Some Remarks on the Distribution of a Sequence Connected with $\zeta\left(\frac{1}{2}\right)$

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CONTENTS

As a complement to a recent paper by Jade Vinson we study the distribution of the sequence $(\sum_{j=1}^{n} j^{-s})_{n\geq 1}$ modulo 1 with the aim of explaining its different behaviour when $s = \frac{1}{2}$ and when $\frac{1}{2} < s < 1$. We tackle this question from a different point of view using the theory of uniformly distributed sequences.

1. INTRODUCTION

In a recent paper, Jade Vinson [Vinson 01] studied the distribution modulo 1 of the sequence $(\sum_{j=1}^{n} j^{-s})_{n\geq 1}$ (where $s \in (0,1)$) with the aim of explaining the striking difference between the distributions when $s = \frac{1}{2}$ and $s \neq \frac{1}{2}$. We try to shed more light on some of the phenomena described in [Vinson 01] using the theory of uniform distribution.

1.1 Notation

If $x \in \mathbb{R}$, then $\{x\} = x - [x]$ denotes the fractional part of x. We use $\omega_n^s = \sum_{j=1}^{n} j^{-s}$ and $\omega^s = (\omega_n^s)_{n\geq 1}$ as convenient shorthand notation. If $(x_n)_{n\geq 1}$ is a sequence in the unit interval $[0,1)$, then

$$D_N(x_1,\ldots,x_N) = \sup_{0\leq a<b\leq 1}\left|\frac{\left|\{n : 1\leq n\leq N, x_n\in[a,b)\}\right|}{N} - (b-a)\right|$$

is called the discrepancy of this sequence. If $\xi = (\xi_n)_{n\geq 1}$ is a sequence of real numbers, we write $D_N(\xi)$ instead of $D_N\big(\{\xi_1\},\ldots,\{\xi_N\}\big)$. We recall that a sequence ξ of reals is uniformly distributed modulo 1 if and only if $D_N(\xi) \to 0$ as $N \to \infty$.

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2. UNIFORMLY DISTRIBUTED SEQUENCES

Theorem 2.1. *Let $s \in (0,1)$. Then ω^s is uniformly distributed modulo 1 and*

$$\left| D_N(\omega^s) - D_N\big((\tfrac{1}{1-s}n^{1-s})_{n\geq 1}\big)\right| \leq 2N^{-s/(s+1)}$$

for all positive integers N.

Proof: Using the Euler summation formula, we find that

$$\sum_{j=1}^{n} j^{-s} = \tfrac{1}{1-s}n^{1-s} + \zeta(s) + s\int_n^\infty \{t\}t^{-s-1}\,dt \quad (2\text{–}1)$$

for all positive integers n (see e.g., Theorem 3.2 in [Apostol 76]. This implies that

$$\tfrac{1}{1-s}n^{1-s} + \zeta(s) < \omega_n^s < \tfrac{1}{1-s}n^{1-s} + \zeta(s) + n^{-s}. \quad (2\text{–}2)$$

The sequence $(\frac{1}{1-s}n^{1-s})_{n\geq 1}$ is known to be uniformly distributed modulo 1 (see e.g., Example 2.7 in Chapter 1 of [Kuipers and Niederreiter 74]). It follows from Lemma 1.1 in Chapter 1 of [Kuipers and Niederreiter 74] that $\big(\frac{1}{1-s}n^{1-s} + \zeta(s)\big)_{n\geq 1}$ is uniformly distributed modulo 1, and from Theorem 1.2 in Chapter 1 of [Kuipers and Niederreiter 74] and Equation (2–2) that ω^s is uniformly distributed modulo 1. The assertion about discrepancies is implied by the fact that

$$D_N\big((\tfrac{1}{1-s}n^{1-s} + \zeta(s))_{n\geq 1}\big) = D_N\big((\tfrac{1}{1-s}n^{1-s})_{n\geq 1}\big)$$

for all positive integers N and the following lemma by taking $\varepsilon = N^{-s/(s+1)}$ there. □

2.1 Notation

From now on, we will use $\tau_n^s = \frac{1}{1-s}n^{1-s} + \zeta(s)$, $\xi_n^s = \frac{1}{1-s}n^{1-s}$, $\tau^s = (\tau_n^s)_{n\geq 1}$ and $\xi^s = (\xi_n^s)_{n\geq 1}$.

Lemma 2.2. *Let s, ω^s and τ^s be as above. Then*

$$\big|D_N(\omega^s) - D_N(\tau^s)\big| \leq \varepsilon + \varepsilon^{-1/s}N^{-1}$$

for all $\varepsilon > 0$.

Proof: Note that if $n > \varepsilon^{-1/s}$, then $n^{-s} < \varepsilon$ and therefore $\tau_n^s < \omega_n^s < \tau_n^s + \varepsilon$ by (2-2). Let $[a,b) \subseteq [0,1)$. If $\{\tau_n^s\} \in [a,b)$, then either $n \leq \varepsilon^{-1/s}$ or $\{\omega_n^s\} \in J$ where

$$J = \begin{cases} [a, b+\varepsilon) & \text{if } b+\varepsilon \leq 1, \\ [0, b+\varepsilon-1) \cup [a,1) & \text{if } 1 < b+\varepsilon < a+1, \\ [0,1) & \text{if } a+1 \leq b+\varepsilon. \end{cases}$$

In all three cases, we have

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\tau_n^s\} \in [a,b)\}\big| \\ \leq \big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in J\}\big| + \varepsilon^{-1/s}.\end{aligned} \quad (2\text{–}3)$$

We claim that

$$\big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in J\}\big| \leq N(b-a+\varepsilon) + ND_N(\omega^s). \quad (2\text{–}4)$$

If $b+\varepsilon \leq 1$, this reduces to

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in [a, b+\varepsilon)\}\big| \\ \leq N(b-a+\varepsilon) + ND_N(\omega^s)\end{aligned}$$

which is obviously true. If $1 < b+\varepsilon < a+1$, this follows from

$$\begin{aligned}&\big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in J\}\big| \\ &= N - \big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in [b+\varepsilon-1, a)\}\big| \\ &\leq N - \big(N(a-b-\varepsilon+1) - ND_N(\omega^s)\big).\end{aligned}$$

Finally, if $a+1 \leq b+\varepsilon$, then $b-a+\varepsilon \geq 1$ and the assertion is trivially fulfilled. Putting Equations (2–3) and (2–4), together we see that

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\tau_n^s\} \in [a,b)\}\big| - N(b-a) \\ \leq N\varepsilon + ND_N(\omega^s) + \varepsilon^{-1/s}.\end{aligned} \quad (2\text{–}5)$$

If $\{\omega_n^s\} \in [a+\varepsilon, b)$, then either $n \leq \varepsilon^{-1/s}$ or $\{\tau_n^s\} \in [a,b)$ which implies that

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in [a+\varepsilon, b)\}\big| \\ \leq \big|\{n : 1 \leq n \leq N, \{\tau_n^s\} \in [a,b)\}\big| + \varepsilon^{-1/s}.\end{aligned} \quad (2\text{–}6)$$

Furthermore, we have

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\omega_n^s\} \in [a+\varepsilon, b)\}\big| \\ \geq N(b-a-\varepsilon) - ND_N(\omega^s).\end{aligned} \quad (2\text{–}7)$$

Both Equations (2–6) and (2–7) remain true if $a+\varepsilon \geq b$. Together they imply that

$$\begin{aligned}\big|\{n : 1 \leq n \leq N, \{\tau_n^s\} \in [a,b)\}\big| - N(b-a) \\ \geq -N\varepsilon - ND_N(\omega^s) - \varepsilon^{-1/s}.\end{aligned} \quad (2\text{–}8)$$

From Equations (2–5) and (2–8), we can now deduce that

$$\begin{aligned}\left|\frac{\big|\{n : 1 \leq n \leq N, \{\tau_n^s\} \in [a,b)\}\big|}{N} - (b-a)\right| \\ \leq D_N(\omega^s) + \varepsilon + N^{-1}\varepsilon^{-1/s}\end{aligned}$$

and therefore $D_N(\tau^s) \leq D_N(\omega^s) + \varepsilon + N^{-1}\varepsilon^{-1/s}$. By an analogous argument, we can prove $D_N(\omega^s) \leq D_N(\tau^s) + \varepsilon + N^{-1}\varepsilon^{-1/s}$ which completes the proof. □

The above theorem tells us that the fractional parts of the sequence ω^s will be spread out evenly over the unit interval in the long run. Furthermore, the deviation from a perfect uniform distribution is comparable to that of the sequence ξ^s. The three papers [Schoißengeier 81], [Baxa and Schoißengeier 98], and [Baxa 98] contain a detailed study of the long-term behaviour of sequences $(\alpha n^\sigma)_{n\geq 1}$ modulo 1 (where $\alpha > 0$ and $0 < \sigma \leq \frac{1}{2}$) and their results can be put to good use.

Corollary 2.3.

(i) If $0 < s \leq (\sqrt{5}-1)/2$, *then* $D_N(\omega^s) = O(N^{-s/(s+1)})$ *as* $N \to \infty$.

(ii) If $(\sqrt{5}-1)/2 < s < 1$, *then* $\lim_{N\to\infty} N^{1-s} D_N(\omega^s) = \frac{1}{8}$.

Proof: (i) First let $1/2 \leq s \leq (\sqrt{5}-1)/2$. As $D_N(\tau^s) = D_N(\xi^s)$ and $D_N(\xi^s) = O(N^{s-1})$ (see [Schoißengeier 81], it follows that

$$\begin{aligned} |D_N(\omega^s)| &\leq |D_N(\omega^s) - D_N(\tau^s)| + |D_N(\xi^s)| \\ &\ll N^{-s/(s+1)} + N^{s-1} \ll N^{-s/(s+1)} \end{aligned}$$

as $-s/(s+1) \geq s-1$ if $1/2 \leq s \leq (\sqrt{5}-1)/2$.

If $0 < s < 1/2$, then $D_N(\xi^s) = O(N^{-s})$ (see Exercise 3.1 in Chapter 2 of [Kuipers and Niederreiter 74] and the assertion can be proved analogously.

(ii) As $s^2 + s - 1 > 0$ for $s > (\sqrt{5}-1)/2$, we get

$$\begin{aligned} N^{1-s}|D_N(\omega^s) - D_N(\tau^s)| &\leq 2N^{1-s-s/(s+1)} \\ &= 2N^{-(s^2+s-1)/(s+1)} \to 0 \end{aligned}$$

as $N \to \infty$ and as $\lim_{N\to\infty} N^{1-s} D_N(\xi^s) = \frac{1}{8}$ (because of Corollary 3 in [Schoißengeier 81] the assertion follows. □

3. CONCLUDING REMARKS

Corollary 2.3 tells us that the sequence ω^s is particularly well-behaved when s is close to 1. As we are mainly concerned with long-term behaviour, some of the phenomena described in Vinson's paper elude us.

Nevertheless, we are able to offer an explanation for the large central spike in the histogram for $\zeta(\frac{1}{2})$ (Figure 1 in [Vinson 01]). During the investigation of sequences of shape $(\alpha n^\sigma)_{n\geq 1}$, it turned out that their behaviour is far more complicated when $\sigma = \frac{1}{2}$ and $\alpha^2 \in \mathbb{Q}$ than when either $0 < \sigma < \frac{1}{2}$ or $\sigma = \frac{1}{2}$ and $\alpha^2 \notin \mathbb{Q}$. As a special instance, the behaviour of $(2\sqrt{n})_{n\geq 1}$ is far more intricate than that of $(\frac{1}{1-s} n^{1-s})_{n\geq 1}$ for $1/2 < s < 1$. Although the sequence $(2\sqrt{n})_{n\geq 1}$ is uniformly distributed modulo 1, its fractional parts attain the value 0 infinitely often as $\{2\sqrt{n}\} = 0$ whenever n is a square. (This behaviour is typical for sequences $(\alpha\sqrt{n})_{n\geq 1}$ with α^2 rational. A detailed description of this phenomenon can be found in Lemma 1 of [Baxa and Schoißengeier 98].)

Because of Equation (2-2), we see that among the N points $\{\omega_1^{1/2}\}, \ldots, \{\omega_N^{1/2}\}$, there are $K := \lceil \sqrt{N} \rceil$ points $\{\omega_1^{1/2}\}, \{\omega_4^{1/2}\}, \{\omega_9^{1/2}\}, \ldots, \{\omega_{K^2}^{1/2}\}$ which form the beginning of a subsequence which will eventually converge to $\{\zeta(\frac{1}{2})\} = \zeta(\frac{1}{2}) + 2$ from above. This should lead to the spike and explains its location.

Surprisingly, the sequence $(\alpha n^\sigma)_{n\geq 1}$ with $\frac{1}{2} < \sigma < 1$ seems to have received far less attention than the case $0 < \sigma \leq \frac{1}{2}$. J. Schoißengeier [Schoißengeier 81] proved that the estimate $D_N\big((\alpha n^\sigma)_{n\geq 1}\big) = O(N^{\sigma-1})$ we used above is not sharp, but this seems to be the last published result on this sequence. It would be an interesting problem to study this case in detail, which should also lead to a better understanding of the behaviour of the sequence ω^s modulo 1 with $0 < s < \frac{1}{2}$.

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REFERENCES

[Apostol 76] T. M. Apostol. *Introduction to Analytic Number Theory.* New York-Heidelberg-Berlin: Springer-Verlag, 1976.

[Baxa 98] C. Baxa. "On the Discrepancy of the Sequence $(\alpha\sqrt{n})$ II." *Arch. Math.* 70 (1998), 366–370.

[Baxa and Schoißengeier 98] C. Baxa and J. Schoißengeier. "On the Discrepancy of the Sequence $(\alpha\sqrt{n})$." *J. London Math. Soc.* 57 (1998), 529–544.

[Kuipers and Niederreiter 74] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* New York-London-Sydney-Toronto: Wiley, 1974.

[Schoißengeier 81] J. Schoißengeier. "On the Discrepancy of Sequences (αn^σ)." *Acta Math. Acad. Sci. Hungar.* 38 (1981), 29–43.

[Vinson 01] J. Vinson. "Partial Sums of $\zeta(\frac{1}{2})$ Modulo 1." *Experiment. Math.* 10 (2001), 337–344.

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