text{vol}(6_3) + \sqrt{-1}\,\text{CS}(6_3)}{2\gamma}$$

holds up to digits shown above.

4. KNOT 8_9

We label the edges of the $(1,1)$-tangle presentation of the knot 8_9 as in Figure 5.
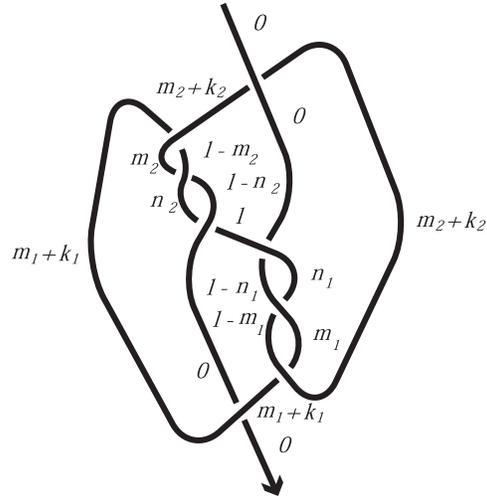

FIGURE 5. Labeling of 8_9.

We obtain the following formula of the colored Jones polynomial of the knot 8_9, where we put $l = m_1 + m_2 + k_1 + k_2$ and use the formula (3–1):

$$J_N(8_9) = \sum_{\substack{0 \le l, m_1, m_2, n_1, n_2 \le N-1 \\ m_1 + n_1, m_2 + n_2 \le l \\ m_1 + m_2 \le l}} \left| \frac{(q)_{l-m_1}(q)_l(q)_{l-m_2}}{(q)_{m_1}(q)_{m_2}(q)_{n_1}(q)_{n_2}} \right|^2$$

$$\times \frac{(\bar{q})_{l-n_1}(q)_{l-n_2}}{(q)_{l-m_1-n_1}(\bar{q})_{l-m_2-n_2}}$$

$$\times q^{(m_2-m_1)(l-m_1-m_2)+(n_2-n_1)(l-n_1-n_2)+m_2-m_1+n_2-n_1},$$

which can be asymptotically approximated by

$$\int \cdots \int \exp\frac{\sqrt{-1}}{2\gamma} V_{8_9}(x, y, z, u, v)\,dx\,dy\,dz\,du\,dv,$$

where x, y, z, u, and v correspond to q^{-l}, q^{m_1}, q^{m_2}, q^{n_1}, and q^{n_2} respectively, and

$$V_{8_9}(x, y, z, u, v)$$

$$= -\text{Li}_2(xy) + \text{Li}_2\left(\frac{1}{xy}\right) - \text{Li}_2(xz) + \text{Li}_2\left(\frac{1}{xz}\right)$$

$$- \text{Li}_2(xu) + \text{Li}_2\left(\frac{1}{xv}\right) - \text{Li}_2(x) + \text{Li}_2\left(\frac{1}{x}\right) - \text{Li}_2(y)$$

$$+ \text{Li}_2\left(\frac{1}{y}\right) - \text{Li}_2(z) + \text{Li}_2\left(\frac{1}{z}\right) - \text{Li}_2(u) + \text{Li}_2\left(\frac{1}{u}\right)$$

$$- \text{Li}_2(v) + \text{Li}_2\left(\frac{1}{v}\right) + \text{Li}_2(xzv) - \text{Li}_2\left(\frac{1}{xyu}\right)$$

$$- \log\frac{y}{z}\log(xzv) - \log\frac{u}{v}\log(xyu).$$

Thus, we have

$$- \text{Im}\, V_{8_9}(x_0, y_0, z_0, u_0, v_0) = 7.5881802\ldots,$$

$$\text{Re}\, V_{8_9}(x_0, y_0, z_0, u_0, v_0) = 0$$

for

$$x_0 = 0.7366011609 - 0.6763273835\sqrt{-1},$$
$$y_0 = 0.4472176075 - 0.1647027124\sqrt{-1},$$
$$z_0 = 1.968989044 - 0.7251455025\sqrt{-1},$$
$$u_0 = 0.3859112582 - 0.0202712198\sqrt{-1},$$
$$v_0 = 2.584139126 - 0.1357401508\sqrt{-1}$$

satisfying

$$\operatorname{Im} V_{8_9}(x_0, y_0, z_0, u_0, v_0) < 0,$$
$$\arg x_0 + \arg y_0 + \arg u_0 \leq 2\pi,$$
$$\arg x_0 + \arg z_0 + \arg v_0 \leq 2\pi,$$
$$\arg x_0 + \arg u_0 + \arg v_0 \leq 2\pi.$$

It follows from the calculation by SnapPea that

$$\exp \frac{\sqrt{-1}}{2\gamma} V_{8_9}(x_0, y_0, z_0, u_0, v_0)$$
$$= \exp \frac{\operatorname{vol}(8_9) + \sqrt{-1}\, \mathrm{CS}(8_9)}{2\gamma},$$

up to digits shown above.

5.   KNOT 8_{20}

In this section, we discuss a relation between the asymptotic behavior of the colored Jones polynomial and the Chern–Simons invariant for the knot 8_{20}. We label each edge in the diagram of the knot in Figure 6.
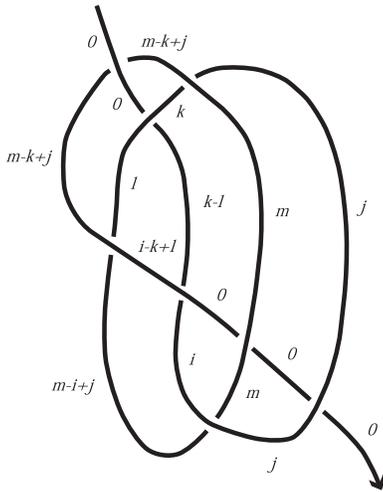

FIGURE 6. Labeling of 8_{20}.

The N-colored Jones polynomial of the knot 8_{20} is given by

$$\sum_{\substack{j,l \leq k \leq i+l \leq j+m \\ j \leq i \\ 0 \leq i,j,k,l,m \leq N-1}} \frac{\{(\bar{q})_i (q)_k (\bar{q})_m\}^2}{\{(\bar{q})_j (q)_l\}^2 (q)_{k-l} (\bar{q})_{i-k+l} (\bar{q})_{j+m-i-l} (q)_{i-j} (q)_{k-j}}$$
$$\times q^{k+m+im+km-il}, \quad (5\text{–}1)$$

which can be rewritten in the integral

$$\int \cdots \int \exp \frac{\sqrt{-1}}{2\gamma} V_{8_{20}}(x, y, z, u, v)\, dx\, dy\, dz\, du\, dv$$

with

$$V_{8_{20}}(x, y, z, u, v) = -2\mathrm{Li}_2(x) + 2\mathrm{Li}_2\left(\frac{1}{y}\right) + 2\mathrm{Li}_2(z)$$
$$- 2\mathrm{Li}_2\left(\frac{1}{u}\right) - 2\mathrm{Li}_2\left(\frac{1}{x}\right) - \mathrm{Li}_2\left(\frac{1}{xy}\right)$$
$$- \mathrm{Li}_2\left(\frac{z}{y}\right) - \mathrm{Li}_2(zu) + \mathrm{Li}_2(xzu)$$
$$+ \mathrm{Li}_2\left(\frac{1}{xyuv}\right) + \log x \log u$$
$$+ \log x \log v - \log z \log v + \frac{\pi^2}{2}.$$

Here x, y, z, u, and v correspond to q^{-i}, q^j, q^k, q^{-l}, and q^m, respectively.

Stationary points are solutions to partial differential equations,

$$\frac{\partial V_{8_{20}}}{\partial x} = \frac{\partial V_{8_{20}}}{\partial y} = \frac{\partial V_{8_{20}}}{\partial z} = \frac{\partial V_{8_{20}}}{\partial u} = \frac{\partial V_{8_{20}}}{\partial v} = 0.$$

From these equations, we have the following system of algebraic equations:

$$(1-x)^2 \left(1 - \frac{1}{xyuv}\right) uv = \left(1 - \frac{1}{xy}\right)(1 - xzu),$$

$$\left(1 - \frac{1}{xy}\right)\left(1 - \frac{z}{y}\right) = \left(1 - \frac{1}{y}\right)^2 \left(1 - \frac{1}{xyuv}\right),$$

$$(1-z)^2 (1 - xzu)\, v = (1 - zu)\left(1 - \frac{z}{y}\right),$$

$$(1 - zu)\left(1 - \frac{1}{xyuv}\right) x = \left(1 - \frac{1}{u}\right)^2 (1 - xzu),$$

$$\left(1 - \frac{1}{v}\right)^2 z = \left(1 - \frac{1}{xyuv}\right) x.$$

Using MAPLE V, we get a stationary point $(x_0, y_0, z_0, u_0, v_0)$ which satisfies the conditions

$$\arg \frac{1}{u_0} \leq \arg z_0, \quad \arg z_0 \leq \arg \frac{1}{x_0} + \arg \frac{1}{u_0}$$

from the range in the summation in (5–1), and

$$\operatorname{Im} V_{8_{20}}(x_0, y_0, z_0, u_0, v_0) < 0,$$

where Im denotes the imaginary part. Note that the range of (5–1) can be read as

$$\arg \frac{1}{u} \le \arg z \le \arg \frac{1}{x} + \arg \frac{1}{u},$$
$$\arg \frac{1}{x} + \arg \frac{1}{y} + \arg \frac{1}{u} \le \arg v,$$
$$0 \le \arg \frac{1}{x} + \arg \frac{1}{y},$$
$$0 \le \arg \frac{1}{x}, \arg z, \arg \frac{1}{u}, \arg v \le 2\pi.$$

To put it concretely,

$$x_0 = 2.878599677 + 2.657408013\sqrt{-1},$$
$$y_0 = \infty,$$
$$z_0 = -0.4425377456 - 0.4544788919\sqrt{-1},$$
$$u_0 = 0.3542198353 - 0.02180673815\sqrt{-1},$$
$$v_0 = 0.1458832937 - 0.3399257634\sqrt{-1}.$$

Then we obtain

$$-\operatorname{Im} V_{8_{20}}(x_0, y_0, z_0, u_0, v_0) = 4.1249032\ldots,$$
$$-\frac{\operatorname{Re} V_{8_{20}}(x_0, y_0, z_0, u_0, v_0) + \pi^2}{2\pi^2} = 0.1033634\ldots.$$

Applying values of $\operatorname{vol}(8_{20})$ and $\operatorname{CS}(8_{20})$ given by Snap-Pea [Weeks 02], we see that the following equation holds up to digits shown above.

$$\exp \frac{\sqrt{-1}}{2\gamma} V_{8_{20}}(x_0, y_0, z_0, u_0, v_0)$$
$$= \exp \frac{\operatorname{vol}(8_{20}) + \sqrt{-1}\operatorname{CS}(8_{20})}{2\gamma}.$$

Note that $\operatorname{CS}(8_{20})$ is defined modulo π^2.

6. WHITEHEAD LINK

For the final example, we calculate the limit of the colored Jones polynomial of the Whitehead link given by (2–1), which can be changed to the formula

$$J_N(L) = \sum_{\substack{0 \le i,j,k \le N-1 \\ k \le i,j}} \frac{\{(\bar{q})_i (\bar{q})_j\}^2}{(q)_k^4 (\bar{q})_{i-k} (\bar{q})_{j-k}} q^{-(N-1)N/2}.$$

This can be asymptotically approximated by

$$\iiint \exp \frac{\sqrt{-1}}{2\gamma} V_L(x, y, z) \, dx \, dy \, dz,$$

where

$$V_L(x, y, z) = -2\operatorname{Li}_2\left(\frac{1}{x}\right) - 2\operatorname{Li}_2\left(\frac{1}{y}\right) - 4\operatorname{Li}_2(z)$$
$$+ \operatorname{Li}_2\left(\frac{z}{x}\right) + \operatorname{Li}_2\left(\frac{z}{y}\right) + \pi^2,$$

and x, y, and z correspond to q^i, q^j, and q^k respectively. For a stationary point $(x_0, y_0, z_0) = (\infty, \infty, 1 + \sqrt{-1})$, we obtain

$$-\operatorname{Im} V_L(x_0, y_0, z_0) = 3.663862\ldots,$$
$$-\frac{\operatorname{Re} V_L(x_0, y_0, z_0)}{2\pi^2} = -0.1250000\ldots.$$

Since these values agree with SnapPea, the equation

$$\exp \frac{\sqrt{-1}}{2\gamma} V_L(x_0, y_0, z_0) = \exp \frac{\operatorname{vol}(L) + \sqrt{-1}\operatorname{CS}(L)}{2\gamma}$$

holds up to digits shown above.

7. TOPOLOGICAL CHERN-SIMONS INVARIANT AND SOME EXAMPLES

We propose a topological definition of the Chern–Simons invariant for links.

For a link L, if there exists the limit

$$2\pi \operatorname{Im} \lim_{N \to \infty} \log \frac{J_{N+1}(L)}{J_N(L)} \quad \mod \pi^2,$$

then we denote it by $\operatorname{CS}_{\operatorname{TOP}}(L)$ and call it the *topological Chern–Simons invariant* of L.

Let us give some numerical examples.

For the knot 5_2, we list some values of $(N, 2\pi \log(J_{N+1}(5_2)/J_N(5_2)))$ by Pari-Gp in Table 1.

By fitting the above data to quadratic functions on $1/N$, we can obtain the limit value

$$2.82813 - 3.02414\sqrt{-1}$$

of $2\pi \log(J_{N+1}(5_2)/J_N(5_2))$ as $N \to \infty$ numerically, which agrees with the value

$$2.8281220 - 3.02412837\sqrt{-1}$$

by SnapPea. We display our data graphically in Figures 7 and 8, which help us to see the limit.

Similarly, for the Whitehead link L, we illustrate our numerical check in Table 2, Figure 9, and Figure 10.

Fitting, we get the numerical limit value $3.66386 + 2.46742\sqrt{-1}$ of $2\pi \log(J_{N+1}(L)/J_N(L))$ as $N \to \infty$, which agrees with our result in Section 6.

$(40, 3.0582237212618427226138859566 - 3.0229246132817202873919749685\sqrt{-1})$
$(50, 3.0130815085301883535738548221 - 3.0233403685175070691348557805\sqrt{-1})$
$(60, 2.9827443187535806968217722991 - 3.0235740428789354296457206401\sqrt{-1})$
$(70, 2.9609554049617391707491141511 - 3.0237173817863748529305746316\sqrt{-1})$
$(80, 2.9445482691704501124469663011 - 3.0238115749684722877186117111\sqrt{-1})$
$(100, 2.9214839061082289301846921212 - 3.0239237190278335556695024806\sqrt{-1})$
$(120, 2.9060464213886660002825423986 - 3.0239853749303072344439866322\sqrt{-1})$
$(150, 2.8905598819075371283720015116 - 3.0240362951439691790287709012\sqrt{-1})$
$(200, 2.8750242342269416203271563506 - 3.0240762665585453408524106316\sqrt{-1})$
$(250, 2.8656792509695385315620990561 - 3.0240949058113493751391493313\sqrt{-1})$

TABLE 1.

$(40, 3.8929203591018110978095255836 + 2.4574839973308660458125047036\sqrt{-1})$
$(50, 3.8481614664029142251545301806 + 2.4610394740180165698697453016\sqrt{-1})$
$(60, 3.8180290133494993127082361536 + 2.4629767486759802547033908556\sqrt{-1})$
$(70, 3.7963625012095376910789445566 + 2.4641471917958816145824764516\sqrt{-1})$
$(80, 3.7800343275600221950820153856 + 2.4649079234047646222743958686\sqrt{-1})$
$(100, 3.7570622589854778572479912396 + 2.4658037859628196792363273396\sqrt{-1})$
$(120, 3.7416746081790236731591442586 + 2.4662910858966602606886061426\sqrt{-1})$
$(150, 3.7262286497265585905078284296 + 2.4666902040110300079621138806\sqrt{-1})$

TABLE 2. $(N, 2\pi \log(J_{N+1}(L)/J_N(L)))$ for the Whitehead link L
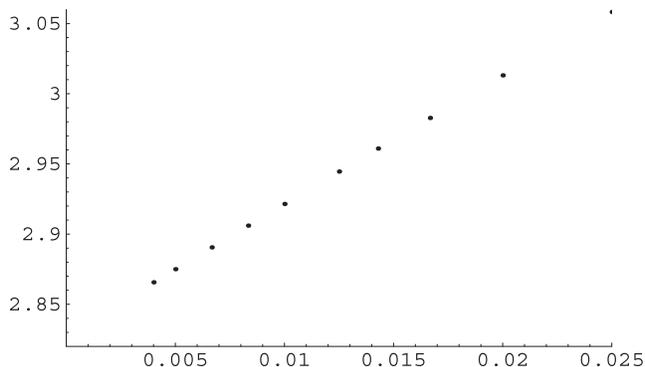

FIGURE 7.
Dots indicate $(1/N, 2\pi \operatorname{Re} \log(J_{N+1}(5_2)/J_N(5_2)))$ for $N = 40, 50, 60, 70, 80, 100, 120, 150, 200, 250$. The origin corresponds to $(0, 2.82)$.
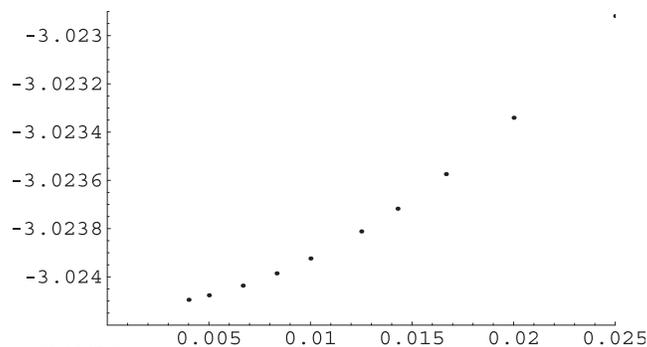

FIGURE 8.
Dots indicate $(1/N, 2\pi \operatorname{Im} \log(J_{N+1}(5_2)/J_N(5_2)))$ for $N = 40, 50, 60, 70, 80, 100, 120, 150, 200, 250$. The origin corresponds to $(0, -3.0242)$.
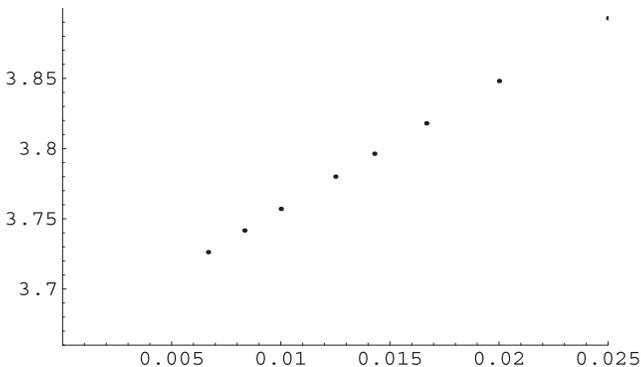
FIGURE 9.
Dots indicate $(1/N, 2\pi \operatorname{Re} \log(J_{N+1}(L)/J_N(L)))$ for $N = 40, 50, 60, 70, 80, 100, 120, 150$. The origin corresponds to $(0, 3.66)$.
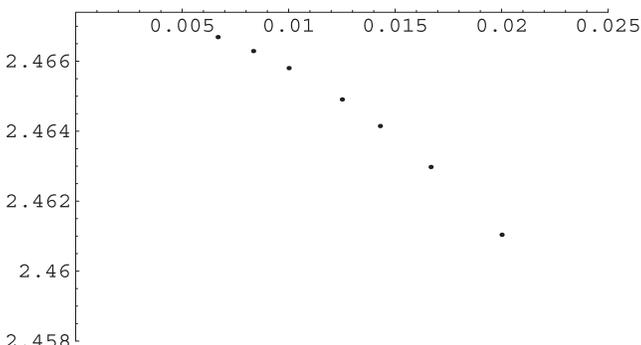

FIGURE 10.
Dots indicate $(1/N, 2\pi \operatorname{Im} \log(J_{N+1}(L)/J_N(L)))$ for $N = 40, 50, 60, 70, 80, 100, 120, 150$. The origin corresponds to $(0, 2.4674)$.

8. CONCLUSION

We have shown the following by direct calculation.

Observation 8.1. Let L be one of the hyperbolic knots 6_3, 8_9, and 8_{20}, or the Whitehead link. Following Kashaev's way, we approximate the colored Jones polynomial $J_N(L)$ of L asymptotically by

$$\int \cdots \int \exp \frac{N\sqrt{-1}}{2\pi} V_L(\mathbf{x}) d\mathbf{x}.$$

Then there exists a stationary point $\mathbf{x}_0$ of V_L such that the formula

$$\exp \frac{N\sqrt{-1}}{2\pi} V_L(\mathbf{x}_0) = \exp \frac{N}{2\pi} (\operatorname{vol}(L) + \sqrt{-1}\, \mathrm{CS}(L))$$

holds up to 6 digits.

Conjecture 8.2. (Complexification of Kashaev's conjecture.) *Let L be a hyperbolic link. Then, it holds that*

$$\operatorname{vol}(L) = 2\pi \lim_{N \to \infty} \frac{\log |\langle L \rangle_N|}{N}$$

with $\operatorname{vol}(L)$ the hyperbolic volume of the complement of L. Moreover, there exists the topological Chern–Simons invariant $\mathrm{CS}_{TOP}(L)$ of L

$$\mathrm{CS}_{TOP}(L) = 2\pi \operatorname{Im} \lim_{N \to \infty} \log \frac{J_{N+1}(L)}{J_N(L)} \mod \pi^2,$$

and $\mathrm{CS}_{TOP}(L)$ equals to $\mathrm{CS}(L)$ modulo π^2. Here $\mathrm{CS}(L)$ is the Chern–Simons invariant of L [Chern and Simons 74, Meyerhoff 86]. Note that the complement of L is a hyperbolic manifold with cusps.

We note that Observation 8.1 also holds for the knots 4_1, 5_2 and 6_1 by calculating Kashaev's examples in [Kashaev 97] using MAPLE V and SnapPea.

Therefore we conclude that the complexified Kashaev conjecture is true, up to several digits, up to choices of contours when we change summations into integrals, and up to choices of saddle (stationary) points when we approximate integrals by the saddle point method, for the six hyperbolic knots above and for the Whitehead link.

Note that if the complexified Kashaev conjecture is true then the topological Chern–Simons invariant of a hyperbolic link coincides with its Chern–Simons invariant associated with the hyperbolic metric. Moreover if the volume conjecture is true then the colored Jones polynomial would give both the simplicial volume and the topological Chern–Simons invariant for any knot.

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