Approximate Squaring

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CONTENTS

We study the "approximate squaring" map $f(x) := x\lceil x \rceil$ and its behavior when iterated. We conjecture that if f is repeatedly applied to a rational number $r = l/d > 1$ then eventually an integer will be reached. We prove this when $d = 2$, and provide evidence that it is true in general by giving an upper bound on the density of the "exceptional set" of numbers which fail to reach an integer. We give similar results for a p-adic analogue of f, when the exceptional set is nonempty, and for iterating the "approximate multiplication" map $f_r(x) := r\lceil x \rceil$, where r is a fixed rational number. We briefly discuss what happens when "ceiling" is replaced by "floor" in the definitions.

1. INTRODUCTION

In this paper we study the "approximate squaring" map $f : \mathbb{Q} \to \mathbb{Q}$ given by

$$f(x) := x\lceil x \rceil \tag{1–1}$$

and consider its behavior when iterated. Although there is an extensive literature on iterated maps (see for example [Collet and Eckmann 80], [Beardon 91], [Lagarias 92]), including the study of various first-order recurrences involving the ceiling function ([Eisele and Hadeler 90], [Graham and Yan 99]), the approximate squaring map seems not to have been treated before and has some interesting features.

The function f behaves qualitatively like iterating the rational function $R(x) = x^2$. Indeed, all points $|x| \leq 1$ have a bounded orbit under $f(x)$, while all points $|x| > 1$ have unbounded orbits and diverge to ∞, just as they do when $R(x)$ is iterated. However, $f(x)$ has the additional feature that it is discontinuous at integer points. It follows that the nth iterate $f^{(n)}$ is discontinuous at a certain set of rational points, namely, those points x where $f^{(n)}(x)$ is an integer.

It is therefore natural to ask: if we start with a rational number r with $|r| > 1$ and iterate f, will we always eventually reach an integer? This question is the subject of our paper.

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Numerical experiments suggest that the answer to our question is "Yes," although it may take many steps, and consequently involve some very large numbers.

For example, starting at $r = \frac{3}{2}$, $f(r) = \frac{3}{2} \cdot 2 = 3$, reaching an integer in one step; and starting at $r = \frac{8}{7}$, we get $f(r) = \frac{16}{7}$, $f^{(2)}(r) = \frac{48}{7}$, and $f^{(3)}(r) = 48$, taking three steps. On the other hand, starting at $r = \frac{6}{5}$, we find

$$\begin{aligned} f(r) &= \frac{12}{5}, \\ f^{(2)}(r) &= \frac{36}{5}, \\ f^{(3)}(r) &= \frac{288}{5}, \\ f^{(4)}(r) &= \frac{16704}{5}, \\ f^{(5)}(r) &= \frac{55808064}{5}, \\ f^{(6)}(r) &= \frac{622908012647232}{5}, \\ f^{(7)}(r) &= \frac{77602878444025201997703040704}{5}, \\ &\vdots, \end{aligned}$$

and we do not reach an integer until $f^{(18)}(r)$, which is a number with 57735 digits.

We note that for any rational starting point r, since $\lceil x \rceil$ is an integer, the denominators d_j of the iterates $f^{(j)}(r)$ must form a nonincreasing sequence with d_{j+1} dividing d_j. For $0 < r \leq 1$, $\lceil r \rceil = 1$ and $f(r) = r$, so there the denominator is fixed. For $-1 < r \leq 0$, $f(r) = 0$, and for $r \leq -1$, $f(r) \geq 1$. So it is sufficient to restrict our attention to the case of rationals $r > 1$.

We make the following conjecture:

Conjecture 1.1. *For each rational $r \in \mathbb{Q}$ with $r > 1$, there is an integer $m \geq 0$ such that $f^{(j)}(r)$ is an integer for all $j \geq m$.*

We establish the conjecture in the special case when the denominator is two, where a complete analysis is possible. This is done in Section 2.

In Section 3, we consider the case of rational starting values r with a fixed denominator $d \geq 3$. We show that the set of starting values that reach an integer after exactly j steps has a limiting density, and that the set of starting values that never reach an integer has density zero. More precisely, at most there is a sparse subset of the rationals $\{r = \frac{l}{d} : d < l \leq x\}$ that fail to become integers under iteration, in the sense that the cardinality of this subset is bounded above by $C(d, \epsilon) x^{1-\alpha_d+\epsilon}$ for a certain positive constant α_d and any positive ϵ, where $C(d, \epsilon)$ is a positive constant depending only on d and ϵ. Showing that this "exceptional set" of starting values that fail to reach integers is in fact empty (or even finite) appears to be a difficult problem, for reasons indicated below. We also show that the set of starting values that reach an integer after exactly j steps has a limiting density.

In Section 4, we consider a p-adic analogue of the approximate squaring map. In this case we show that there is a nonempty exceptional set of elements in $\frac{1}{p^k}\mathbb{Z}_p$ which under iteration never "escape" to the smaller invariant set $\frac{1}{p^{k-1}}\mathbb{Z}_p$. This set has Hausdorff dimension exactly $1 - \alpha_{p^k}$, where α_{p^k} is the same constant that appears in Section 3. The existence of this exceptional set is one reason why it may be a difficult problem to obtain better upper bounds on the cardinality of the exceptional set in Section 3.

In Section 5, we study similar questions concerning the "approximate multiplication" map

$$f_r(x) := r\lceil x \rceil \ , \qquad (1\text{–}2)$$

where r is a fixed rational number. For $r = \frac{1}{b}$, this map is a special case of the map $x \mapsto a + \lceil \frac{x}{b} \rceil$, where $a, b \in \mathbb{Z}$ and $b \geq 2$, studied by P. Eisele and K. P. Hadeler [Eisele and Hadeler 90]. Recently, J. S. Tanton [Tanton 02], together with Charles Adler, formulated a game-theoretic problem "Survivor" and noted that its analysis leads to the study of the sequence of rational numbers

$$a_0 = r, \ a_n = r\lceil a_{n-1} \rceil \text{ for } n \geq 1 \ ,$$

for $r > 1$, which is the trajectory of r under the map $f_r(x)$. He raised the question, "Must some a_n be an integer?" and conjectured that the answer is "Yes." This question differs from the case of the approximate squaring map in that the denominators of successive iterates, though bounded by the denominator of r, may increase or decrease. We note that the long-term dynamics of iterating this map differ according to whether $|r| < 1$, $|r| = 1$, or $|r| > 1$, with the case $r > 1$ being most analogous to the approximate squaring map.

The approximate multiplication maps have some resemblance to the map occurring in the $3x + 1$ problem. Setting $r = \frac{l}{d}$, we observe that $f_r(x)$ maps the domain $\frac{1}{d}\mathbb{Z}$ into itself and, on this domain, is conjugate to the map $g_r : \mathbb{Z} \to \mathbb{Z}$ given by

$$g_r(n) = \begin{cases} \frac{1}{d}n & \text{if } n \equiv 0 \pmod{d}, \\ \frac{1}{d}(n + l(d-b)) & \text{if } n \equiv b \pmod{d}, \\ & 1 \leq b \leq d-1, \end{cases}$$

(see (5–2) of Section 5). In terms of the conjugated map the question that we consider becomes whether for most starting values some iterate of g_r is an integer divisible by d. For $r = \frac{3}{2}$ the conjugated map is

$$g_{3/2}(n) = \begin{cases} \frac{3}{2}n & \text{if } n \equiv 0 \pmod 2, \\ \frac{3}{2}n + \frac{3}{2} & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

This is similar in form to the $3x+1$ function

$$T(n) = \begin{cases} \frac{1}{2}n & \text{if } n \equiv 0 \pmod 2, \\ \frac{3}{2}n + \frac{1}{2} & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

as given in [Lagarias 85] and [Wirsching 98], although the long-term dynamics of $g_{3/2}$ and T are different.

We formulate a conjecture for approximate multiplication maps analogous to the one above for the approximate squaring map. We define the exceptional set $E(r)$ for the map f_r to be

$$E(r) := \left\{ \begin{matrix} n: \; n \in \mathbb{Z} \text{ and} \\ \text{no iterate } f_r^{(j)}(n) \in \mathbb{Z} \text{ for } j \geq 1 \end{matrix} \right\}. \qquad (1\text{–}3)$$

Then we have:

Conjecture 1.2. *For each nonintegral rational $r \in \mathbb{Q}$ with $|r| > 1$, the exceptional set $E(r)$ for the approximate multiplication map f_r is finite.*

The "expanding map" hypothesis, $|r| > 1$, is necessary in the statement of this conjecture, for the conclusion fails for $r = \frac{1}{d}$ with $d \geq 3$, as remarked below. In parallel to the results for the approximate squaring map, we prove Conjecture 1.2 for rational r having denominator 2; it remains open for all rationals with $|r| > 1$ having denominator $d \geq 3$. This conjecture may also be difficult, as indicated in Section 5 by an analogy with the problem of showing that there exist no Mahler Z-numbers, a notorious problem connected with powers of $\frac{3}{2}$ ([Mahler 68], [Flatto, 1991]; also [Choquet 80], [Lagarias 85]).

Our analysis in Section 5 applies more generally to the family $\mathcal{P}_r$ of maps $h_r : \mathbb{Z} \to \mathbb{Z}$ having the form

$$h_r(n) = \frac{1}{d}(ln + l_b) \text{ when } n \equiv b \pmod d, \qquad (1\text{–}4)$$

where the integers l_b satisfy $l_b \equiv -lb \pmod d$. We obtain for all functions $h_r \in \mathcal{P}_r$ an explicit upper bound on the cardinality of the exceptional set $E(h_r, x)$ consisting of all integers $|n| \leq x$ that do not have some iterate that is a multiple of d. We show that for all rationals r,

$$\#E(h_r, x) \leq 4dx^{\beta_d},$$

with $\beta_d = \frac{\log(d-1)}{\log d}$. We complement this upper bound with a result showing that it is of the correct order of magnitude (to within a multiplicative constant) for certain values of r lying in $0 < r < 1$. This is the case for the function g_r for $r = \frac{1}{d}$ with $d \geq 3$. It follows that Conjecture 1.2 cannot hold for these values of r.

Section 6 gives some numerical results related to these questions.

As a final point we observe that one can also consider similar functions obtained by replacing the "ceiling" function with the "floor" function. Exactly the same set of questions can be asked for such functions. Our methods carry over to such functions, and there appear to be analogues of all our results and conjectures. For example:

Conjecture 1.3. *Let $F(x) := x\lfloor x \rfloor$. For each rational $r \in \mathbb{Q}$ with $r \geq 2$, there is an integer $m \geq 0$ such that $F^{(j)}(r)$ is an integer for all $j \geq m$.*

We will not say any more about the "floor" function versions, apart from one result at the end of Section 5.

We use the following notation: $\lceil\ \rceil$ denotes the ceiling function, $\lfloor\ \rfloor$ the floor function, and $\{\!\{\ \}\!\}$ the fractional part. For a prime p, $|\ |_p$ denotes the p-adic valuation. (If $r \in \mathbb{Q}$, $r = p^a \frac{b}{c}$ with $a, b, c \in \mathbb{Z}, c \neq 0$, and $\gcd(p, b) = \gcd(p, c) = 1$, then $|r|_p = a$.) $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic rationals and integers, respectively. For integers r, s, and i, $r|s$ means r divides s, and $r^i||s$ means r^i divides s but r^{i+1} does not.

2. DENOMINATOR 2

In this section we investigate the case where the starting value r has denominator 2. Here we are able to give a complete analysis. The following table shows what happens for the first few values of r. It gives the initial term, the number of steps to reach an integer, and the integer that is reached:

start	steps	reaches
3/2	1	3
5/2	2	60
7/2	1	14
9/2	3	268065
11/2	1	33
13/2	2	2093
15/2	1	60
17/2	4	1204154941925628
19/2	1	95
⋮	⋮	⋮

The number of steps appears to match sequence A001511 in [Sloane 95–03] (and the numbers reached now form sequence A081853 in that database). Indeed, we have the following:

Theorem 2.1. *Let $r = \frac{2l+1}{2}$, with $l \geq 1$. Then $f^{(m)}(r)$ reaches an integer for the first time when $m = |l|_2 + 1$.*

Proof: Note that if $x \in \mathbb{Q}$ has denominator 2 and is not an integer, then $\lceil x \rceil = x + \frac{1}{2}$.

We use induction on $|l|_2 = v$. If $v = 0$ then l is odd, $\lceil r \rceil = r + \frac{1}{2} = l + 1$ is even, and $r\lceil r \rceil$ has become an integer in one step, as claimed.

Suppose $v \geq 1$ and

$$l = 2^v + l_{v+1}2^{v+1} + l_{v+2}2^{v+2} + \cdots$$

is the binary expansion of l, where each $l_i = 0$ or 1. Then,

$$\begin{aligned} r\lceil r \rceil &= \left(l + \frac{1}{2}\right)(l+1) \\ &= \frac{1}{2} + \frac{l}{2} + l + l^2 \\ &= \frac{1}{2} + 2^{v-1} + (l_{v+1}+1)2^v \\ &\quad +(l_{v+1}+l_{v+2})2^{v+1} + \cdots + 2^{2v} + \cdots \\ &= \frac{2l'+1}{2}, \end{aligned}$$

where

$$l' = 2^{v-1} + (l_{v+1}+1)2^v + (l_{v+1}+l_{v+2})2^{v+1} + \cdots + 2^{2v} + \cdots$$

and $|l'|_2 = v - 1$. By the induction hypothesis, this will reach an integer in $v - 1$ steps, so we are done. □

Remark 2.2. The numbers $2l+1$ for which $|l|_2 = v$ are precisely the numbers that are congruent to $2^{v+1} + 1 \pmod{2^{v+2}}$. For example, if $v = 0$, $2l+1 \in \{3, 7, 11, 15, \ldots\}$, of the form $3 \pmod 4$; if $v = 1$, $2l + 1 \in \{5, 13, 21, 29, \ldots\}$, of the form $5 \pmod 8$; and so on.

Corollary 2.3. *Let $r = \frac{2l+1}{2}$, for $l \geq 1$ and $|l|_2 = v$. Then, the first integer value taken by $f^{(m)}(r)$ is*

$$\frac{1}{2}\,\theta^{(v+1)}(2l+1)\ ,$$

where $\theta(y) = y(y+1)/2$.

Proof: This is now a straightforward calculation, again using the fact that if $x \in \mathbb{Q} \setminus \mathbb{Z}$ has denominator 2 then $\lceil x \rceil = x + \frac{1}{2}$. □

For example, if $v = 0$, and $r = (4k+3)/2 = y/2$ (say), then in one step we reach the integer $\frac{1}{2}\theta(y) = y(y+1)/4$; if $v = 1$, and $r = (8k+5)/2 = y/2$, then in two steps we reach the integer

$$\frac{1}{2}\theta(\theta(y)) = \frac{y(y+1)(y^2+y+2)}{16}\ ;$$

if $v = 2$, and $r = (16k+9)/2 = y/2$, then in three steps we reach the integer

$$\frac{1}{2}\theta(\theta(\theta(y))) = \frac{y\,(y+1)\left(y^2+y+2\right)\left(y^2-y+2\right)\left(y^2+3\,y+4\right)}{256}\ ;$$

and so on.

3. DENOMINATOR d

We now analyze the case of rationals with a general denominator d, obtaining less complete results. The next theorem shows that most rationals will eventually reach an integer. In particular, it gives an upper bound on the number of such rationals below x that never reach an integer. Given an integer $d \geq 2$ and a bound $x \geq 1$, we study the "exceptional set"

$$\mathcal{M}_d(x) \ := \ \left\{ \begin{array}{l} l \ : \ 1 \leq l \leq x, \\ f^{(m)}(\frac{l}{d}) \notin \mathbb{Z} \text{ for each } m \geq 1 \end{array} \right\}, \qquad (3\text{–}1)$$

and let $M_d(x) = |\mathcal{M}_d(x)|$. The finite set $[1, d-1] := \{1, 2, \ldots, d-1\}$ is contained in $\mathcal{M}_d(x)$, and Conjecture 1.1 asserts that $\mathcal{M}_d(x) = [1, d-1]$.

Theorem 3.1. *Let $d \geq 2$ be a fixed integer.*

1. *For each finite $j \geq 0$ the set of positive integers l for which the approximate squaring map with initial value $r = \frac{l}{d}$ first iterates to an integer after exactly j steps is a union of arithmetic progressions* (mod d^{j+1}) *restricted to positive integers.*

2. *There is a positive exponent α_d such that, for each $\varepsilon > 0$ and all $x > 1$,*

$$M_d(x) \leq C(d, \varepsilon)\, x^{1-\alpha_d+\varepsilon} \qquad (3\text{–}2)$$

for a positive constant $C(d, \varepsilon)$, with α_d given by

$$\alpha_d \ = \ \min_{d'|d,\, d'>1} \log_{d'}\left(\frac{d'}{\phi(d')}\right)\ , \qquad (3\text{–}3)$$

where ϕ is the Euler totient function. In fact,

$$\alpha_d = \min_{p^j||d} \frac{\log\left(1+\frac{1}{p-1}\right)}{j\log p}\ . \qquad (3\text{–}4)$$

Note that it follows immediately from (3–3) that $0 < \alpha_d \leq 1$ and $\alpha_d = 1$ only for $d = 2$.

To prove the theorem we shall first prove a preliminary lemma. We need some notation concerning the pattern of denominators in a sequence of iterates. We write $\frac{l}{d} = \frac{l_0}{d_0}$ with $\gcd(l_0, d_0) = 1$ and set

$$f^{(j)}\left(\frac{l}{d}\right) = \frac{l_j}{d_j} \ , \ j \geq 1 \ ,$$

where $\gcd(l_j, d_j) = 1$ and $d_j | d$ with $d_j \geq 1$. The pair (l, d) determines the sequence $d_0, d_1, d_2, \ldots$.

We consider the first m iterates for a given input $r = \frac{l}{d}$, and let

$$\mathcal{Y}(l, d, m) := (d_0, d_1, ..., d_m) \tag{3–5}$$

denote the vector of denominators of the first m iterates. We call this a *chain* of length $m+1$. This information in $\mathcal{Y}$ can be described in another way. We set $d_{-1} = d$ and define r_j by

$$r_j = \frac{d_{j-1}}{d_j}$$

for $j = 0, 1, 2, \ldots, m$. We call j a *break-point* if $r_j > 1$. Let the break-points up to the mth iterate be $0 \leq j_1 < j_2 < \ldots < j_t \leq m$. The data $\mathcal{Y}(l, d, m)$ is completely determined by knowledge of the pairs $(j_1, r_{j_1}), ..., (j_t, r_{j_t})$ and m. Note that, since each break-point removes a divisor of d from the denominator, in any chain of length m there are at most s break-points, where s is the number of prime factors of d, counted with multiplicity.

Now consider an arbitrary chain $\mathcal{Y}$, which consists of a sequence $(d_0, d_1, ..., d_m)$ with d_{j+1} dividing d_j and d_0 dividing d. A chain is *complete* if $d_m = 1$, and is *incomplete* otherwise. We will only consider complete chains with $d_{m-1} \geq 2$, i.e., with $j_t = m$.

The following lemma characterizes the set of positive integers l that have a given associated chain $\mathcal{Y}$ of length $m + 1$.

Lemma 3.2. *Given a fixed chain $\mathcal{Y} = (d_0, d_1, ..., d_m)$ of length $m+1$, the set of positive integers l such that $r = \frac{l}{d}$ has $\mathcal{Y}(l, d, m) = \mathcal{Y}$ consists of a collection of arithmetic progressions* (mod $d_{-1}d_0d_1 \cdots d_{m-1}$)*, restricted to the positive integers. There are exactly*

$$\phi(d_0)\phi(d_1)\phi(d_2) \cdots \phi(d_m) \tag{3–6}$$

such arithmetic progressions.

Proof: We study the set of $r = \frac{l}{d}$ having a fixed chain $\mathcal{Y}$ for its initial m iterates.

The set of allowable l with $(l, d) = d_0$ consists of all integers $r_1 l_0$ with $gcd(l_0, d_0)$=1. This consists of $\phi(d_0)$ arithmetic progressions (mod d) $\equiv$ (mod d_{-1}).

We now write $\frac{l_k}{d_k}$ (for $k = 0, 1, \ldots$) in a mixed-radix expansion where the radices depend on k:

$$\frac{l_k}{d_k} = \frac{a_{-1}(k)}{d_k} + a_0(k) + \sum_{j=1}^{\infty} a_j(k) \prod_{l=0}^{j-1} d_{k+l} \ , \tag{3–7}$$

in which the "digits" $a_j(k)$ satisfy

$$0 < a_{-1}(k) < d_k \ , \text{ and } 0 \leq a_j(k) < d_{j+k} \text{ for each } j \geq 0 \ .$$

Here we set $d_{m+j} := d_m$ for all $j \geq 1$. The sum on the right-hand side of (3–7) is actually a finite sum. By definition,

$$\begin{aligned} \frac{l_{k+1}}{d_{k+1}} \ &= \ f\left(\frac{l_k}{d_k}\right) \ = \\ &\left(\frac{a_{-1}(k)}{d_k} + a_0(k) + \sum_{j=1}^{\infty} a_j(k) d_k d_{k+1} \cdots d_{k+j-1}\right) \\ &\times \left(1 + a_0(k) + \sum_{j=1}^{\infty} a_j(k) d_k d_{k+1} \cdots d_{k+j-1}\right) \ . \end{aligned} \tag{3–8}$$

We use induction on $k \geq 0$ to simultaneously establish four properties of this mixed-radix expansion:

(i) We have

$$r_{k+1} \ = \ \gcd(a_0(k) + 1, d_k) \ . \tag{3–9}$$

(ii) We have

$$a_{-1}(k+1) \ \equiv \ a_{-1}(k) \frac{a_0(k) + 1}{r_{k+1}} \ (\text{mod } d_{k+1}) \ . \tag{3–10}$$

(iii) We have

$$\gcd(a_{-1}(k+1), d_{k+1}) \ = \ 1 \ . \tag{3–11}$$

(iv) For $0 \leq j \leq m$,

$$\begin{aligned} a_j(k+1) \ &\equiv \ a_{-1}(k) a_{j+1}(k) \\ &+ G(a_{-1}(k), a_0(k), a_1(k), \ldots, a_j(k)) \ (\text{mod } d_{j+k+1}) \ , \end{aligned} \tag{3–12}$$

for some function $G(a_{-1}(k), a_0(k), a_1(k), \ldots, a_j(k))$. This function includes all the necessary information about "carries" in the multiple-radix expansion.

The base case $k = 0$ is checked directly. Since d_0 divides all terms in the sum on the right-hand side of

(3–7), the right-hand side of (3–8) (when $k = 0$) has a single term $\frac{a_{-1}(0)(a_0(0)+1)}{d_0}$ having a denominator, and this term equals $\frac{a_{-1}(1)}{d_1}$ (mod 1). Since $\gcd(a_{-1}(0), d_0) = 1$ by hypothesis, we must have

$$\gcd(a_0(0)+1, d_0) = \frac{d_0}{d_1} = r_1 \ ,$$

which is (i) for $k = 0$. The term with a denominator in (3–8) is then

$$\frac{a_{-1}(0)\frac{a_0(0)+1}{r_1}}{d_1} \ ;$$

hence

$$a_{-1}(1) \equiv a_{-1}(0)\left(\frac{a_0(0)+1}{r_1}\right) \pmod{d_1} \ , \tag{3–13}$$

which is (ii). Now $\gcd(a_{-1}(1), d_1)$ divides $\gcd(a_{-1}(0), d_1)\gcd(\frac{a_0(0)+1}{r_1}, d_1)$, both terms of which are 1, so (iii) follows. Finally, to establish (iv) when $k = 0$, we drop the terms involving $d_1 d_2 \cdots d_{j+1}$ from (3–8) and observe that there is a term

$$a_{-1}(0)a_{j+1}(0)d_1 d_2 \cdots d_j \ , \tag{3–14}$$

while all the other terms containing any $a_{j+l}(0)$ for $l \geq 1$ are divisible by $d_1 d_2 \cdots d_{j+1}$. (Note that $d_1 d_2 \cdots d_{j+1}$ divides $d_0 d_1 d_2 \cdots d_j$.) This establishes that a congruence of the form (3–12) holds for the digit $a_j(1)$ in the expansion (3–8) for $\frac{l_1}{d_1}$, completing the proof of the base case.

The induction step for general k follows using exactly the same reasoning.

Next, we will deduce from (iv) that for each $k \geq 0$,

$$\begin{aligned} a_0(k+1) \ &\equiv \ a_{k+1}(0)\prod_{l=0}^{k} a_{-1}(l) \\ &+ \widetilde{G}(a_{-1}(0), a_0(0), a_1(0), \ldots, a_k(0)) \pmod{d_{k+1}} \end{aligned} \tag{3–15}$$

for some function $\widetilde{G}$ depending on the indicated variables. The important point about (3–15) is that the dependence on $a_{k+1}(0)$ is linear, even though the dependence on the other initial terms $a_{-1}(0), a_0(0), \ldots, a_k(0)$ is nonlinear.

To prove (3–15), we again use induction on k. For $k = 0$ the assertion is that

$$\begin{aligned} a_0(1) \ &\equiv \ a_1(0)a_{-1}(0) \\ &+ \widetilde{G}(a_{-1}(0), a_0(0)) \pmod{d_1} \ , \end{aligned} \tag{3–16}$$

which is (3–12) with $k = j = 0$. For $k = 1$ we wish to show that

$$\begin{aligned} a_0(2) \ &\equiv \ a_2(0)a_{-1}(0)a_{-1}(1) \\ &+ \widetilde{G}(a_{-1}(0), a_0(0), a_1(0)) \pmod{d_2} \ . \end{aligned} \tag{3–17}$$

Setting $k = 1, j = 0$ and $k = 0, j = 1$ in (3–12), we obtain

$$a_0(2) \ \equiv \ a_{-1}(1)a_1(1) + \widetilde{G}(a_{-1}(1), a_0(1)) \pmod{d_2}$$

and

$$\begin{aligned} a_1(1) \ &\equiv \ a_{-1}(0)a_2(0) \\ &+ \widetilde{G}(a_{-1}(0), a_0(0), a_1(0)) \pmod{d_2} \ , \end{aligned}$$

hence,

$$\begin{aligned} a_0(2) \ &\equiv \ a_2(0)a_{-1}(0)a_{-1}(1) \\ &+ a_{-1}(1)\widetilde{G}(a_{-1}(0), a_0(0), a_1(0)) \\ &+ \widetilde{G}(a_{-1}(1), a_0(1)) \pmod{d_2} \ . \end{aligned}$$

However, from (3–13) and the fact that $0 < a_{-1}(1) < d_1$, $a_{-1}(1)$ is uniquely determined by $a_{-1}(0)$ and $a_0(0)$. Also the induction hypothesis allows us to use (3–16) to eliminate $a_0(1)$. Equation (3–17) follows. The case of general k follows in the same way; we leave the details to the reader.

We have already seen that

$$\gcd(a_{-1}(0), d_1) = 1 \ .$$

From (3–11) and (3–10) ,

$$\gcd(a_{-1}(2), d_2) = 1 \text{ and}$$

$$a_{-1}(2) \ \equiv \ a_{-1}(1)\frac{a_0(1)+1}{r_2} \pmod{d_2} \ .$$

Therefore, $\gcd(a_{-1}(1), d_2) = 1$, and so

$$\gcd(a_{-1}(0)a_{-1}(1), d_2) = 1 \ .$$

Continuing in this way, we obtain

$$\gcd(\prod_{l=0}^{k} a_{-1}(l), d_{k+1}) \ = \ 1 \ , \tag{3–18}$$

for $k = 0, 1, \ldots, m$.

Then (3–15) shows that the congruence class of $a_0(k+1) \pmod{d_{k+1}}$ is uniquely determined by the congruence class of $a_{k+1}(0) \pmod{d_{k+1}}$, once $a_{-1}(0), a_0(0), \ldots, a_k(0)$ are specified.

In particular, for each $k \geq 0$ there are exactly $\phi(d_{k+1})$ congruence classes of $a_k(0) \pmod{d_k}$ that give

$$\gcd(\frac{a_0(k)+1}{r_{k+1}}, d_{k+1}) \ = \ 1 \ ,$$

or in other words that give

$$\gcd(a_0(k)+1, d_k) \ = \ r_{k+1} \ .$$

At each iteration we impose one such condition. Combining this with the $\phi(d_0)$ congruence classes $(\bmod\ d_{-1})$ allowed initially, (3–6) follows. $\square$

Proof of Theorem 3.1: Part (1) follows from Lemma 3.2. Indeed, the set of integers which iterate to an integer in exactly j steps are precisely those integers that belong to some complete chain $\mathcal{Y}$ having $j = j_t$. There are finitely many complete chains of length $m+1$ with $m = j_t = j$, and by Lemma 3.2 the set of positive l such that $\frac{l}{d}$ belongs to a given such chain is a union of arithmetic progressions (mod $d_{-1}d_0d_1\dots d_{m-1}$). Each such progression subdivides into a finite union of arithmetic progressions (mod d^{m+1}), since $d_{-1}d_0d_1\dots d_{m-1}$ divides d^{m+1}.

To establish part (2), let the cutoff value x be given. We call a value $l \le x$ "bad" at size x if there is an iterate $m \ge 0$ such that

$$\prod_{j=0}^{m-1} d_j \le x < \prod_{j=0}^{m} d_j \ , \tag{3–19}$$

with $d_m > 1$; here m depends on l, d, and x. The chain $\mathcal{Y}$ associated to the data $(d_0, d_1, ..., d_m)$ is necessarily an incomplete chain, and we say it is *incomplete at size* x if (3–19) holds. Conversely, all elements $l \le x$ belonging to such an incomplete chain are "bad." We let $N_d(x)$ denote the number of "bad" elements at size x, and we clearly have

$$M_d(x) \le N_d(x) \ .$$

We will establish part (2) by showing the stronger result that

$$N_d(x) \le C(d,\varepsilon)\, x^{1-\alpha_d+\varepsilon} \ . \tag{3–20}$$

To bound $N_d(x)$ we first bound the number of distinct incomplete chains at size x, and then bound the maximal number of elements below x falling in any such chain. It is easy to bound the number of incomplete chains $\mathcal{Y}$ at size x as follows. Since each $d_j \ge 2$, we have $m \le \log_2\ x$. There are $\binom{m}{t}$ choices for the break-points, and the bound $t \le s$ then yields that there are at most

$$\sum_{t=0}^{s} \binom{\log_2 x}{t} \le s(\log_2 x)^s$$

break-point patterns. Since a nontrivial divisor r_i of d_{i-1} is removed at each break-point, we conclude that the number of distinct chains is at most $d!s(\log_2 x)^s$. For fixed d (and hence s) this gives, for any fixed $\epsilon > 0$ and $x \ge 2$, that

$$\#(\mathcal{Y}) \le C_1(d,\varepsilon) x^{\varepsilon} \ , \tag{3–21}$$

for some positive constant $C_1(d,\varepsilon)$.

Now, let $N_d(\mathcal{Y},x)$ count the number of bad l at size x having a given chain $\mathcal{Y} = (d_0, d_1, ..., d_m)$ that is incomplete at size x. In view of (3–21), it suffices to prove an upper bound of the same form as (3–20) for each such chain. So, consider a fixed such chain $\mathcal{Y}$. Now, (3–19) holds and shows that each arithmetic progression (mod $d_0d_1\cdots d_{m-1}$) contains at most d_m elements below x. From (3–6), $N_d(\mathcal{Y},x)$, the number of bad elements l at size x with chain $\mathcal{Y}$, satisfies

$$N_d(\mathcal{Y},x) \le d_m \phi(d_1)\phi(d_2)\cdots\phi(d_m) \ .$$

Now, $x \ge d_0d_1\cdots d_{m-1} \ge d_1d_2\cdots d_{m-1}$, hence

$$\frac{N_d(\mathcal{Y},x)}{x} \le d_m \prod_{j=1}^{m} \frac{\phi(d_j)}{d_j} \ ,$$

and therefore

$$\begin{aligned} \frac{N_d(\mathcal{Y},x)}{x} &\le d_m\, e^{\sum_{j=1}^{m} \log \frac{\phi(d_j)}{d_j}} \\ &\le d\, x^{\frac{\sum_{j=0}^{m} \log \frac{\phi(d_j)}{d_j}}{\log x}} \ . \end{aligned} \tag{3–22}$$

Now, $x \le d_0d_1\cdots d_m$, so that

$$\begin{aligned} \frac{\sum_{j=0}^{m} \left|\log \frac{\phi(d_j)}{d_j}\right|}{\log x} &\ge \frac{\sum_{j=0}^{m} \left|\log \frac{\phi(d_j)}{d_j}\right|}{\sum_{j=0}^{m} \log d_j} \\ &\ge \min_{0\le j\le m} \frac{\left|\log \frac{\phi(d_j)}{d_j}\right|}{\log d_j} \\ &\ge \min_{0\le j\le m} \left(\log_{d_j}\left(\frac{d_j}{\phi(d_j)}\right)\right) \\ &\ge \alpha_d \ , \end{aligned}$$

where we used the definition (3–3). Substituting this in (3–22) yields

$$\frac{N_d(\mathcal{Y},x)}{x} \le d\, x^{-\alpha_d} \ ,$$

since $\frac{\phi(d_j)}{d_j} < 1$ gives $\log \frac{\phi(d_j)}{d_j} < 0$, and so

$$N_d(\mathcal{Y},x) \le d\, x^{1-\alpha_d} \ ,$$

as required. Combined with (3–21), this gives (3–20), hence (3–2).

Finally, we establish the equivalence of (3–3) and (3–4). Since ϕ is multiplicative, for a general $d' > 1$ we have

$$\begin{aligned} \log_{d'}\left(\frac{d'}{\phi(d')}\right) &= \frac{\sum_{p^j\|d'} \log\left(\frac{p^j}{\phi(p^j)}\right)}{\sum_{p^j\|d} \log p^j} \\ &\ge \min_{p^j\|d'} \left\{\log_{p^j}\left(\frac{p^j}{\phi(p^j)}\right)\right\} \ . \end{aligned} \tag{3–23}$$

Thus the minimum in (3–3) is attained when d' is a prime power. Now

$$\log_{p^j}\left(\frac{p^j}{\phi(p^j)}\right) = \frac{\log\left(\frac{1}{1-\frac{1}{p}}\right)}{\log\ p^j} = \frac{\log\left(1+\frac{1}{p-1}\right)}{j\log p} \tag{3–24}$$

is minimized by making j as large as possible, so we obtain the formula (3–4). □

The upper bound in (3–20) in Theorem 3.1 has an exponent that is essentially the best possible. Indeed, if l, d, x, and m are such that

$$d_1 = d_2 = \cdots = d_m = d' \ ,$$

where d' is the value that minimizes (3–3), so that $t = 1$ and the chain is simply $\mathcal{Y} = \{(1, d')\}$, we have

$$N_d(\mathcal{Y}, x) \geq C_d^{'} x^{1-\alpha_d} \ ,$$

for some positive constant $C_d^{'}$.

We next give a probabilistic interpretation of Theorem 3.1, following a suggestion of Daniel Berend. Given a denominator d and an integer cutoff value x, let X be a random integer chosen uniformly on the interval $[1, x]$, and let $s = X/d$. We iterate f starting at s, and let $\mathrm{Stop}(X)$ be the random variable giving the number of iterates needed to reach an integer. $\mathrm{Stop}(X)$ takes nonnegative integer values (including 0 and ∞). There is an associated probability distribution

$$\{\mathrm{Prob}_x[\mathrm{Stop} = j] : j = 0, 1, \ldots, \infty\}$$

which depends on the interval $[1, x]$.

Theorem 3.3. *For each fixed denominator d, there exists a limiting distribution as $x \to \infty$ of the stopping times of the approximate squaring map f on rationals $\frac{l}{d}$. More precisely, for each finite j the limit*

$$Prob_\infty[Stop(X) = j] := \lim_{x\to\infty} Prob_x[Stop(X) = j] \tag{3–25}$$

exists and is a rational number with denominator dividing d^{j+1}, and for $j = \infty$ we have

$$Prob_\infty[Stop(X) = \infty] := \lim_{x\to\infty} Prob_x[Stop(x) = \infty] = 0 \ . \tag{3–26}$$

In the case when $d = p$ is a prime the limiting distribution is a geometric distribution:

$$Prob_\infty[Stop(X) = j] = p(1-p)^j \ . \tag{3–27}$$

Proof: The assertion (3–25) follows directly from part (1) of Theorem 3.1, which shows that the set with stopping time j is a union of complete arithmetic progressions (mod d^{j+1}). The assertion (3–26) follows from part (2) of Theorem 3.1. The computation of the density (3–27) follows from the fact that for $d = p$ and each $j \geq 0$ there is a unique complete chain $\mathcal{Y}$ with stopping time j, having the single break-point $(j, 1)$, giving one arithmetic progression (mod p^{j+1}). □

Remark 3.4.

1. This stopping time formulation is similar to that arising in the $3x+1$ problem and studied in [Terras 76, 79].
2. In principle the limiting distribution for any specific composite d is computable from the proof of Theorem 3.1.
3. For all d we have $\mathrm{Prob}_\infty[\mathrm{Stop}(X) = 0] = 1/d$.

4. *p*-ADIC ITERATION

We now consider an approximate squaring map defined on the p-adic numbers analogous to the approximate squaring map on $\mathbb{Q}$ and study the question of whether some iterate will eventually become a p-adic integer. We show that now there is always a nonempty exceptional set of p-adic numbers which never become p-adic integers.

Let p be a prime ≥ 2 and let $\mathbb{Q}_p$ denote the p-adic numbers, with typical element $\alpha = \sum_{j=-k}^{\infty} a_j p^j$, where $k \in \mathbb{Z}$ and the a_j satisfy $0 \leq a_j \leq p-1$. The p-adic integral part of α is given by the function $F_p : \mathbb{Q}_p \to \mathbb{Q}_p$, where

$$F_p(\alpha) := \sum_{j=\max\{0,k\}}^{\infty} a_j p^j \ ,$$

while the p-adic fractional part (or "principal part") of α is

$$P_p(\alpha) := \sum_{j<0} a_j p^j \ ,$$

which is a finite sum (possibly empty); thus, $\alpha = P_p(\alpha) + F_p(\alpha)$. We investigate the function

$$f_p(\alpha) := \alpha(F_p(\alpha) + 1) \ ,$$

which is a p-adic analogue of the approximate squaring map defined in (1–1). If we regard the rationals $\mathbb{Q}$ as embedded in $\mathbb{Q}_p$, then for nonintegral r in the subring $\mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{Q} \subseteq \mathbb{Q}_p$, we have $f_p(r) = r\lceil r \rceil$, so the iterates there agree with the approximate squaring map.

For each $k \geq 0$ the set $\frac{1}{p^k}\mathbb{Z}_p$ is invariant under the action of $f_p(x)$, and

$$\mathbb{Z} \subseteq \frac{1}{p}\,\mathbb{Z} \subseteq \frac{1}{p^2}\,\mathbb{Z} \subseteq \cdots \ .$$

We define the exceptional set

$$\Omega_k(p) := \left\{ \begin{array}{l} \alpha \in \frac{1}{p^k}\,\mathbb{Z}_p \ : \ f^{(m)}(\alpha) \notin \frac{1}{p^{k-1}}\,\mathbb{Z}_p \\ \text{for each} \ \ m \geq 1 \end{array} \right\} , \quad (4\text{–}1)$$

for $k \geq 1$. The set $\Omega_k(p)$ is an analogue of the exceptional sets $\mathcal{M}_d(x)$ studied in the last section, corresponding to the denominator $d = p^k$. (Note that $\Omega_k(p) \cap \frac{1}{p^k}\mathbb{Z}_{>0}$ is contained in $\frac{1}{p^k}\mathcal{M}_{p^k}(\infty)$, as defined in (3–1)). We will show that these sets are nonempty, determine their Hausdorff dimension s, and get upper and lower bounds on their Hausdorff s-dimensional measure.

The s-dimensional p-adic Hausdorff measure $\mathcal{H}_p^s(\Omega)$ of a closed set Ω in $\mathbb{Q}_p$ is defined by the general prescription in [Falconer 90, Chapter 2] or [Federer 69, Section 2.10]. Here, $0 < s \leq 1$. The *diameter* of a measurable set $S \subseteq \mathbb{Q}_p$ is

$$|S| := \sup\{|\alpha - \beta|_p \ : \ \alpha,\, \beta \in S\} \ ,$$

and its *p-adic measure* $\mu_p(S)$ is the Haar measure with the normalization $\mu_p(\mathbb{Z}_p) = 1$. A p-adic ball

$$B(\alpha; p^l) := \{\beta \in \mathbb{Q}_p \ : \ |\beta - \alpha|_p \leq p^l\}$$

is both closed and open and has the property that its diameter equals its measure:

$$|B(\alpha; p^l)| = \mu(B(\alpha; p^l)) = \frac{1}{p^l} \ .$$

For each $\delta > 0$ we define

$$\mathcal{H}^s(\Omega, \delta) := \inf \left\{ \begin{array}{l} \displaystyle\sum_{j=1}^{\infty} |I_j|^s \ : \ \Omega \subseteq \bigcup_{j=1}^{\infty} I_j \ , \\ |I_j| \leq \delta \text{ for all } j \end{array} \right\} , \qquad (4\text{–}2)$$

and[1]

$$\mathcal{H}_p^s(\Omega) := \lim_{\delta \to 0} \mathcal{H}^s(\Omega, \delta) \ . \qquad (4\text{–}3)$$

The *p-adic Hausdorff dimension* of Ω is the unique value s_0 such that $\mathcal{H}_p^s(\Omega) = \infty$ for $s < s_0$ and $\mathcal{H}_p^s(\Omega) = 0$ for $s > s_0$. The value of $\mathcal{H}_p^{s_0}(\Omega)$ may be zero, finite, or infinite.

[1] The limit (which may be ∞) exists since $\mathcal{H}^s(\Omega, \delta_1) \geq \mathcal{H}^s(\Omega, \delta_2)$ if $\delta_1 \leq \delta_2$.

A closed set $\Omega \subset \mathbb{Z}_p$ is called *weakly self-similar* (mod p^k) *with branching ratio* b, where b is an integer ≥ 2, if the following "equal branching" property holds for $l = 1, 2, \ldots$. Let $W_l(\Omega)$ denote the set of initial sequences of digits of length lk in Ω, i.e.,

$$W_l(\Omega) = \left\{ \begin{array}{l} \beta = \displaystyle\sum_{j=0}^{lk-1} a_j p^j \ : \ \text{there exists some } \alpha \in \Omega \\ \quad \text{whose "initial part" } p^{lk} P_p(p^{-lk}\alpha) = \beta \end{array} \right\} . \qquad (4\text{–}4)$$

Then each sequence in $W_l(\Omega)$ should extend to exactly b sequences in $W_{l+1}(\Omega)$. That is, if the digits of an element $\beta \in \Omega$ are grouped in blocks of size k, once the first l blocks of digits are specified, there are exactly b allowable choices for the next block of digits.

Theorem 4.1. *For each $k \geq 1$ the set $p^k\Omega_k(p)$ is weakly self-similar (mod p^k) with branching ratio $b = \phi(p^k) = p^k - p^{k-1}$.*

Proof: The set $W_l(\Omega_k(p))$ specifies the conditions under which the first l iterates $f(\alpha)$, $f^{(2)}(\alpha)$, $\ldots$, $f^{(l)}(\alpha) \notin \frac{1}{p^{k-1}}\,\mathbb{Z}_p$. Each such condition is a congruence (mod p^{lk}), which has exactly $\phi(p^k) = p^k - p^{k-1}$ solutions for the next digit (compare (3–12) in the proof of Theorem 3.1, taking each $d_j = p^k$). Finally the definition of $\Omega_k(p)$ implies it is a closed set. Thus, $p^k\Omega_k(p)$ is weakly self-similar. $\square$

We can determine the Hausdorff dimension of weakly self-similar sets in $\mathbb{Z}_p$, together with upper and lower bounds for the Hausdorff measure at this dimension.

Theorem 4.2. *Let $\Omega \subseteq \mathbb{Z}_p$ be a weakly self-similar set (mod p^k) with branching ratio b satisfying $2 \leq b < p^k$. Then Ω is a compact set and has Hausdorff dimension $s(\Omega)$ given by*

$$\dim_H(\Omega) = \frac{\log b}{\log p^k} \ .$$

Its $s(\Omega)$-dimensional Hausdorff measure satisfies

$$\left(\frac{b}{p^k}\right)^{1-\frac{1}{k}} \ \leq \ \mathcal{H}_p^{s(\Omega)}(\Omega) \ \leq \ 1 \ . \qquad (4\text{–}5)$$

Proof: The compactness of Ω is established similarly to Theorem 4.1. We define $W_l(\Omega)$ as in (4–4). This set has cardinality b^l by hypothesis, and

$$\Omega = \bigcap_{l=1}^{\infty} \widetilde{W}_l(\Omega) \ ,$$

where $\widetilde{W}_l(\Omega)$ is the compact set

$$\widetilde{W}_l(\Omega) = \left\{ \begin{array}{l} \tilde{\beta} \in \mathbb{Z}_p \ : \ \tilde{\beta} \equiv \beta \pmod{p^{lk}} \\ \text{for some } \beta \in W_l(\Omega) \end{array} \right\} .$$

To determine the Hausdorff dimension, it suffices to establish the inequalities (4–5), since the fact that the Hausdorff measure is positive and finite determines the Hausdorff dimension. Set $s = (\log b)/(\log p^k)$, so that $p^{ks} = b$. Also $0 < s < 1$.

For the upper bound in (4–5), we consider the sets $\widetilde{W}_l(\Omega)$. Now b^l balls of diameter p^{lk} cover $\widetilde{W}_l(\Omega)$, hence cover Ω. Thus, for $\delta = p^{-lk}$ this covering gives

$$\mathcal{H}^s(\Omega, p^{-lk}) \le \frac{b^l}{p^{lks}} = \frac{b^l}{b^l} = 1 ,$$

which implies

$$\mathcal{H}_p^s(\Omega)) \le 1 .$$

The lower bound argument is similar in spirit to that used for Cantor sets in [Falconer 90, pages 31–32]. By the compactness of Ω, we need only prove that the lower bound holds for finite coverings. The nonarchimedean property of the valuation $|\ |_p$ means that each I_j has diameter p^m for some m, and hence we can enlarge I_j to a ball $B(\alpha; p^m) \supseteq I_j$ without changing its diameter. But $B(\alpha; p^m)$ gives an open cover, so it has a finite subcover:

$$\Omega \subseteq \bigcup_{j=1}^{m} B(\alpha_j; p^{m_j}) ,$$

and

$$\sum_{j=1}^{m} |B(\alpha_j; p^{m_j})|^s \ = \ \sum_{j=1}^{m} p^{m_j s} .$$

We want to show

$$\sum_{j=1}^{m} |B(\alpha_j; p^{m_j})|^s \ \ge \ \left(\frac{b}{p^k}\right)^{\frac{k-1}{k}} . \tag{4–6}$$

We first replace these balls with balls of diameter p^{-kl_j} for integers l_j. Write

$$m_j = -kl_j + k_j \,, \ 0 \le k_j \le k-1 .$$

Then, we can cover $B(\alpha_j; p^{m_j})$ with p^{k_j} balls $B(\alpha_{j'}; p^{-kl_j})$. We claim that

$$\sum_{j=1}^{m} |B(\alpha_j; p^{m_j})|^s \ \ge \ \left(\frac{b}{p^k}\right)^{\frac{k-1}{k}} \sum_{j'} |B(\alpha_{j'}; p^{-kl_j})|^s . \tag{4–7}$$

This will follow if we show that

$$\left(p^{-kl_j+k_j}\right)^s \ge \left(\frac{b}{p^k}\right)^{\frac{k-1}{k}} p^{k_j} \left(p^{-kl_j}\right)^s$$

holds for each j. This in turn is equivalent to showing

$$p^{k_j(s-1)} \ge \left(\frac{b}{p^k}\right)^{\frac{k-1}{k}} .$$

Since $p^{ks} = b$, we have

$$p^{k_j(s-1)} = \left(\frac{b}{p^k}\right)^{\frac{k_j}{k}} \ge \left(\frac{b}{p^k}\right)^{\frac{k-1}{k}} ,$$

which proves (4–7). Thus (4–6) will follow from showing that

$$\sum_{j=1}^{m} |B(\alpha_j; p^{-kl_j})|^s \ \ge \ 1 \tag{4–8}$$

for any set of such balls that covers Ω. We may suppose $l_1 \le l_2 \le \cdots \le l_m$. By the weak self-similarity of Ω, there are b^{l_m} principal parts to cover with balls of diameter p^{-kl_m} in $\widetilde{W}_m(\Omega)$. Weak self-similarity also says that each ball of radius p^{-kl_j} covers either none of or else exactly $b^{l_m - l_j}$ such principal parts in $\widetilde{W}_m(\Omega)$. Since the balls cover Ω, we must have

$$\sum_{j=1}^{m} b^{l_m - l_j} \ge b^{l_m} .$$

Dividing by b^{l_m} yields

$$\sum_{j=1}^{m} b^{-l_j} \ge 1 . \tag{4–9}$$

Using this bound, we obtain

$$\begin{aligned} \sum_{j=1}^{m} |B(\alpha_j; p^{-kl_j})|^s \ &= \ \sum_{j=1}^{m} p^{-kl_j s} \\ &= \ \sum_{j=1}^{m} b^{-l_j} \\ &\ge \ 1 , \end{aligned}$$

which is (4–8). Thus (4–7) holds. $\square$

Corollary 4.3. *The set $\Omega_k(p)$ has Hausdorff dimension*

$$\dim_H(\Omega_k(p)) = s(p^k) \ := \ 1 \ - \ \frac{\log\left(1 + \frac{1}{p-1}\right)}{k \log p} \tag{4–10}$$

for $k \ge 1$. Furthermore, its $s(p^k)$-dimensional Hausdorff measure satisfies

$$\left(1 - \frac{1}{p}\right)^{1-\frac{1}{k}} (p^k - p^{k-1}) \ \le \ \mathcal{H}_p^{s(p^k)}(\Omega_k(p)) \ \le \ p^k - p^{k-1} . \tag{4–11}$$

Proof: This follows by applying Theorems 4.1 and 4.2 to the set $p^k\Omega_k(p)$ and using the fact that

$$\mathcal{H}_p^{s(p^k)}(\Omega_k(p)) = b\mathcal{H}_p^{s(p^k)}(p^k\Omega_k(p)) \ ,$$

since $(p^k)^s = b$. Using the branching ratio $b = \phi(p^k) = p^k - p^{k-1}$, we have

$$\frac{\log b}{\log p^k} = \frac{\log p^k + \log(1-1/p)}{\log p^k}$$

which gives (4–10). □

Remark 4.4. For the branching ratio $b = \phi(p^k) = p^k - p^{k-1}$, one can show that equality may occur on either side of the Hausdorff measure bounds in (4–5) (or (4–11)). For the lower bound, take the $p^k - p^{k-1}$ allowed digit sets in each layer to be $\sum_{j=lk}^{k(l+1)-1} a_j\, p^j$ with the restriction that $a_{lk} \neq 0$. Then we can cover Ω with $(p^k - p^{k-1})^{l+1}(p-1)$ balls of radius $p^{-(lk+1)}$ and get

$$\begin{aligned}\mathcal{H}_p^s(\Omega_k(p)) \ &\leq \ (p^k - p^{k-1})(p^k - p^{k-1})^l p^{-lks}[(p-1)p^{-s}] \\ &\leq \ (p^k - p^{k-1}) \left[\left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p}\right)^{-\frac{1}{k}} \right] .\end{aligned}$$

Since this coincides with the lower bound in (4–11), $\mathcal{H}_p^s(\Omega_k(p))$ must equal this bound.

For the upper bound, choose the $p^k - p^{k-1}$ allowed digits in each layer to be $\sum_{j=lk}^{k(l+1)-1} a_j\, p^j$ with the restriction that $a_{k(l+1)-1} \neq 0$. Then each residue class (mod $p^{-k(l+1)+r}$) covers exactly $(p-1)p^{r-1}$ classes of Ω (mod p^{-lk}), and we can do no better than the upper bound.

It would be interesting to know how many elements $r = \frac{l}{p^k}$ have no iterate $f_p^n(r) \in \frac{1}{p^{k-1}}\mathbb{Z}_p$. We know that this set contains the $p^k - p^{k-1}$ elements $1 \leq l \leq p^k - 1$ with $(l,p) = 1$. We conjecture that these are the only such elements.

5. APPROXIMATE MULTIPLICATION MAPS

We can use similar methods to study iteration of the approximate multiplication map $f_r : \mathbb{Q} \to \mathbb{Q}$ given by

$$f_r(x) = r\lceil x \rceil \ , \tag{5–1}$$

where r is a fixed rational number, say $r = \frac{l}{d}$ with $\gcd(l,d) = 1$. In this case, since x enters into the iteration only as the integer $\lceil x \rceil$, we may restrict attention to initial values $x \in \mathbb{Z}$. We consider the case where the denominator $d > 1$, and study the question of whether some iterate $f_r^{(j)}(x)$ will be an integer for some $j \geq 1$. Note that all iterates lie in $\frac{1}{d}\mathbb{Z}$.

Unlike the case of approximate squaring, the iterates do not remain integral once they become integral. However, the truth of Conjecture 1.2 would imply that infinitely many members of a sequence of iterations $\{f_r^{(j)}(n) \ : \ j \geq 1\}$ will be integers, provided $|r| > 1$.

It is convenient to rescale the map to eliminate the denominators d, by conjugating f_r by the dilation $\Phi_d(x) = d\,x$. The result is the map $g_r : \mathbb{Z} \to \mathbb{Z}$ given by $g_r(x) := \Phi_d \circ f_r \circ \Phi_d^{-1}(x)$. Thus,

$$g_r(x) = d\, f_r(\frac{x}{d}) = l\lceil \frac{x}{d} \rceil \ , \tag{5–2}$$

with

$$g_r^{(j)}(x) = d\, f_r^{(j)}(\frac{x}{d})$$

for $j = 1, 2, \ldots$. We have

$$g_r(n) = \frac{1}{d}(ln + l_b) \text{ when } n \equiv b \pmod{d} \ ,$$

where $l_0 = 0$ and

$$l_b = l(d-b) \ \text{ for } \ 1 \leq b \leq d-1 \ .$$

For example, when $r = \frac{3}{2}$, we have

$$g_{3/2}(n) = \begin{cases} \frac{3}{2}n & \text{if } \ n \equiv 0 \pmod{2}, \\ \frac{3}{2}n + \frac{3}{2} & \text{if } \ n \equiv 1 \pmod{2}. \end{cases}$$

Our question then becomes: when does the sequence of iterations $\{g_r^{(j)}(n) \ : \ j \geq 1\}$ contain a term which is divisible by d?

The map g_r belongs to a general class of functions, which we will denote by $\mathcal{P}_r$, consisting of those "periodically linear" functions $h_r : \mathbb{Z} \to \mathbb{Z}$ of the form

$$h_r(n) = \frac{1}{d}(ln + l_b) \text{ when } n \equiv b \pmod{d} \ , \tag{5–3}$$

where $r = l/d$ is rational and the integers $\{l_b : \ 0 \leq b \leq d-1\}$ satisfy the conditions

$$l_b \equiv -lb \pmod{d} \tag{5–4}$$

needed to give an integer-valued map. Thus, although the notation does not reflect this, h_r is defined by specifying $r = l/d$ and constants $l_0, l_1, \ldots, l_{d-1}$ satisfying (5–4). We note that any map h_r in $\mathcal{P}_r$ is "self-similar" in the sense that its linear part $\frac{l}{d}n$ is independent of the residue class. More general classes of periodically linear functions have been studied in connection with the $3x+1$ problem (see Section 3.2 of [Lagarias 85]); those classes differ from $\mathcal{P}_r$ in allowing the linear part of the map to depend on the residue class (mod d).

Our methods apply generally to the question of whether a particular function $h_r \in \mathcal{P}_r$ has an iterate that is divisible by d (the denominator of r). The behavior of the map h_r depends on whether $|r| > 1$, the "expanding map" case; $r = \pm 1$, the "indifferent map" case; or $|r| \leq 1$, the "contracting map" case. Our motivation comes from the expanding map case, but the results proved below apply to all cases.

We formulate a general conjecture concerning functions in this class that are expanding maps, of which Conjecture 1.2 is a special case.

Conjecture 5.1. *Let $r = \frac{l}{d}$ with $\gcd(l, d) = 1$ and $|r| > 1$. Let $h_r : \mathbb{Z} \to \mathbb{Z}$ be a function in the class $\mathcal{P}_r$. Then for each integer n, with at most a finite number of exceptions, there is some iterate $j \geq 1$ such that $h_r^{(j)}(n) \equiv 0 \pmod{d}$.*

The "expanding" condition on r is necessary, for the conjecture fails for certain r with $0 < r < 1$, as shown in Theorem 5.4 below.

An interesting map in the class $\mathcal{P}_r$ is

$$\tilde{g}_r(x) = \lceil rx \rceil \ . \tag{5–5}$$

This map has $l_0 = 0$ and

$$l_b = d - (lb \mod d) \text{ for } 1 \leq b \leq d-1 \ .$$

The function $\tilde{g}_{3/2}(x)$ appears in Mahler's study of Z-numbers [Mahler 68], as explained below. We have

$$\tilde{g}_{3/2}(n) = \begin{cases} \frac{3}{2}n & \text{if } n \equiv 0 \pmod{2}, \\ \frac{3}{2}n + \frac{1}{2} & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Mahler's study of Z-numbers led to questions similar to Conjecture 5.1. A *Z-number* is a positive real number ξ with the property that

$$0 \ \leq \ \{\{(\frac{3}{2})^n \xi\}\} \ \leq \ \frac{1}{2} \text{ for all } \ n \geq 1 \ ,$$

where $\{\{x\}\} = x - \lfloor x \rfloor$ denotes the fractional part of x. Mahler conjectured that Z-numbers do not exist and showed that a necessary and sufficient condition for their nonexistence is that for each $n \geq 1$ there exists some $j \geq 1$ (depending on n) such that

$$\tilde{g}_{3/2}^{(j)}(n) \equiv 3 \pmod{4} \ . \tag{5–6}$$

Mahler's conjecture remains open. Mahler obtained a nontrivial upper bound on the number of Z-numbers smaller than x, and [Flatto 91] improved the upper bound to $O(x^{0.59})$ for $x \to \infty$.

In comparison to (5–6), Conjecture 5.1 for $\tilde{g}_{3/2}$ (with $r = \frac{3}{2}$) asserts that for each $n \in \mathbb{Z}$ there exists some $j \geq 1$ (depending on n) with

$$\tilde{g}_{3/2}^{(j)}(n) \equiv 0 \pmod{2} \ , \tag{5–7}$$

with at most a finite number of exceptions. This special case of Conjecture 5.1 is true, as a consequence of the next theorem. There is exactly one exceptional integer, $n = -1$, whose iterates never satisfy (5–7).

More generally, for the case of rational numbers r with denominator $d = 2$, Conjecture 5.1 is provable for all functions in the class $\mathcal{P}_r$ in a fashion analogous to that used for the approximate squaring map in Section 2.

Theorem 5.2. *Let h_r be a function in the class $\mathcal{P}_r$, for fixed $r = \frac{2t+1}{2}$ where t is an integer. Then for each $n \in \mathbb{Z}$, with at most two exceptions, there exists some iterate $k \geq 1$ with*

$$h_r^{(k)}(n) \equiv 0 \pmod{2} \ . \tag{5–8}$$

Proof: We have

$$h_r(n) = \begin{cases} r\,n + l_0 & \text{if } n \equiv 0 \pmod{2}, \\ r\,n + \frac{1}{2} + l_1 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

for some integers l_0, l_1.

We claim that the set of integers n satisfying

$$h_r^{(j)}(n) \not\equiv 0 \pmod{2} \text{ for } 1 \leq j \leq k \tag{5–9}$$

consists of the integers in exactly two arithmetic progressions $b \pmod{2^{k+1}}$, one consisting of even integers and one of odd integers. We prove the claim by induction on $k \geq 1$. For the base case, let $b \equiv a_0 \pmod{2}$ with $a_0 = 0$ or 1 fixed, and consider the arithmetic progression $n = b + 2m$, with $m \in \mathbb{Z}$. Then,

$$h_r(n) = h_r(b) + (2t+1)m \ ,$$

and the condition $h_r(n) \equiv 0 \pmod{2}$ restricts m to lie in a single congruence class $m \equiv a_1(b) \pmod{2}$. We conclude that exactly two congruence classes $b \equiv a_0 + 2a_1 \pmod{4}$ satisfy (5–9) for $k = 1$, one consisting of even integers and one of odd integers, completing the base case.

For the induction step, supposing (5–9) true for k, let $b \pmod{2^{k+1}}$ run over the two allowed congruence classes for the given k. Consider the arithmetic progression $n = b + 2^{k+1}m$, with $m \in \mathbb{Z}$. Then, we have

$$h_r^{(j)}(n) \equiv h_r^{(j)}(b) \pmod{2} \ \text{ for } \ 1 \leq j \leq k$$

and

$$h_r^{(k+1)}(n) = h_r^{k+1}(b) + (2t+1)^{k+1} m \ .$$

The condition that

$$h_r^{(k+1)}(n) \equiv 0 \pmod 2$$

is equivalent to saying that $m \equiv a_{k+1}(b) \pmod 2$, which excludes the congruence class

$$b' \equiv b + a_{k+1}(b) 2^{k+1} \pmod{2^{k+2}} \ .$$

Thus two congruence classes (mod 2^{k+2}) remain which satisfy (5–9) for $1 \le j \le k+1$. Since each of the previous classes $b \pmod{2^{k+1}}$ contributed one of these classes, one contains even integers and the other contains odd integers. This completes the induction step.

Denote these two classes by $b_0(k+1) \pmod{2^{k+2}}$ and $b_1(k+1) \pmod{2^{k+2}}$, respectively. It follows that (5–8) holds except for integers in the sets

$$\bigcup_{k=1}^{\infty} \{n \equiv b_0(k) \pmod{2^{k+1}}\}$$

and

$$\bigcup_{k=1}^{\infty} \{n \equiv b_1(k) \pmod{2^{k+1}}\} \ .$$

Each of these sets contains at most one element, so there are at most two exceptional elements. □

Concerning Theorem 5.2, there exist examples of functions $h_r \in \mathcal{P}_r$ with denominator $d = 2$ such that there are zero, one, or two elements in the exceptional set. For example, for $r = \frac{3}{2}$ the function $\tilde{g}_{3/2}$ has the single exceptional element $n = -1$. The map

$$h_{3/2}(n) = \begin{cases} \frac{3}{2}n + \frac{1}{2} & \text{if } n \equiv 1 \pmod 2, \\ \frac{3}{2}n - 1 & \text{if } n \equiv 0 \pmod 2 \end{cases} \tag{5–10}$$

has two exceptional points, 0 and -1, all the iterates of which are odd.

We also note that Theorem 5.2 applies to the two values $r = \pm\frac{1}{2}$ where the maps in $\mathcal{P}_r$ are contracting.

We next prove a result in the direction of Conjecture 5.1 for denominators $d \ge 3$. We bound the number of exceptions below x for a general function in the class $\mathcal{P}_r$, using an argument similar to that for the approximate squaring map studied in Section 3. Given a rational number $r = \frac{l}{d}$ with $\gcd(l,d) = 1$ and a function h_r in the class $\mathcal{P}_r$, we define the exceptional set by

$$E(h_r; x) := \left\{ \begin{array}{l} n \in \mathbb{Z}: \ |n| \le x, \\ h_r^{(k)}(n) \not\equiv 0 \pmod d \\ \text{for all } k \ge 1 \end{array} \right\}, \tag{5–11}$$

and let

$$N(h_r; x) := \#(E(h_r; x)) \ .$$

(The exceptional set in (1–3) is $E(r) = E(g_r; \infty)$.) The following result holds for all rational r, including those with $|r| \le 1$.

Theorem 5.3. *Let $r = \frac{l}{d}$ be a rational number with $\gcd(l,d) = 1$, and suppose that $d \ge 2$.*

1. *For each function $h_r \in \mathcal{P}_r$, the set of integers l such that the function h_r iterated starting from the initial value $x = l$ reaches an integer for the first time at iterate $j \ge 1$ is a union of $d(d-1)^j$ complete arithmetic progressions* (mod d^{j+1}).

2. *There is a constant $0 \le \beta_d < 1$ depending only on d such that for every function $h_r \in \mathcal{P}_r$ we have*

$$N(h_r; x) \le 4 d x^{\beta_d} \ . \tag{5–12}$$

The precise value of the constant is

$$\beta_d = \frac{\log(d-1)}{\log d} = 1 - \log_d\left(1 + \frac{1}{d-1}\right) \ . \tag{5–13}$$

Proof: The arguments to establish this result are simpler than those for the approximate squaring map because we can use radix expansions to the fixed base d. If $n \equiv b \pmod d$, then

$$h_r(n) = \frac{l}{d} n + \frac{l_b}{d} \ , \tag{5–14}$$

where $0 \le l_b \le d-1$ with $l_b \equiv -lb \pmod d$.

We claim that the elements $n \in \mathbb{Z}$ such that

$$h_r^{(j)}(n) \not\equiv 0 \pmod d \text{ for } 1 \le j \le k \tag{5–15}$$

consist of a certain set of $d(d-1)^k$ residue classes (mod d^{k+1}). We proceed by induction on $k \ge 1$. For the base case $k = 1$, given $b \pmod d$ the elements of the arithmetic progression $n = b + dm$ with $m \in \mathbb{Z}$ have

$$h_r(n) = h_r(b) + lm \ .$$

Since $\gcd(l,m) = 1$, as $m \in \mathbb{Z}$ varies these numbers cycle through every residue class $(\bmod\ d)$. In particular, there is one class $m \equiv a_1(b) \pmod d$, say, that gives $g_r(n) \equiv 0 \pmod d$. This arithmetic progression $b + a_1(b)d \pmod{d^2}$ is ruled out and all elements of the remaining $d-1$ arithmetic progressions (mod d^2) satisfy (5–15) with $k = 1$. This completes the base case.

For the induction step, suppose (5–15) holds for k, and there are $d(d-1)^k$ allowed residue classes $b \pmod{d^{k+1}}$. For each of these residue classes, consider the arithmetic progression $n = b + d^{k+1}m$ with $m \in \mathbb{Z}$. Using (5–14) repeatedly, we have

$$h_r^{k+1}(n) = h_r^{(k+1)}(b) + l^{k+1}m \ .$$

Since $\gcd(l, m) = 1$, this progression cycles through all residue classes (mod d), and the condition $h_r^{k+1}(n) \equiv 0 \pmod{d}$ rules out one residue class, say $b + a_{k+1}(b)d^{k+1} \pmod{d^{k+2}}$. Thus imposing the additional condition

$$h_r^{k+1}(n) \not\equiv 0 \pmod{d}$$

leaves $d(d-1)^{k+1}$ allowed residue classes (mod d^{k+2}) whose elements satisfy (5–15) for $k+1$. This completes the induction step, proving the claim.

The proof of the claim establishes part (1), since the elements that first have an integer iterate at the kth step for $k \geq 1$ were shown to form $d(d-1)^{k-1}$ complete arithmetic progressions (mod d^{k+1}).

To establish part (2), we will prove a stronger result than stated above. Set

$$E^*(h_r; x) := \left\{ \begin{array}{l} n \in \mathbb{Z} : \ |n| \leq x, \\ h_r^{(k)}(n) \not\equiv 0 \pmod{d} \\ \text{for } 1 \leq k \leq \log_d x \end{array} \right\} \tag{5–16}$$

and let $N^*(h_r; x) = \#(E^*(h_r; x))$. Certainly $N^*(h_r; x) \geq N(h_r; x)$, so it suffices to establish

$$N^*(h_r; x) \leq 4dx^{\beta_d} \ , \tag{5–17}$$

where β_d is given in (5–13).

Now suppose $d^k \leq x < d^{k+1}$. We observe that applying (5–15), with k replaced by $k-1$, shows that the elements in $E^*(h_r; x)$ are necessarily contained in a set of $d(d-1)^{k-1}$ residue classes (mod d^k), since $\log_d x > k-1$. It follows that

$$\begin{aligned} N^*(h_r; x) &\leq d(d-1)^{k-1} \cdot 2\left\lceil \frac{x}{d^k} \right\rceil \\ &\leq 2d^2(d-1)^{k-1} \leq 4d(d-1)^k \ , \end{aligned}$$

since $d \geq 2$. Now $x \geq d^k$, so $k \leq \frac{\log x}{\log d}$ and we have

$$\begin{aligned} N^*(h_r; x) &\leq 4de^{k \log(d-1)} \\ &\leq 4dx^{\frac{\log(d-1)}{\log d}} \ , \end{aligned}$$

as asserted. □

Theorem 5.3 has a probabilistic interpretation analogous to that of Theorem 3.3. We leave its formulation to the reader.

We now show that the upper bound of part (2) of Theorem 5.3 is nearly best possible for certain values of r in the interval $0 < r < 1$, where the map is a contracting map. That is, for suitable maps in the class $\mathcal{P}_r$, the upper bound (5–12) of Theorem 5.3 is within a multiplicative constant of the best possible upper bound.

Theorem 5.4. *Let $r = \frac{1}{d}$ with $d \geq 3$. Then, for the conjugated approximate multiplication map g_r of (5–2), the exceptional set $E(g_r; x)$ has cardinality*

$$N(g_r; x) \geq \frac{1}{d} x^{\beta_d} \ , \ \textit{for all } x \geq d \ , \tag{5–18}$$

with $\beta_d = \frac{\log(d-1)}{\log d}$. In particular, the full exceptional set $E(g_r; \infty)$ is infinite.

Proof: Note that for $r = \frac{1}{d}$ the functions g_r and $\tilde{g}_r$ coincide. We claim that, for each $k \geq 1$, the subset Σ_k of $[1, d^k]$ given by

$$\left\{ \begin{array}{l} n : 1 \leq n \leq d^k, \\ n = a_0 + a_1 d + \ldots + a_{k-1}d^{k-1} \\ \text{with all } a_i \not\equiv -1 \pmod{d} \ \ \text{for } i \geq 1 \end{array} \right\}$$

is contained in $E(g_r; d^k)$. (The cardinality of Σ_k is $d(d-1)^{k-1}$.) We prove this by induction on $k \geq 1$.

For the base case $k = 1$ we have

$$\Sigma_1 = \{1, 2, \ldots, d-1\} \subseteq E(g_r; d)$$

because each $g_r(n) = 1 \not\equiv 0 \pmod{d}$ and 1 is a fixed point of g_r. Next,

$$g_r(\Sigma_k) \subset \Sigma_{k-1} \ ,$$

because

$$g_r(n) = (a_1 + 1) + a_2 d + a_3 d^2 + \cdots + a_{k-1}d^{k-2} \in \Sigma_{k-1} \ ,$$

and $1 \leq a_1 + 1 \leq d - 1$ by hypothesis. Thus, $\Sigma_k \subset E(g_r; d^k)$, which completes the induction step.

For $d^k \leq x < d^{k+1}$, with $k \geq 1$, we have

$$\begin{aligned} N(g_r; x) &\geq N(g_r; d^k) \\ &\geq \#\Sigma_k = d(d-1)^{k-1} \geq \frac{1}{d} x^{\beta_d} \ , \end{aligned} \tag{5–19}$$

as asserted. □

Since the exceptional set $E(r)$ of the approximate multiplication map f_r has $E(r) = E(g_r; \infty)$ and $E(g_r, x) \subset E(g_r; \infty)$, Theorem 5.4 shows that the conclusion of Conjecture 1.2 does not hold for these values of r.

It seems plausible that for all values $-1 < r < 1$ (except $r = 0$) there is some function in the class $\mathcal{P}_r$ for which the conclusion of Conjecture 5.1 does not hold. However, we do not attempt to construct such functions here.

We end this section with a result that shows that in at least one case the approximate multiplication map based on the floor function,

$$F_r(x) := r\lfloor x \rfloor \ ,$$

behaves in exactly the same way as $f_r(x)$. A version of this result was communicated to us by Benoit Cloitre.

Theorem 5.5. *Let $r = \frac{d+1}{d}$ with $d \geq 1$. Then, for any integer $m \geq 1$, the sequence $F_r^{(j)}(m+d)$, $j \geq 0$, takes exactly as long to reach an integer as the sequence $f_r^{(j)}(m)$, $j \geq 0$.*

Proof: Let $y_j = f_r^{(j)}(m)$ and $Y_j = F_r^{(j)}(m+d)$. An easy induction argument shows that $Y_j = y_j + d + 1$ for $j \geq 0$. □

6. NUMERICAL RESULTS

The simplest case where we do not know if the approximate squaring map f of (1–1) will always reach an integer is when the starting value $r = l/d$ has denominator $d = 3$. We wish to determine $\theta(r)$ (say), the smallest value of $k \geq 0$ for which $f^{(k)}(r)$ is an integer.

Testing any particular value of r is complicated by the fact—already illustrated in Section 1—that the iterates grow so rapidly. This difficulty can be overcome by writing the kth iterate $l_k/d_k := f^{(k)}(r)$ in "base d":

$$\frac{l_k}{d_k} = \sum_{j=-1}^{\infty} a_j(k) d^k \ , \tag{6–1}$$

where the "digits" $a_j(k)$ satisfy $0 \leq a_j(k) < d$ (compare (3–7)), but storing only the terms in (6–1) with $j \leq M$. That is, we work mod d^{M+1}. As long as $\theta(r) \leq M - 1$, we get the correct answer by finding the smallest k for which $a_{-1}(k) = 0$. If this has not happened by the time k reaches M, we increase M and repeat.

For denominator 3 the value $M = 25$ is sufficient to show that $\theta(l/3)$ is finite for $3 \leq l \leq 2000$. The following table shows what happens for the first few values. It gives the initial term, the number of steps to reach an integer, and the integer that is reached.

start	θ	reaches
3/3	0	1
4/3	2	8
5/3	6	1484710602474311520
6/3	0	2
7/3	1	7
8/3	1	8
9/3	0	3
10/3	5	1484710602474311520
11/3	2	220
⋮	⋮	⋮

(These are sequences A072340 and A085276 in [Sloane 95–03].) In the range $l \leq 2000$, large values of $\theta(l/3)$ are scarce. The first few record values are $\theta(l/3) = 0, 2, 6, 22, 23$, reached at $l = 3, 4, 5, 28, 1783$ respectively.

Starting values $r = \frac{d+1}{d}$ take longer to converge–we discussed the cases $d \leq 8$ in Section 1. The initial values of $\theta((d+1)/d)$ can be found in sequence A073524 in [Sloane 95–03]. The first few record values are $0, 1, 2, 3, 18, 26, 56, 79, 200, 225, 388, 1444$, reached at $d = 1, 2, 3, 4, 5, 11, 19, 31, 37, 67$, 149, 199 respectively (sequences A073529, A073528). R. G. Wilson, v. [Wilson 02] has checked that $\theta((d+1)/d)$ is finite for $d \leq 500$.

It is amusing to note that the record value 1444 has the following interpretation: starting with $\frac{200}{199}$ and repeatedly approximately squaring, the first integer reached is

$$200^{2^{1444}} \ ,$$

a number with about 10^{435} digits.

The approximate multiplication map $f_r(x)$ of (5–1) is easier to compute since it grows more slowly. Let $\theta_r(n)$ denote the smallest value of $k \geq 1$ for which $f_r^{(k)}(n)$ is an integer. We give just one example. This table shows what happens when $f_{4/3}(n)$ is iterated with starting value n:

n:	0	1	2	3	4	5	6	7	8	9	10	11	12	⋯
$\theta_{4/3}(n)$:	1	3	2	1	2	9	1	8	3	1	7	2	1	⋯
reaches:	0	4	4	4	8	84	8	84	20	12	84	20	16	⋯

(sequences A085068 and A085071). Large values of $\theta_{4/3}(n)$ are again scarce. The first few record values are $\theta_{4/3}(n) = 1, 3, 9, 15, 17, 18, 24, 27, 28, 30, 40$, reached at $0, 1, 5, 161, 1772, 3097, 3473$, 23084, 38752, 335165, 491729 respectively (sequences A085328 and A085330). We thank J. Earls [Earls 03] for computing the last six terms in these two sequences.

All the evidence supports the conjectures made here; it would be nice to know more.

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