Small Hyperbolic 3-Manifolds With Geodesic Boundary

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CONTENTS

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We classify the orientable finite-volume hyperbolic 3-manifolds having nonempty compact totally geodesic boundary and admitting an ideal triangulation with at most four tetrahedra. We also compute the volume of all such manifolds, describe their canonical Kojima decomposition, and discuss manifolds having cusps.

The eight manifolds built from one or two tetrahedra were previously known. There are 151 different manifolds built from three tetrahedra, realizing 18 different volumes. Their Kojima decomposition always consists of tetrahedra (but occasionally requires four of them). There is a single cusped manifold, which we can show to be a knot complement in a genus-2 handlebody. Concerning manifolds built from four tetrahedra, we show that there are 5,033 different ones, with 262 different volumes. The Kojima decomposition consists either of tetrahedra (as many as eight of them in some cases), of two pyramids, or of a single octahedron. There are 30 manifolds having a single cusp and one having two cusps.

Our results were obtained with the aid of a computer. The complete list of manifolds (in SnapPea format) and full details on their invariants are available on the world wide web.

1. INTRODUCTION

This paper is devoted to the class of all orientable finite-volume hyperbolic 3-manifolds having nonempty compact totally geodesic boundary and admitting a minimal ideal triangulation with either three or four but no fewer tetrahedra. We describe the theoretical background and experimental results of a computer program that has enabled us to classify all such manifolds. (The case of manifolds obtained from two tetrahedra was previously dealt with in [Fujii 90]). We also provide an overall discussion of the most important features of all these manifolds, namely of:

- their volumes;
- the shape of their canonical Kojima decomposition;
- the presence of cusps.

These geometric invariants have all been determined by our computer program. The complete list of manifolds in SnapPea format and detailed information on the invariants is available from [Petronio 04].

2. PRELIMINARIES AND STATEMENTS

We consider in this paper the class $\mathcal{H}$ of orientable 3-manifolds M having compact nonempty boundary ∂M and admitting a complete finite-volume hyperbolic metric with respect to which ∂M is totally geodesic. It is a well-known fact (see [Kojima 90]) that such an M is the union of a compact portion and some cusps based on tori, so it has a natural compactification obtained by adding some tori. The elements of $\mathcal{H}$ are regarded up to homeomorphism, or equivalently isometry (by Mostow's rigidity).

2.1 Candidate Hyperbolic Manifolds

Let us now introduce the class $\widetilde{\mathcal{H}}$ of 3-manifolds M such that:

- M is orientable, compact, boundary-irreducible, and acylindrical (see [Fomenko and Matveev 97] for the terminology we use about 3-manifolds);
- ∂M consists of some tori (possibly none of them) and at least one surface of negative Euler characteristic.

The basic theory of hyperbolic manifolds [Thurston 78] implies that, up to identifying a manifold with its natural compactification, the inclusion $\mathcal{H} \subset \widetilde{\mathcal{H}}$ holds. We note that, by Thurston's hyperbolization, an element of $\widetilde{\mathcal{H}}$ actually lies in $\mathcal{H}$ if and only if it is atoroidal. However we do not require atoroidality in the definition of $\widetilde{\mathcal{H}}$, for a reason that will be mentioned later in this section and explained in detail in Section 3.

Let Δ denote the standard tetrahedron, and let Δ^* be Δ minus open stars of its vertices. Let M be a compact 3-manifold with $\partial M \neq \emptyset$. An *ideal triangulation* of M is a realization of M as a gluing of a finite number of copies of Δ^*, induced by a simplicial face-pairing of the corresponding Δ's. We denote by $\mathcal{C}_n$ the class of all orientable manifolds admitting an ideal triangulation with n, but no fewer, tetrahedra, and we set

$$\mathcal{H}_n = \mathcal{H} \cap \mathcal{C}_n \quad \text{and} \quad \widetilde{\mathcal{H}}_n = \widetilde{\mathcal{H}} \cap \mathcal{C}_n.$$

We can now quickly explain why we did not include atoroidality in the definition of $\widetilde{\mathcal{H}}$. The point is that there is a general notion [Matveev 90] of *complexity* $c(M)$ for a compact 3-manifold M, and $c(M)$ coincides with the minimal number of tetrahedra in an ideal triangulation precisely when M is boundary-irreducible and acylindrical. This property makes it feasible to enumerate the elements of $\widetilde{\mathcal{H}}_n$.

To summarize our definitions, *we can interpret $\mathcal{H}_n$ as the set of 3-manifolds that have complexity n and are hyperbolic with nonempty compact geodesic boundary, while $\widetilde{\mathcal{H}}_n$ is the set of complexity-n manifolds which are only "candidate hyperbolic."*

2.2 Enumeration Strategy

The general strategy of our classification result is then as follows:

- we employ the technology of standard spines [Matveev 90] (and more particularly o-graphs [Benedetti and Petronio 95]), together with certain *minimality tests* (see Section 3 below), to produce for $n = 3, 4$ a list of triangulations with n tetrahedra, such that every element of $\widetilde{\mathcal{H}}_n$ is represented by some triangulation in the list. Note that the same element of $\widetilde{\mathcal{H}}_n$ is represented by several distinct triangulations. Moreover, there could a priori be in the list triangulations representing manifolds of complexity lower than n, but the result of the classification itself actually shows that our minimality tests are sophisticated enough to ensure this does not happen;
- we write down and numerically solve the hyperbolicity equations (see [Frigerio and Petronio 04] and Section 4 below) for all the triangulations, finding solutions in the vast majority of cases (all of them for $n = 3$);
- we numerically compute the tilts (see [Frigerio and Petronio 04, Ushijima 02a] and Section 4) of each of the geometric triangulations thus found, whence determining whether the triangulation (or maybe a partial assembling of the tetrahedra of the triangulation) gives Kojima's canonical decomposition. When it does not, we modify the triangulation according to the strategy described in [Frigerio and Petronio 04], eventually finding the canonical decomposition in all cases;
- we compare the canonical decompositions to each other, thus finding precisely which pairs of triangulations in the list represent identical manifolds; we then build a list of distinct hyperbolic manifolds, which coincides with $\mathcal{H}_n$ because of the next point;

- we prove that, when the hyperbolicity equations have no solution, then indeed the manifold is not a member of $\mathcal{H}_n$, because it contains an incompressible torus (this is shown in Section 3).

Even if the next point is not really part of the classification strategy, we single it out as an important one:

- we compute the volume of all the elements of $\mathcal{H}_n$ using the geometric triangulations already found and the formulae from [Ushijima 02b].

2.3 One-Edged Triangulations

Before turning to the description of our discoveries, we must mention another point. Let us denote by Σ_g the orientable surface of genus g and by $\mathcal{K}(M)$ the set of blocks of the canonical Kojima decomposition of $M \in \mathcal{H}$. We have introduced in [Frigerio et al. 03a] the class $\mathcal{M}_n$ of orientable manifolds having an ideal triangulation with n tetrahedra and a single edge, and we have shown that for $n \geqslant 2$ and $M \in \mathcal{M}_n$:

- M is hyperbolic with geodesic boundary Σ_n;
- M has a unique ideal triangulation with n tetrahedra, which coincides with the canonical decomposition; moreover, $c(M) = n$ and $\mathcal{M}_n = \{M \in \widetilde{\mathcal{H}}_n : \partial M = \Sigma_n\}$;
- the volume of M depends only on n and can be computed explicitly.

These facts imply in particular that $\mathcal{M}_n$ is contained in $\mathcal{H}_n$.

2.4 Nature of the Results

Since we have employed computers, it seems appropriate to underline the experimental nature of our results, to indicate the possible sources of errors, and to explain how we have dealt with them. The enumeration of the potential hyperbolic manifolds $\widetilde{\mathcal{H}}_n$ relies on purely combinatorial methods, so there is no numerical approximation at this stage. The computer program implementing the enumeration is a variation on one that proved to be efficient in the closed case, where our results [Martelli and Petronio 01] were independently checked by Matveev and his collaborators.

Assume now that our enumeration of $\widetilde{\mathcal{H}}_n$ is correct, and note that we discard from $\widetilde{\mathcal{H}}_n$ only manifolds that we can prove theoretically to be nonhyperbolic. In addition, the techniques of Lackenby [Lackenby 00] imply, as described in [Costantino et al. 04], that already an approximate solution of the *angle* equations only [Frigerio and Petronio 04] is sufficient to guarantee hyperbolicity. Therefore, our list for $\mathcal{H}_n$ is sure to contain hyperbolic manifolds, even if their hyperbolic structures are computed only approximately. So the list could differ from the right one only for containing duplicates.

Duplicates were removed by computing Kojima's canonical decomposition via tilts [Frigerio and Petronio 04], which in turn requires the knowledge of the exact hyperbolic structure, so indeed numerical issues could arise here. Hyperbolic structures were computed solving the equations of [Frigerio and Petronio 04] by Newton's method with partial pivoting. This method of course requires rounding of real numbers, but in all cases convergence to the solution was extremely fast and stable; moreover, a number of cases were checked by hand, so we are very confident that our approximations of the solutions are accurate. The computation of tilts also involves rounding, but the Kojima decomposition was formally verified to be exact in all cases involving polyhedra different from the tetrahedron, and in many other cases. Moreover all the tilts found were many orders of magnitudes away from 0 than the precision we were using. For these reasons we think that our list for $\mathcal{H}_n$ actually does not contain any duplicates.

2.5 Results

We can now state our main results, recalling first [Fujii 90] that $\mathcal{H}_1 = \emptyset$ and $\mathcal{H}_2 = \mathcal{M}_2$ has eight elements, and pointing out that all the values of volumes in our statements are approximate, not exact ones. More accurate approximations are available on the web [Petronio 04].

2.5.1 Results in complexity 3. We have discovered that:

- $\mathcal{H}_3$ coincides with $\widetilde{\mathcal{H}}_3$ and has 151 elements;
- $\mathcal{M}_3$ consists of 74 elements of volume 10.428602;
- all the 77 elements of $\mathcal{H}_3 \setminus \mathcal{M}_3$ have boundary Σ_2, and one of them also has one cusp.

Moreover, the elements M of $\mathcal{H}_3 \setminus \mathcal{M}_3$ split as follows:

- 73 compact M with $\mathcal{K}(M)$ consisting of three tetrahedra; vol(M) attains 15 different values, ranges from 7.107592 to 8.513926, and has maximal multiplicity nine, with distribution according to number of manifolds as shown in Table 3 (see the Appendix);
- three compact M with $\mathcal{K}(M)$ consisting of four tetrahedra; they all have the same volume of 7.758268;

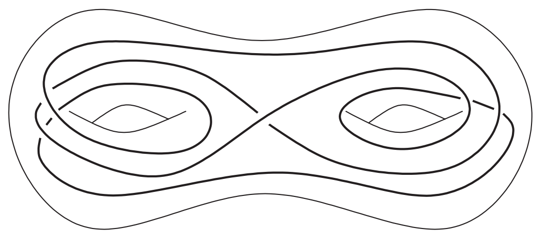

FIGURE 1. The cusped manifold having complexity three and nonempty boundary is the complement of a knot in the genus-2 handlebody.

- one noncompact M; it has a single toric cusp, $\mathcal{K}(M)$ consists of three tetrahedra, and $\mathrm{vol}(M) = 7.797637$.

The cusped element of $\mathcal{H}_3$ turns out to be a very interesting manifold. In [Frigerio et al. 03b] we analyzed all the Dehn fillings of its toric cusp, showing that precisely six of them are nonhyperbolic and improving previously known bounds on the distance between nonhyperbolic fillings. In particular, we have shown that there are fillings giving the genus-2 handlebody, so the manifold in question is a knot complement, as shown in Figure 1.

2.5.2 Results in complexity 4. We have discovered that:

- $\mathcal{H}_4$ has 5,033 elements, and $\widetilde{\mathcal{H}}_4$ has six more;
- 5,002 elements of $\mathcal{H}_4$ are compact; more precisely:
  - 2,340 have boundary Σ_4 (i.e., they belong to $\mathcal{M}_4$);
  - 2,034 have boundary Σ_3;
  - 628 have boundary Σ_2;
- 31 elements of $\mathcal{H}_4$ have cusps; more precisely:
  - 12 have one cusp and boundary Σ_3;
  - 18 have one cusp and boundary Σ_2;
  - one has two cusps and boundary Σ_2.

More detailed information about the volume and the shape of the canonical Kojima decomposition of these manifolds is described in Tables 1 and 2. In these tables each box corresponds to the manifolds M having a prescribed boundary and type of $\mathcal{K}(M)$. The first information we provide (in boldface) within the box is the number of distinct such M. When all the M in the box have the same volume, we indicate its value. Otherwise, we indicate the minimum, the maximum, the number of different values, and the maximal multiplicity of the values of the volume function, and we refer to one of the tables in the Appendix where more accurate information can be found. We emphasize here that, just as above, $\mathcal{K}(M)$ only describes the *blocks* of the Kojima decomposition, not the combinatorics of the gluing.

In addition to what is described in the tables, we have the following extra information on the geometric shape of $\mathcal{K}(M)$ when it is given by an octahedron:

- the group of 56 manifolds in Table 1 is built from an octahedron with all dihedral angles equal to $\pi/6$;
- the group of 14 manifolds in Table 1 is built from an octahedron with all dihedral angles equal to $\pi/3$;
- the group of 8 manifolds in Table 1 is built from an octahedron with three dihedral angles $2\pi/3$ along a triple of pairwise disjoint edges and two more complicated angles (one repeated three times, one six times).

A careful analysis of the values of volumes found leads to the following consequences:

Remark 2.1. For $n = 3, 4$, the maximum of the volume on $\mathcal{H}_n$ is attained at the elements of $\mathcal{M}_n$.

Remark 2.2. With the only exceptions discussed below in Remarks 2.4 and 2.5, if two manifolds in $\mathcal{H}_3 \cup \mathcal{H}_4$ have the same volume, then they also have the same complexity, boundary, and number of cusps. Moreover, they typically also have the same geometric shape of the blocks of the Kojima decomposition (but of course not the same combinatorics of gluings).

Remark 2.3. There are 280 distinct values of volume we have found in our census, and the vast majority of them correspond to more than one manifold. As a matter of fact, only 25 values are attained just once: 22 are in Tables 6 and 7, two in Table 9, and one is the volume of the cusped element of $\mathcal{H}_3$.

Remark 2.4. As stated above, there are three elements of $\mathcal{H}_3$ with canonical decomposition made of four tetrahedra. The set of geometric shapes of these four tetrahedra is actually the same in all three cases, and it turns out that the same tetrahedra can also be glued to give five different elements of $\mathcal{H}_4$. This gives the only example we have of elements $\mathcal{H}_3$ having the same volume as elements of $\mathcal{H}_4$. The volume in question is 7.758268.

Remark 2.5. The double-cusped manifold in $\mathcal{H}_4$ has the same volume 9.134475 as two of the single-cusped ones (see Table 9), and it is probably worth mentioning a

	Σ_4	Σ_3	Σ_2
4 tetra	**2,340** vol = 14.238170	**1,936** min(vol) = 11.113262 max(vol) = 12.903981 values = 59 max mult = 138 (Tables 4 and 5)	**555** min(vol) = 7.378628 max(vol) = 10.292422 values = 169 max mult = 27 (Tables 6 and 7)
5 tetra		**42** vol = 11.796442	**41** min(vol) = 8.511458 max(vol) = 9.719900 values = 16 max mult = 6 (Table 8)
6 tetra			**3** vol = 8.297977
8 tetra			**3** vol = 8.572927
1 octa (regular)		**56** vol = 11.448776	**14** vol = 9.415842
1 octa (non-reg)			**8** vol = 8.739252
2 square pyramids			**4** vol = 9.044841

TABLE 1. Number of compact elements of $\mathcal{H}_4$, subdivided according to the boundary (columns) and shape of the canonical Kojima decomposition (rows); "tetra" and "octa" mean tetrahedron and octahedron, respectively, and "square pyramid" means pyramid with square basis.

	1 cusp, Σ_3	1 cusp, Σ_2	2 cusps, Σ_2
4 tetra	**12** vol = 11.812681	**16** min(vol) = 8.446655 max(vol) = 9.774939 values = 8 max mult = 3 (Table 9)	**1** vol = 9.134475
2 square pyramids		**2** vol = 8.681738	

TABLE 2. Number of cusped elements of $\mathcal{H}_4$, subdivided according to cusps and boundary (columns), and the shape of the canonical Kojima decomposition (rows).

heuristic explanation for this fact. Recall first that an ideal triangulation of a manifold induces a triangulation of the basis of the cusps. For 28 of the single-cusped manifolds in $\mathcal{H}_4$, this triangulation involves two triangles, but for two of them it involves four, just as it does with the double-cusped manifold (both tori contain two triangles). In addition, the geometric shapes of the four triangles are the same in all three cases. In other words, one sees here that four Euclidean triangles can be used to build either two "small" Euclidean tori or a single "big" Euclidean torus (in two different ways). So, in some sense, the three manifolds in question have the same "total cuspidal geometry" (even if two manifolds have one cusp and one has two). This phenomenon already occurs in the case of manifolds without boundary [Weeks], and also in this case leads to equality of volumes. In the present case equality is also explained by the fact that the three manifolds in question have Kojima decomposition with the same geometric shape of the blocks. In fact, each of them is the gluing of four isometric partially truncated tetrahedra with three dihedral angles $\pi/3$ and three $\pi/6$.

The next information may also be of some interest:

Remark 2.6. We will show further on that the six manifolds in $\widetilde{\mathcal{H}}_4 \setminus \mathcal{H}_4$ split along an incompressible torus into two blocks, one homeomorphic to the twisted interval bundle over the Klein bottle and the other one to the cusped manifold that belongs to $\mathcal{H}_3$. These blocks give the JSJ decomposition of the manifolds involved. We will also show that the manifolds are indeed distinct by analyzing the gluing matrix of the JSJ decomposition.

Remark 2.7. As an ingredient of our arguments, we have completely classified the combinatorially inequivalent ways of building an orientable manifold by gluing together in pairs the faces of an octahedron. This topic was already mentioned in [Thurston 78] as an example of how difficult classifying 3-manifolds could be (note that there are as many as 8,505 gluings to be compared for combinatorial equivalence). For instance, the group of 56 manifolds that appears in Table 1 arises from the gluings of the octahedron such that all the edges get glued together. The groups of 14 and 8 arise similarly, requiring two edges and restrictions on their valence.

Remark 2.8. We have never included data about homology, because this invariant typically gives much coarser information than the geometric invariants that we have computed (only 14 different homology groups arise for our 5,184 manifolds). We note, however, that it occasionally happens that two manifolds having the same complexity, boundary, volume, and geometric blocks of the canonical decomposition have different homology. The homology groups we have found are $\mathbb{Z}^2 \oplus \mathbb{Z}/_n$ for $n = 1, \ldots, 8$; $\mathbb{Z}^3 \oplus \mathbb{Z}/_n$ for $n = 1, 2, 3, 5$; $\mathbb{Z}^4$; and $\mathbb{Z}^2 \oplus \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$.

Remark 2.9. Even if we have not yet introduced the hyperbolicity equations that we use to find the geometric structures, we point out a remarkable experimental discovery. The equations to be used in the cusped case are qualitatively different (and a lot more complicated) than those to be used in the compact case. However, for all the 32 cusped manifolds of the census, the hyperbolic structure was first found as a limit of approximate solutions of the compact equations.

Remark 2.10. For each M in $\mathcal{H}_3 \cup \mathcal{H}_4$, each of the (often multiple) minimal triangulations of M has been found to be geometric, i.e., the corresponding set of hyperbolicity equations has been proved to have a genuine solution. This strongly supports the conjecture that "minimal implies geometric," which one could already guess from the cusped case [Weeks].

Remark 2.11. For each M in $\mathcal{H}_3 \cup \mathcal{H}_4$, the Kojima decomposition has been obtained by merging some tetrahedra of a geometric triangulation of M. It follows that the Kojima decomposition of every manifold in $\mathcal{H}_3 \cup \mathcal{H}_4$ admits a subdivision into tetrahedra.

3. SPINES AND THE ENUMERATION METHOD

If M is a compact orientable 3-manifold, let $t(M)$ be the minimal number of tetrahedra in an ideal triangulation of either M, when $\partial M \neq \emptyset$, or M minus any number of balls, when M is closed. The function t thus defined has only one nice property: it is finite-to-one. In [Matveev 90] Matveev has introduced another function c, which he called *complexity*, having many remarkable properties not satisfied by t. For instance, c is additive on connected sums, and it does not increase when cutting along an incompressible surface. Moreover, it was proved in [Matveev 90, Matveev 98] that c equals t on the most interesting 3-manifolds, namely $c(M) = t(M)$ when M is ∂-irreducible and acylindrical, and $c(M) < t(M)$ otherwise. Therefore, if $\chi(\partial M) < 0$, we have $c(M) = t(M)$ if and only if $M \in \widetilde{\mathcal{H}}$.

3.1 Definition of Complexity

We work in the piecewise linear category and use its customary terminology [Rourke and Sanderson 82], which includes the notions of *link* (of a point in a polyhedron) and *collapse* (of a polyhedron onto a subpolyhedron). A compact polyhedron P is called *simple* if the link of every point in P is contained in the one-skeleton $\Delta^{(1)}$ of the tetrahedron. A point, a compact graph, and a compact surface are thus simple. Three important possible kinds of neighbourhoods of points are shown in Figure 2. A point having the whole of $\Delta^{(1)}$ as a link is called a *vertex*, and its regular neighbourhood is as shown in Figure 2(c). The set $V(P)$ of the vertices of P consists of

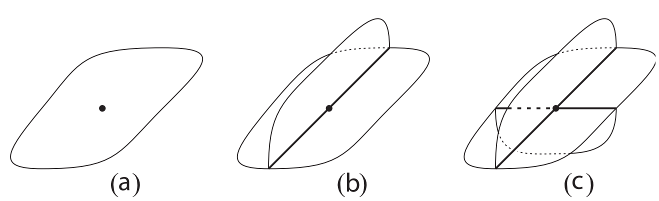

FIGURE 2. Neighbourhoods of points in a standard polyhedron.

isolated points, so it is finite. Points, graphs and surfaces of course do not contain vertices. A compact polyhedron P contained in the interior of a compact manifold M with $\partial M \neq \emptyset$ is a *spine* of M if M collapses onto P, i.e., if $M \setminus P \cong \partial M \times [0,1)$. The *complexity* $c(M)$ of a 3-manifold M is now defined as the minimal number of vertices of a simple spine of either M, when $\partial M \neq \emptyset$, or M minus some balls, when M is closed.

Since a point is a spine of the ball, a graph is a spine of a handlebody, and a surface is a spine of an interval bundle, and these spines do not contain vertices, the corresponding manifolds have complexity zero. This shows that c is not finite-to-one on manifolds containing essential discs or annuli.

In general, to compute the complexity of a manifold one must look for its *minimal* spines, i.e., the simple spines with the lowest number of vertices. It turns out [Matveev 90, Matveev 98] that M is ∂-irreducible and acylindrical if and only if it has a minimal spine that is *standard*. A polyhedron is standard when every point has a neighbourhood of one of the types (a)–(c) shown in Figure 2, and the sets of such points induce a cellularization of P. That is, defining $S(P)$ as the set of points of type (b) or (c), the components of $P \setminus S(P)$ should be open discs—the *faces*—and the components of $S(P) \setminus V(P)$ should be open segments—the *edges*.

The spines we are interested in are, therefore, standard and minimal. A standard spine is naturally dual to an ideal triangulation of M, as suggested in Figure 3. Moreover, by definition of $\widetilde{\mathcal{H}}$ and the results of Matveev just cited, a manifold M with $\chi(\partial M) < 0$ belongs to $\widetilde{\mathcal{H}}$ if and only if it has a standard minimal spine. These two facts imply the assertion already stated that $c = t$ on $\widetilde{\mathcal{H}}$ and $c < t$ outside $\widetilde{\mathcal{H}}$ on manifolds with boundary of negative χ.

3.2 Enumeration

A naive approach to the classification of all manifolds in $\widetilde{\mathcal{H}}_n$ for a fixed n would be as follows:

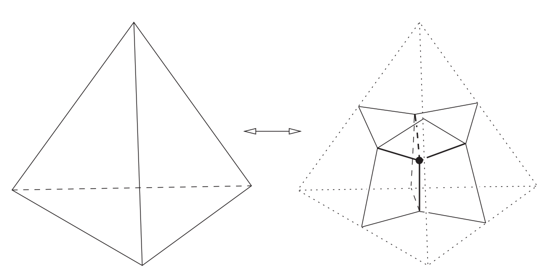

FIGURE 3. Duality between ideal triangulations and standard spines.

1. construct the finite list of all standard polyhedra with n vertices that are spines of some orientable manifold with boundary as prescribed (each such polyhedron is the spine of a unique manifold [Casler 65], and computing the boundary is a routine matter [Benedetti and Petronio 95]);

2. check which of these spines are minimal, and discard the nonminimal ones;

3. compare the corresponding manifolds for equality.

Step 1 is feasible (even if the resulting list is very long), but Step 2 is not, because there is no general algorithm to tell if a given spine is minimal or not. In our classification of $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$, we have only performed some *minimality tests*. Our tests are based on the moves shown in Figure 4, which are easily seen to transform a spine of a manifold into another spine of the same manifold. Namely, we have used the following fact:

- if a spine P of the list transforms into another one with less than n vertices via a combination of the moves of Figure 4, then P is not minimal, so it can be discarded.

For our enumeration of $\widetilde{\mathcal{H}}_3$ and $\widetilde{\mathcal{H}}_4$, an important computational stratagem was to construct the candidate spines portion after portion, following the branches of a tree, and to "cut the dead branches" at their bases. This means that the nonminimality test just described was applied also to partially constructed spines, which makes sense because the moves of Figure 4 have a local nature, so a spine containing a nonminimal portion cannot be minimal.

Remark 3.1. Starting from a standard spine, move (1) of Figure 4 always leads to a simple but nonstandard spine, and move (2) also does on some spines, whereas moves (3) and (4) always give standard spines. In particular, only moves (3) and (4) have counterparts at the level of triangulations. This extra flexibility of simple spines compared to triangulations is crucial for the enumeration.

Having obtained a list of candidate minimal spines with n vertices, we conclude the classification of $\widetilde{\mathcal{H}}_n$ for $n = 3, 4$ as follows:

- for each spine in the list, we write down and try to solve numerically the hyperbolicity equations, and if we find a solution, we compute the canonical Kojima decomposition, as discussed in Section 4. Solutions

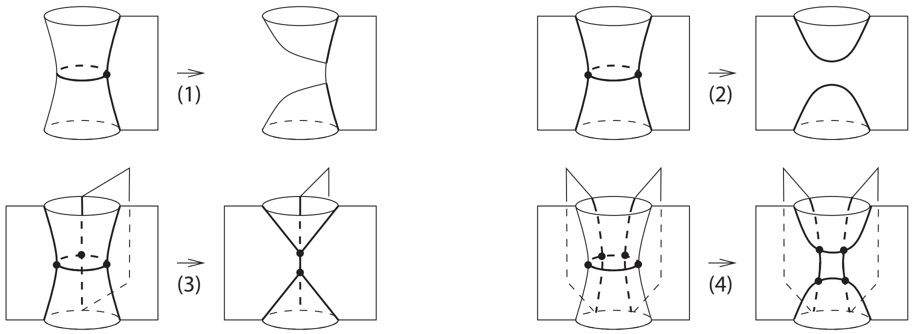

FIGURE 4. Moves on simple spines.

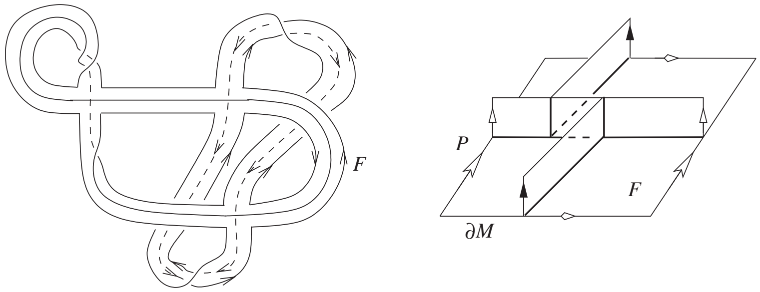

FIGURE 5. Left: a regular neighbourhood of $S(P)$; the rest of P is obtained by attaching two discs. Right: a regular neighbourhood in P of the torus $T = \overline{F}$; arrows indicate gluings.

are found in all cases for $n = 3$ and in all but six cases for $n = 4$. All six nonhyperbolic spines contain Klein bottles, so the corresponding manifolds cannot be hyperbolic;

- comparing the canonical decompositions of the hyperbolic manifolds thus found and making sure they do not belong to $\mathcal{H}_m$ for $m < n$, we classify $\mathcal{H}_n$. This gives $\widetilde{\mathcal{H}}_3 = \mathcal{H}_3$ and $\mathcal{H}_4$;
- we show that the six nonhyperbolic spines give distinct manifolds, whose complexity cannot be less than four, proving that $\widetilde{\mathcal{H}}_4 \setminus \mathcal{H}_4$ contains six manifolds.

The rest of this section is devoted to proving the last step and the assertions of Remark 2.6.

3.3 Classification of $\widetilde{\mathcal{H}}_4 \setminus \mathcal{H}_4$

To analyze the six nonhyperbolic spines with four vertices, we need more information on the cusped element M of $\mathcal{H}_3$. Its unique minimal spine P (described in Figure 5 (left) has two faces, one of which, denoted by F and marked in the picture, is an open hexagon whose closure in P is a torus T. Since a neighbourhood of T in P is as in Figure 5 (right), $P \setminus F$ is incident to T on one side.

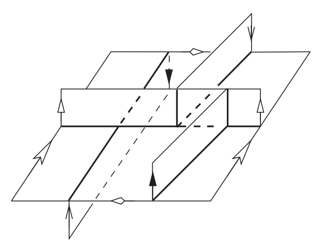

FIGURE 6. A simple polyhedron with θ-shaped boundary.

Moreover, the cusp of M lies on the other side of T, so T can be viewed as the torus boundary component of the compactification of M.

Let us now consider the polyhedron Q of Figure 6, which one easily sees to be a spine of the twisted interval bundle $K \mathbin{\widetilde{\times}} I$ over the Klein bottle. Note also that Q has a natural θ-shaped boundary ∂Q (a graph with two vertices and three edges) that we can assume lies on $\partial(K \mathbin{\widetilde{\times}} I)$. Now, if P and F are those of Figure 5, $P \setminus F$ also has a θ-shaped boundary, and it turns out that all the six nonhyperbolic candidate minimal spines with four vertices have the form $(P \setminus F) \cup_\psi Q$, for some homeomorphism $\psi : \partial Q \to \partial(P \setminus F)$. It easily follows that the associated manifold is $M \cup_\Psi (K \mathbin{\widetilde{\times}} I)$, where $\Psi : \partial(K \mathbin{\widetilde{\times}} I) \to T$ is the only homeomorphism extending ψ.

Let us now choose a homology basis on $\partial(K \tilde{\times} I)$ so that the three slopes contained in ∂Q are $0, 1, \infty \in \mathbb{Q} \cup \{\infty\}$. Doing the same on T, we see that Ψ must map $\{0, 1, \infty\}$ to itself, so its matrix in $\mathrm{GL}_2(\mathbb{Z})$ must be one of the following 12 ones:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pm\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Moreover, the six spines in question realize up to sign all these matrices. Now the JSJ decomposition of $M \cup_\Psi (K \tilde{\times} I)$ consists of M and $K \tilde{\times} I$, so $M \cup_\Psi (K \tilde{\times} I)$ is classified by the equivalence class of Ψ under the action of the automorphisms of M and $K \tilde{\times} I$ [Fomenko and Matveev 97]. But we can prove that M has no automorphisms (see below), and it is easily seen that the only automorphism of $K \tilde{\times} I$ acts as minus the identity on $\partial(K \tilde{\times} I)$ (or, to be precise, on its first homology). Therefore, the six spines represent different manifolds. Moreover they are ∂-irreducible, acylindrical, and nonhyperbolic, so they cannot belong to $\widetilde{\mathcal{H}}_m = \mathcal{H}_m$ for $m < 4$, and the classification is complete.

4. HYPERBOLICITY EQUATIONS AND THE TILT FORMULA

In this section we recall how an ideal triangulation can be used to construct a hyperbolic structure with geodesic boundary on a manifold and how an ideal triangulation can be promoted to become the canonical Kojima decomposition of the manifold. We first treat the compact case and then sketch the variations needed for the case where also some cusps exist. For all details and proofs (and for some very natural terminology that we use here without giving actual definitions), we address the reader to [Frigerio and Petronio 04].

4.1 Moduli and Equations

The basic idea for constructing a hyperbolic structure via an ideal triangulation is to realize the tetrahedra as special geometric blocks in $\mathbb{H}^3$ and then to require that the structures match when the blocks are glued together. To describe the blocks to be used, we first recall that we denote by Δ^* a truncated tetrahedron, that is a tetrahedron minus open stars of its vertices. Then, we call the *hyperbolic truncated tetrahedron* a realization of Δ^* in $\mathbb{H}^3$ such that the truncation triangles and the lateral faces of Δ^* are geodesic triangles and hexagons, respectively, and the dihedral angle between a triangle and a hexagon is always $\pi/2$. Now one can show that:

- a hyperbolic structure on a combinatorial truncated tetrahedron is determined by the 6-tuple of dihedral angles along the internal edges;
- the only restriction on this 6-tuple of positive reals comes from the fact that the angles of each of the four truncation triangles sum up to less than π;
- the lengths of the internal edges can be computed as explicit functions of the dihedral angles;
- a choice of hyperbolic structures on the tetrahedra of an ideal triangulation of a manifold M gives rise to a hyperbolic structure on M if and only if all matching edges have the same length and the total dihedral angle around each edge of M is 2π.

Given a triangulation of M consisting of n tetrahedra, one then has the *hyperbolicity equations*: a system of $6n$ equations with unknown varying in an open set of $\mathbb{R}^{6n}$ that, by Mostow's rigidity, admits one solution at most. We have solved these equations using Newton's method with partial pivoting, after having explicitly written the derivatives of the length function. Convergence to the solution was always extremely fast, and it was checked to be stable under modifications of the numerical parameters involved in the implementation of Newton's method.

4.2 Canonical Decomposition

Epstein and Penner [Epstein and Penner 88] have proved that cusped hyperbolic manifolds without boundary have a *canonical decomposition*, and Kojima [Kojima 90, Kojima 92] has proved the same for hyperbolic 3-manifolds with nonempty geodesic boundary. This gives the following very powerful tool for recognizing manifolds: *M_1 and M_2 are isometric (or, equivalently, homeomorphic) if and only if their canonical decompositions are combinatorially equivalent.* We have always checked equality and inequality of the manifolds in our census using this criterion, and we have proved that the cusped element of $\mathcal{H}_3$ has no nontrivial automorphism (a result used at the end of Section 3) by showing that its canonical decomposition has no combinatorial automorphism.

Before explaining the lines along which we have found the canonical decomposition of our manifolds, let us spend a few more words on the decomposition itself. In the cusped case its blocks are *ideal polyhedra*, whereas in the geodesic boundary case they are *hyperbolic truncated polyhedra* (an obvious generalization of a truncated tetrahedron). In both cases the decomposition is obtained by projecting first to $\mathbb{H}^3$ and then to the manifold M the faces of the convex hull of a certain family $\mathcal{P}$ of points in Minkowsky 4-space. In the cusped case these points lie on the light-cone, and they are the duals of the horoballs projecting in M to Margulis neighbourhoods of the cusps. In the geodesic boundary case the points lie on the hyperboloid of equation $\|x\|^2 = +1$, and they are the duals of the hyperplanes giving $\partial\widetilde{M}$, where $\widetilde{M} \subset \mathbb{H}^3$ is a universal cover of M.

4.3 Tilts

Assume M is a hyperbolic 3-manifold, either cusped without boundary or compact with geodesic boundary, and let a geometric triangulation $\mathcal{T}$ of M be given. One natural issue is then to decide if $\mathcal{T}$ is the canonical decomposition of M and, if not, to promote $\mathcal{T}$ to become canonical. These matters are faced using the *tilt formula* as in [Weeks 93, Ushijima 02a], that we now describe.

If σ is a d-simplex in $\mathcal{T}$, the ends of its lifting to $\mathbb{H}^3$ determine (depending on the nature of M) either $d+1$ Margulis horoballs or $d+1$ components of $\partial\widetilde{M}$, whence $d+1$ points of $\mathcal{P}$. Now let two tetrahedra Δ_1 and Δ_2 share a 2-face F, and let $\widetilde{\Delta}_1, \widetilde{\Delta}_2$, and $\widetilde{F}$ be liftings of Δ_1, Δ_2, and F to $\mathbb{H}^3$ such that $\widetilde{\Delta}_1 \cap \widetilde{\Delta}_2 = \widetilde{F}$. Let $\overline{F}$ be the 2-subspace in Minkowsky 4-space that contains the three points of $\mathcal{P}$ determined by $\widetilde{F}$. For $i = 1, 2$, let $\overline{\Delta}_i^{(F)}$ be the half-3-subspace bounded by $\overline{F}$ and containing the fourth point of $\mathcal{P}$ determined by $\widetilde{\Delta}_i$. Then one can show that $\mathcal{T}$ is canonical if and only if, whatever F, Δ_1, Δ_2, the convex hull of the half-3-subspaces $\overline{\Delta}_1^{(F)}$ and $\overline{\Delta}_2^{(F)}$ does not contain the origin of Minkowsky 4-space, and the half-3-subspaces themselves lie on distinct 3-subspaces. Moreover, if the first condition is met for all triples F, Δ_1, Δ_2, the canonical decomposition is obtained by merging together the tetrahedra along which the second condition is not met.

The tilt formula defines a real number $t(\Delta, F)$ describing the "slope" of $\overline{\Delta}^{(F)}$. More precisely, one can translate the two conditions of the previous paragraph into the inequalities $t(\Delta_1, F) + t(\Delta_2, F) \leqslant 0$ and $t(\Delta_1, F) + t(\Delta_2, F) \neq 0$, respectively. Since we can compute tilts explicitly in terms of dihedral angles, this gives a very efficient criterion to determine whether $\mathcal{T}$ is canonical or a subdivision of the canonical decomposition. Even more, it suggests where to change $\mathcal{T}$ in order to make it more likely to be canonical, namely along 2-faces where the total tilt is positive. This is achieved by 2-to-3 moves along the offending faces, as discussed in [Frigerio and Petronio 04]. We only note here that the evolution of a triangulation toward the canonical decomposition is not quite sure to converge in general, but it always does in practice, and it always did for us. We also mention that our computer program is only able to handle triangulations: whenever some mixed negative and "zero" tilts were found, the canonical decomposition was later worked out by hand and actually proved not to be a triangulation. Here "zero" is of course just a numerical approximation of the exact value, but we mention again that all the "nonzero" values of tilts we have found were reasonably large, many orders of magnitude larger than the tolerance we used.

4.4 Cusped Manifolds with Boundary

When one is willing to accept both compact geodesic boundary and toric cusps (but not annular cusps), the same strategy for constructing the structure and finding the canonical decomposition applies, but many subtleties and variations have to be taken into account. Let us quickly mention which ones.

4.4.1 Moduli. To parametrize tetrahedra one must consider that if a vertex of some Δ lies in a cusp, then the corresponding truncation triangle actually disappears into an ideal vertex (a point of $\partial\mathbb{H}^3$). At the level of moduli, this translates into the condition that the triangle be Euclidean, i.e., that its angles sum up to precisely π.

4.4.2 Equations. If an internal edge ends in a cusp, then its length is infinity; so some of the length equations must be dismissed when there are cusps. There are no consistency issues connected with half-infinite edges, but, when an edge is infinite at both ends, one must make sure that the gluings around the edge do not induce a sliding along the edge. This translates into the condition that the *similarity moduli* of the Euclidean triangles around the edge have product 1. This ensures existence of the hyperbolic structure, but one still has to impose completeness of cusps. Just as in the case where there are cusps only, this amounts to requiring that the similarity tori on the boundary be Euclidean, which translates into the *holonomy equations* involving the similarity moduli.

4.4.3 Canonical decomposition. When there are cusps, the set of points $\mathcal{P}$ to take the convex hull of consists of the duals of the planes in $\partial\widetilde{M}$ and of some points on the light-cone dual to the cusps. The precise discussion on how to choose these extra points is too complicated to be reproduced here (see [Frigerio and Petronio 04]), but the implementation of the choice was actually very easy in the (not many) cusped members of our census. The computation of tilts and the discussion on how to find the canonical decomposition are basically unaffected by the presence of cusps.

5. APPENDIX: TABLES OF VOLUMES

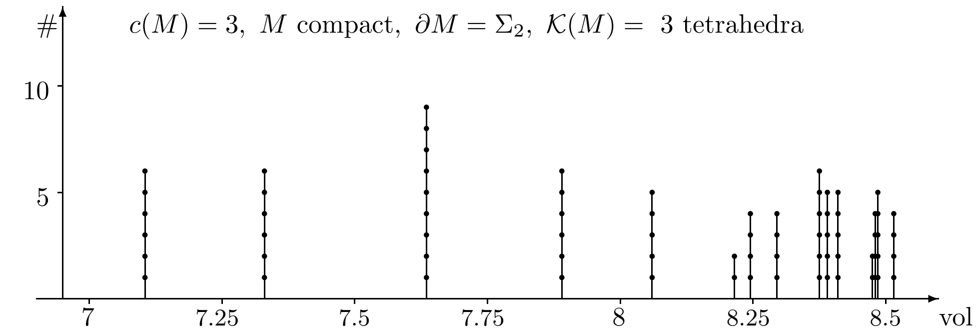

TABLE 3. Number of manifolds per value of volume for the compact elements of $\mathcal{H}_3$ with boundary of genus 2 and canonical decomposition into thee tetrahedra.

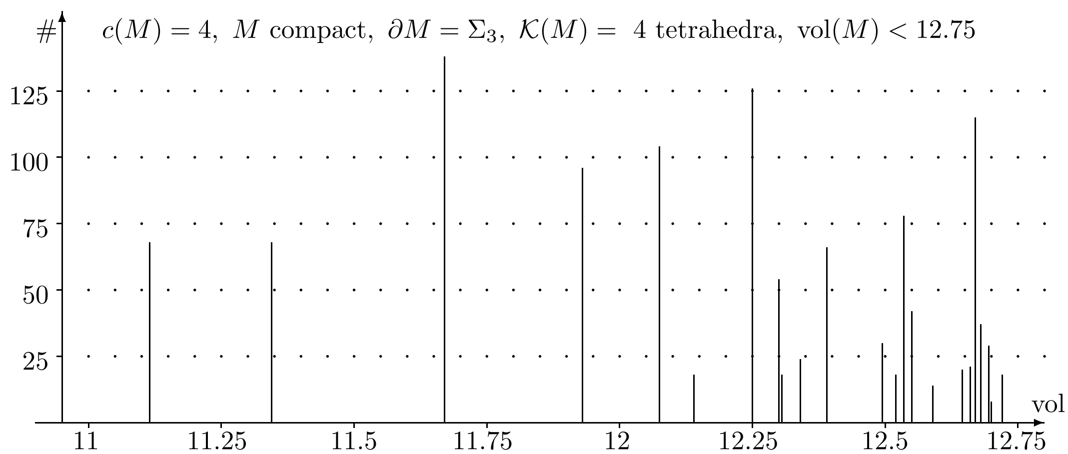

TABLE 4. Number of manifolds per value of volume for compact elements of $\mathcal{H}_4$ with boundary Σ_3 and canonical decomposition into four tetrahedra—first part.

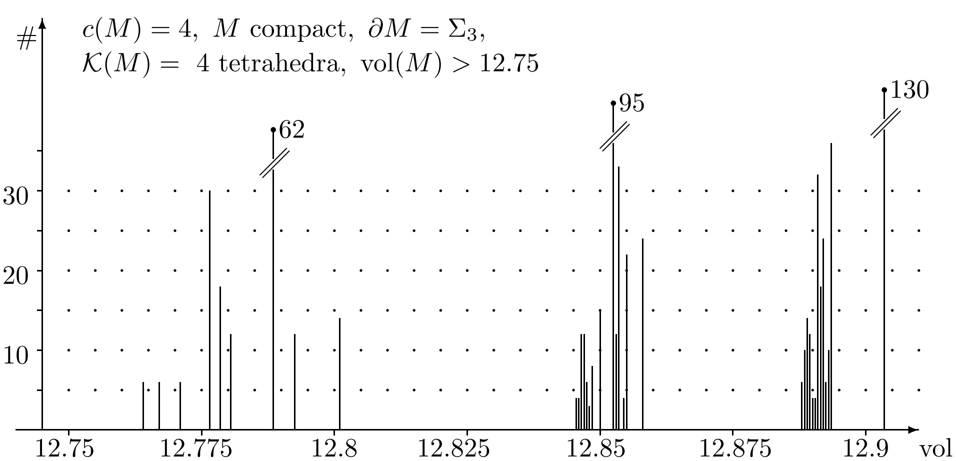

TABLE 5. Number of manifolds per value of volume for compact elements of $\mathcal{H}_4$ with boundary Σ_3 and canonical decomposition into four tetrahedra—second part. Note the changes of scale.

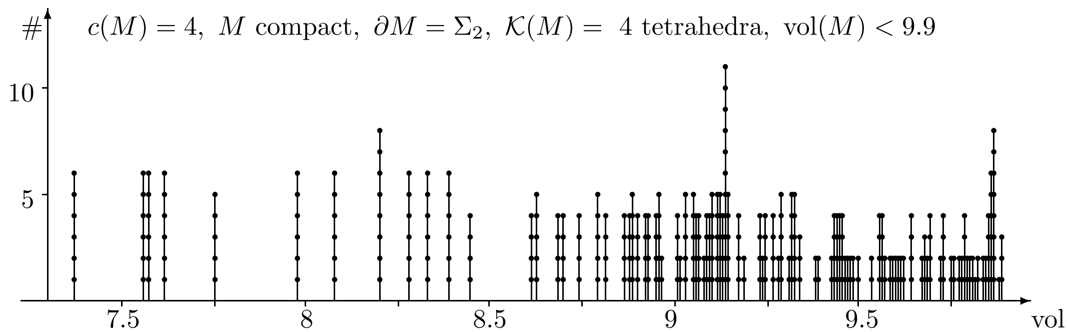

TABLE 6. Number of manifolds per value of volume for compact elements of $\mathcal{H}_4$ with boundary Σ_2 and canonical decomposition into four tetrahedra—first part.

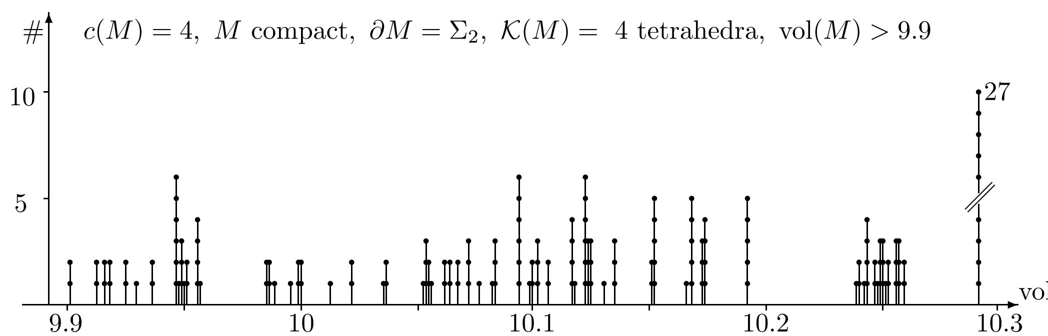

TABLE 7. Number of manifolds per value of volume for compact elements of $\mathcal{H}_4$ with boundary Σ_2 and canonical decomposition into four tetrahedra—second part. Note the change of scale on volumes.

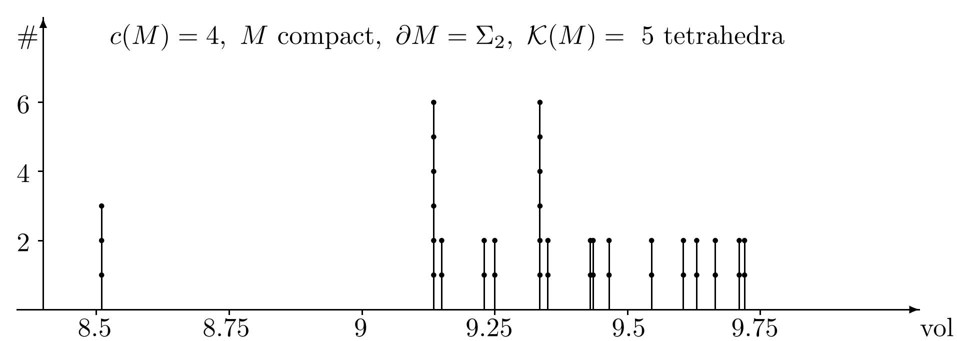

TABLE 8. Number of manifolds per value of volume for compact elements of $\mathcal{H}_4$ with boundary Σ_2 and canonical decomposition into five tetrahedra.

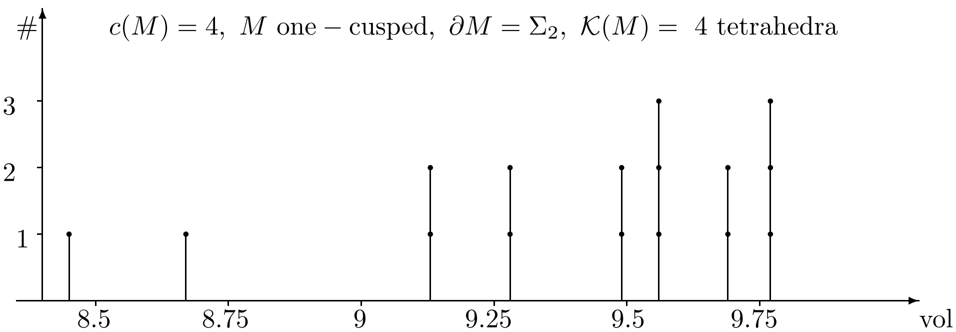

TABLE 9. Number of manifolds per value of volume for one-cusped elements of $\mathcal{H}_4$ with boundary Σ_2 and canonical decomposition into four tetrahedra.

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