Congruence Subgroups Associated to the Monster

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CONTENTS

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Let $\Delta = \{G : g(G) = 0, \Gamma_0(m) \le G \le N(\Gamma_0(m))$ for some $m\}$, where $N(\Gamma_0(m))$ is the normaliser of $\Gamma_0(m)$ in $PSL_2(\mathbb{R})$ and $g(G)$ is the genus of $\mathbb{H}^*/G$. In this article, we determine all the m. Further, for each m, we list all the intermediate groups G of $\Gamma_0(m) \le N(\Gamma_0(m))$ such that $g(G) = 0$. All the intermediate groups of width 1 at ∞ are also listed in a separate table (see www.math.nus.edu.sg/~matlml/).

1. INTRODUCTION

Let $m \in \mathbb{N}$ and let h be the largest divisor of 24 such that $m = nh^2$. The normaliser of $\Gamma_0(m)$ is given by

$$N(\Gamma_0(m)) = \Gamma_0^+(nh|h) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0^+(n) \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix},$$

where $\Gamma_0^+(n)$ is the group generated by $\Gamma_0(n)$ and all the Atkin-Lehner involutions associated to $\Gamma_0(n)$ (see [Atkin and Lehner 70, Akbas and Singerman 90, Conway 79, Conway 96]). Recall that for each exact divisor e of n (e is an exact divisor of n if $\gcd(e, n/e) = 1$), an Atkin-Lehner involution w_e associated to $\Gamma_0(n)$ is of the form

$$w_e = \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cn/\sqrt{e} & d\sqrt{e} \end{pmatrix}.$$

In [Conway 79], Conway and Norton raised the question: which groups between $\Gamma_0(m)$ and its normaliser have genus zero? The purpose of this article is to give an answer to this question. Define

$$\Delta = \{G : g(G) = 0,\ \Gamma_0(m) \le G \le N(\Gamma_0(m)) \text{ for some } m\},$$
$$\Omega = \{m : g(N(\Gamma_0(m))) = 0\}. \qquad (1\text{–}1)$$

It is clear that

$$\Delta = \{G : g(G) = 0,\ \Gamma_0(m) \le G \le N(\Gamma_0(m)),\ m \in \Omega\}. \qquad (1\text{–}2)$$

Our main results include (i) the determination of Ω, and (ii) the determination of the set of all intermediate groups

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G of $\Gamma_0(m) \leq N(\Gamma_0(m))$ of genus zero (see Section 4.1). This answers the question of Conway and Norton. Note that groups in Δ have relevance for the Monster simple group (see [Conway 79, Thompson 80] for examples) and such groups of $n|h$-type have been determined by C. R. Ferenbaugh [Ferenbaugh 93]. The set Δ can be found on our web site. As for the set Ω, we may write each m in Ω into $m = nh^2$, where h is the largest divisor of 24 such that $h^2|m$. Then the pair (n, h) can be found in Table 1 (see Section 4.2). There are 419 of them. In particular, the n (64 of them) can be found in the following: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39,41, 42, 44, 45, 46, 47, 49, 50, 51, 54, 55, 56, 59, 60, 62, 66, 69, 70, 71, 78, 87, 92, 94, 95, 105, 110, 119. Table 3 also provides us with the number of conjugacy classes, the numbers of intermediate groups, and the number of groups having width 1 at infinity. Congruence subgroups of $PSL_2(\mathbb{Z})$ of genus up to 24 have been determined by C. Cummins and S. Pauli [Cummins and Pauli 03]. Torsion-free genus zero congruence subgroups of $PSL_2(\mathbb{R})$ have been determined by A. Sebbar [Sebbar 01].

The main fact used in our study is a simple theorem that determines the genera of all the intermediate groups of $\Gamma_0(m) \leq N(\Gamma_0(m))$ (see Theorem 2.5).

The rest of this article is organised as follows: in Section 2, we determine the genus formula of G, where G is an intermediate group of $\Gamma_0(m) \leq N(\Gamma_0(m))$, for any m. In particular, this formula can be implemented using the software package GAP (Groups, Algorithms, and Programming). Section 3 proves that Ω and Δ are finite. Section 4 provides us with a list of some intermediate groups of $\Gamma_0(16) \leq N(\Gamma_0(16))$ of genus 0 and a web site (www.math.nus.edu.sg/∼matlml/) that gives the set Δ and all the subgroups in Δ of width 1 (at ∞). Table 4 gives the signature of $\Gamma_0^+(n)$, where $n \in E$ (see Section 3 for the definition of E). Table 5 gives all the intermediate groups G of $\Gamma_0(n) \leq \Gamma_0^+(n)$ such that $g(G) = 0$. Section 5 gives a set of representatives of nonconjugate elliptic subgroups of orders 2, 3, 4, 6 of $\Gamma_0^+(n)$, $n \in E$. Section 6 gives a systematic description of cusps of $\Gamma_0(nh^2)$ and $\Gamma(n)$. The permutation representations of $\Gamma_0^+(nh|h)$ and $PSL_2(\mathbb{Z})$ on the sets of cusps of $\Gamma_0(nh^2)$ and $\Gamma(n)$, respectively, are also determined in Section 6.

It is our pleasure and duty to report that congruence subgroups of $PSL_2(\mathbb{R})$ of genus 0 and 1 have been determined by Cummins [Cummins 04]. This is achieved by studying the quotient groups $\Gamma_0^+(f)/\Gamma_0(nf) \cap \Gamma(n)$, where f is square-free. It is clear that every group in our list (www.math.nus.edu.sg/∼matlml/) must conjugate to one of the groups listed in Table 3 of Cummins. However, it is not an easy task to compare these two lists as we are working in different quotient groups ($\Gamma_0^+(f)/\Gamma_0(nf) \cap \Gamma(n)$ for Cummins and $N(\Gamma_0(m))/\Gamma_0(m)$ for Chua and Lang) and the groups are presented differently (*invariants* of the groups for Cummins and matrices modulo $\Gamma_0(nh^2)$ for Chua and Lang). Comparison is possible only when groups admit the following properties:

(i) groups of Chua and Lang of width 1 (at ∞),

(ii) groups of Cummins that normalise $\Gamma_0(m)$ for some m.

The result of this comparison is given in Table 7 of Cummins and Table 3 of Chua and Lang separately and is in agreement. To be more precise, let (c_1, c_2, c_3) (of Cummins) and (cl_1, cl_2, cl_3) (of Chua and Lang) be the corresponding triples, then

$$c = c_3 = cl_3,$$

where c is the number of intermediate groups of genus 0, width 1 (at ∞) of $\Gamma_0(nh^2) \leq N(\Gamma_0(nh^2)) = \Gamma_0^+(nh|h)$.

2. GENUS FORMULA

The main purpose of this section is to give a genus formula of intermediate groups of $\Gamma_0(m) \leq N(\Gamma_0(m))$ that can be implemented in GAP. Recall first that if G is a subgroup of $PSL_2(\mathbb{R})$ commensurable with $PSL_2(\mathbb{Z})$, then

$$\chi(G) = 2(g(G) - 1) + c + \sum_{i=1}^{r}(1 - 1/d_i), \qquad (2\text{–}1)$$

where $-\chi(G)$ is the Euler characteristic, c is the number of cusps of G, r is the number of nonconjugating elliptic subgroups of G, and $d_1, d_2, \cdots, d_r$ are their orders.

Recall that $\langle\sigma\rangle$ is an elliptic subgroup of Γ if $\langle\sigma\rangle$ is a maximal cyclic subgroup of Γ. An element e is called an elliptic element if $\langle e\rangle$ is an elliptic subgroup.

Remark 2.1. Let $m \in \mathbb{N}$. Then $\chi(\Gamma_0^+(m)) = \frac{m}{6}\prod_{p|m}(p + 1)/2p$, where p runs through all the prime divisors of m.

Let G be an intermediate group of $\Gamma_0(m) = \Gamma_0(nh^2) \leq N(\Gamma(m)) = \Gamma_0^+(nh|h)$. Applying (2–1), the genus $g(G)$ satisfies

$$\chi(\Gamma_0^+(n))[\Gamma_0^+(nh|h) : G] = 2(g(G) - 1) + c + \frac{5v_6}{6} + \frac{3v_4}{4} + \frac{2v_3}{3} + \frac{v_2}{2}, \qquad (2\text{–}2)$$

where c is the number of cusps of G and $v_n = v_n(G)$ is the number of nonconjugating elliptic subgroups of order n of G. The set $\{g(G), c, v_2, v_3, v_4, v_6\}$ is called the signature of G.

2.1 Signature of $\Gamma_0^+(n)$, Where n Is Square-Free

The purpose of this section is to determine the signature of $\Gamma_0^+(n)$, where n is square-free. Recall first that since n is square-free, $\Gamma_0^+(n)$ has a unique cusp. Applying results of Maclachlan [Maclachlan 81] and Akbas and Singerman [Akbas and Singerman 92, Theorem 2], we have:

Theorem 2.2. *Let n be an integer (not necessarily square-free). Then $v_6(\Gamma_0^+(n))$, $v_3(\Gamma_0^+(n))$, and $v_4(\Gamma_0^+(n))$ are either 1 or 0 and*

(a) $v_6(\Gamma_0^+(n)) = 1$ iff $3|n$ and all the divisors of $n/3$ are of the form $3k+1$,

(b) $v_4(\Gamma_0^+(n)) = 1$ iff $2|n$ and all the divisors of $n/2$ are of the form $4k+1$,

(c) $v_3(\Gamma_0^+(n)) = 1$ iff all the divisors of n are of the form $3k+1$.

Applying Theorem 2.2, representatives of nonconjugate elliptic subgroups of orders 3, 4, and 6 can be obtained easily (see Section 5). Since n is square-free, $v_2(\Gamma_0^+(n))$ is given by Maclachlan [Maclachlan 81]. The determination of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(n)$ can be reduced to a simple study of certain positive definite quadratic forms. In particular, a complete list of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(n)$, where $g(\Gamma_0^+(n)) = 0$, can be found in Section 5. Note that Theorem 2.2 implies that

$$\begin{aligned} v_6(\Gamma_0^+(n)) \cdot v_4(\Gamma_0^+(n)) &= v_6(\Gamma_0^+(n)) \cdot v_3(\Gamma_0^+(n)) \\ &= v_3(\Gamma_0^+(n)) \cdot v_4(\Gamma_0^+(n)) \\ &= 0. \end{aligned} \tag{2–3}$$

2.2 Signature of $\Gamma_0^+(n)$, Where n Is Not Necessarily Square-Free

The purpose of this section is to determine the signature of $\Gamma_0^+(n)$ for any n. Note first that $v_k(\Gamma_0^+(n))$, $k \geq 3$, can be determined by Theorem 2.2. As a consequence, representatives of nonconjugate elliptic subgroups of orders 3, 4, and 6 can be obtained easily. The number of cusps of $\Gamma_0^+(n)$ can be determined by applying results of [Akbas and Singerman 92]. In order to determine a set of nonconjugate elliptic subgroups of 2, we recall the following results of [Lang 01, Section 4]. For readers' convenience, a proof of (i) of the following can be found in Section 7.

Suppose that G is a subgroup of $\Gamma_0^+(fB|B)$. Let $\Gamma_0^+(fB|B) = \cup g_i G$ and let $\{\tau_1, \tau_2, \cdots, \tau_s\}$ be a complete set of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(fB|B)$.

(i) If $v_6(\Gamma_0^+(fB|B)) = 1$, then $v_4(\Gamma_0^+(fB|B)) = v_3(\Gamma_0^+(fB|B)) = 0$. Let u be an element of order 6 of $\Gamma_0^+(fB|B)$. Suppose that $|\{g_i : g_i^{-1}ug_i \in G\}| = r$, $|\{g_i : g_i^{-1}u^2g_i \in G\}| = k$, $|\{g_i : g_i^{-1}\tau_j g_i \in G\}| = e_j$, $|\{g_i : g_i^{-1}u^3 g_i \in G\}| = e$. Then $v_4(G) = 0$, $v_6(G) = r$, $v_3(G) = (k-r)/2$, $v_2(G) = e_1 + e_2 + \cdots + e_s + (e-r)/3$.

(ii) If $v_4(\Gamma_0^+(fB|B)) = 1$, then $v_3(\Gamma_0^+(fB|B)) = v_6(\Gamma_0^+(fB|B)) = 0$. Let u be an element of order 4 of $\Gamma_0^+(nh|h))$. Suppose that $|\{g_i : g_i^{-1}ug_i \in G\}| = r$, $|\{g_i : g_i^{-1}u^2g_i \in G\}| = k$, $|\{g_i : g_i^{-1}\tau_j g_i \in G\}| = e_j$. Then $v_4(G) = r$, $v_6(G) = v_3(G) = 0$, and $v_2(G) = e_1 + e_2 + \cdots + e_s + (k-r)/2$.

(iii) If $v_3(\Gamma_0^+(fB|B)) = 1$, then $v_4(\Gamma_0^+(fB|B)) = v_6(\Gamma_0^+(fB|B)) = 0$. Let u be an element of order 3 of $\Gamma_0^+(fB|B)$. Suppose that $|\{g_i : g_i^{-1}ug_i \in G\}| = r$, $|\{g_i : g_i^{-1}\tau_j g_i \in G\}| = e_j$. Then $v_4(G) = v_6(G) = 0$, $v_3(G) = r$, $v_2(G) = e_1 + e_2 + \cdots + e_s$.

(iv) Suppose that $v_3(\Gamma_0^+(fB|B)) = v_4(\Gamma_0^+(fB|B)) = v_6(\Gamma_0^+(fB|B)) = 0$. Suppose further that $|\{g_i : g_i^{-1}\tau_j g_i \in G\}| = e_j$. Then $v_4(G) = v_6(G) = v_3(G) = 0$, $v_2(G) = e_1 + e_2 + \cdots + e_s$.

Let $G = \Gamma_0^+(n)$ and let $n = fB^2$, where f is square-free. Since f is square-free, a complete list of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(fB|B)$ can be determined by our results in Section 5. Applying the above results, the number of nonconjugating elliptic subgroups of order 2 of $\Gamma_0^+(n)$ can be determined. The set of representatives of nonconjugating elliptic subgroups of order 2 of $\Gamma_0^+(n)$ can now be determined by applying our results in Section 5.

2.3 Genus of Intermediate Groups of $\Gamma_0(m) \leq N(\Gamma_0(m))$

Throughout the section, $m = nh^2$, where h is the largest divisor of 24 such that $h^2|m$. Since $\Gamma_0(m)$ is a normal subgroup of $N(\Gamma_0(m)) = \Gamma_0^+(nh|h)$, $\Gamma_0^+(nh|h)$ acts on the set of cusps of $\Gamma_0(m)$. Denote the action by ρ.

Lemma 2.3. *Let $K = \Gamma_0^+(nh|h)$. Then $\rho(K) \cong K/\Gamma_0(nh^2)$.*

Proof: Suppose not. Let $\sigma \in K - \Gamma_0(nh^2)$ be chosen such that $\rho(\sigma) = 1$. This implies that $\rho(\sigma)$ fixes the cusp $[\infty]$. It follows that $\sigma(\infty) = \tau(\infty)$ for some $\tau \in \Gamma_0(nh^2)$. Hence

$$\tau^{-1}\sigma = \begin{pmatrix} 1 & x/h \\ 0 & 1 \end{pmatrix}$$

for some $x \in \mathbb{Z}$. Since $\tau^{-1}\sigma \in K - \Gamma_0(nh^2)$, x is not a multiple of h. As a consequence, $\tau^{-1}\sigma$ does not fix the cusp $[0] = \{g(0) : g \in \Gamma_0(nh^2)\} = \{x/y : gcd(y, nh^2) = 1\}$. As $\tau \in \Gamma_0(nh^2)$, we conclude that σ does not fix $[0]$. Hence $\rho(\sigma) \neq 1$. A contradiction. Hence $\rho(K) \cong K/\Gamma_0(nh^2)$. □

Lemma 2.4. *Let $A \leq B$ be finite groups and let $\{b_1, b_2, \cdots, b_k\}$ be a set of left coset representatives of A in B. Let $g \in B$. Then*

$$|\{b_i : b_i^{-1}gb_i \in A\}| = [B : A]|Cl_B(g) \cap A|/|Cl_B(g)|.$$

Proof: Let $\Delta = \{b_i : b_i^{-1}gb_i \in A\}$. Suppose that $m = |\{b_i^{-1}gb_i : b_i \in \Delta\}|$. For our convenience, we may assume that $\{b_i^{-1}gb_i : b_i \in \Delta\} = \{b_i^{-1}gb_i : i = 1, 2, \cdots, m\}$. It is clear that

$$\Delta = \cup_{i=1}^{m}\{b_{i_n} : b_{i_n}^{-1}gb_{i_n} = b_i^{-1}gb_i\} \text{ (disjoint union).} \tag{2–4}$$

Let $\Psi = \{b_i^{-1}gb_i : i = 1, 2, \cdots, m\}$. For each $b^{-1}gb \in \Psi$, it is clear that

$$\begin{aligned} b_i^{-1}gb_i = b^{-1}gb \text{ iff } & b_i^{-1}bb^{-1}gbb^{-1}b_i = b^{-1}gb \\ \text{iff } & b^{-1}b_i \in C_B(b^{-1}gb) = \cup_{t=1}^{s}x_tC_A(b^{-1}gb). \end{aligned} \tag{2–5}$$

One can now prove that there are exactly $|C_B(b^{-1}gb)/C_A(b^{-1}gb)|$ choices for b_i such that (2–5) holds. It follows that

$$\{b_{i_n} : b_{i_n}^{-1}gb_{i_n} = b_i^{-1}gb_i\} = |C_B(b_i^{-1}gb_i)/C_A(b_i^{-1}gb_i)|. \tag{2–6}$$

Applying (2–4) and (2–6), we have

$$\begin{aligned} |\Delta| &= \sum_{i=1}^{m} |C_B(b_i^{-1}gb_i)/C_A(b_i^{-1}gb_i)| \\ &= [B : A]|Cl_B(g) \cap A|/|Cl_B(g)|. \end{aligned}$$

This completes the proof of the lemma. □

By Lemma 2.3, there is a one-to-one correspondence between the intermediate groups of $\Gamma_0(nh^2) \leq \Gamma_0^+(nh|h)$ and subgroups of $\rho(\Gamma_0^+(nh|h))$. As a consequence, (i), (ii), (iii), and (iv) of Section 2.2 and the genus formula (2–2) can be evaluated in the finite group $\rho(\Gamma_0^+(nh|h))$. By Lemma 2.4, we have the following:

Theorem 2.5. *Let $\{\tau_1, \tau_2, \cdots, \tau_s\}$ be a complete set of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(nh|h)$ and let G be an intermediate group of $\Gamma_0(m) \leq N(\Gamma_0(m)) = \Gamma_0^+(nh|h)$. Then G possesses c cusps, where c is the number of $\rho(G)$ orbits.*

$$\begin{aligned} g(G) = 1 + \chi(\Gamma_0^+(n))[\Gamma_0^+(nh|h) : G]/2 - c/2 \\ - \frac{5v_6(G)}{12} - \frac{3v_4(G)}{8} - \frac{2v_3(G)}{6} - \frac{v_2(G)}{4}, \end{aligned}$$

where $-\chi(\Gamma_0^+(n))$ is the Euler characteristic. Further, $v_n(G)$ is given as follows:

(i) Suppose that $v_6(\Gamma_0^+(nh|h)) = 1$. Then

$$v_3(\Gamma_0^+(nh|h)) = v_4(\Gamma_0^+(nh|h)) = 0.$$

Let u be an element of $\Gamma_0^+(nh|h)$ of order 6. Denote by $Cl(u)$ the conjugacy class of u in $\rho(\Gamma_0^+(nh|h)$, then

$$\begin{aligned} r &= [\Gamma_0^+(nh|h) : G]|Cl(\rho(u)) \cap \rho(G)|/|Cl(\rho(u))|, \\ k &= [\Gamma_0^+(nh|h) : G]||Cl(\rho(u^2)) \cap \rho(G)|/|Cl(\rho(u^2)), \\ e_j &= [\Gamma_0^+(nh|h) : G]|Cl(\rho(\tau_j)) \cap \rho(G)|/|Cl(\rho(\tau_j)), \\ e &= [\Gamma_0^+(nh|h) : G]|Cl(\rho(u^3)) \cap \rho(G)|/|Cl(\rho(u^3))|, \\ v_4(G) &= 0, \\ v_6(G) &= r, \\ v_3(G) &= (k - r)/2, \\ v_2(G) &= e_1 + e_2 + \cdots + e_s + (e - r)/3. \end{aligned}$$

(ii) Suppose that $v_4(\Gamma_0^+(nh|h)) = 1$. Then

$$v_3(\Gamma_0^+(nh|h)) = v_6(\Gamma_0^+(nh|h)) = 0.$$

Let u be an element of $\Gamma_0^+(nh|h)$ of order 4. Then

$$\begin{aligned} r &= [\Gamma_0^+(nh|h) : G]|Cl(\rho(u)) \cap \rho(G)|/|Cl(\rho(u))|, \\ k &= [\Gamma_0^+(nh|h) : G]||Cl(\rho(u^2)) \cap \rho(G)|/|Cl(\rho(u^2)), \\ e_j &= [\Gamma_0^+(nh|h) : G]|Cl(\rho(\tau_j)) \cap \rho(G)|/|Cl(\rho(\tau_j)), \\ v_4(G) &= r, \\ v_6(G) &= v_3(G) = 0, \\ v_2(G) &= e_1 + e_2 + \cdots + e_s + (k - r)/2. \end{aligned}$$

(iii) Suppose that $v_3(\Gamma_0^+(nh|h)) = 1$. Then

$$v_4(\Gamma_0^+(nh|h)) = v_6(\Gamma_0^+(nh|h)) = 0.$$

Let u be an element of $\Gamma_0^+(nh|h)$ of order 4. Then

$$\begin{aligned} r &= [\Gamma_0^+(nh|h):G]|Cl(\rho(u))\cap\rho(G)|/|Cl(\rho(u))|,\\ e_j &= [\Gamma_0^+(nh|h):G]|Cl(\rho(\tau_j))\cap\rho(G)|/|Cl(\rho(\tau_j))|,\\ v_3(G) &= r,\\ v_6(G) &= v_4(G)=0,\\ v_2(G) &= e_1+e_2+\cdots+e_s. \end{aligned}$$

(iv) Suppose that

$$v_3(\Gamma_0^+(nh|h)) = v_4(\Gamma_0^+(nh|h)) = v_6(\Gamma_0^+(nh|h)) = 0.$$

Then

$$\begin{aligned} e_j &= [\Gamma_0^+(nh|h):G]|Cl(\rho(\tau_j))\cap\rho(G)|/|Cl(\rho(\tau_j))|,\\ v_3(G) &= v_6(G)=v_4(G)=0,\\ v_2(G) &= e_1+e_2+\cdots+e_s. \end{aligned}$$

3. THE FINITENESS OF Δ

Recall first that

$$\Delta = \{G : g(G)=0, \Gamma_0(m)\le G\le N(\Gamma_0(m)) \text{ for some } m\},$$

$$\Omega = \{m : g(N(\Gamma_0(m))) = g(\Gamma_0^+(nh|h)) = 0\}.$$

The main purpose of this section is to show that Ω and Δ are finite (see Sections 3.1 and 3.2). In order to achieve this, we define the following set:

$$E = \{n : g(\Gamma_0^+(n)) = 0\}.$$

We use the following results by Zograf [Zograf 91].

Theorem 3.1. [Zograf 91] *Let Γ be a subgroup of $PSL_2(\mathbb{R})$ commensurable with $PSL_2(\mathbb{Z})$ and let G be a congruence subgroup of Γ. Then $g(G)+1 > 3\chi(\Gamma)[\Gamma:G]/64$, where $-\chi(\Gamma)$ is the Euler characteristic,*

$$\chi(\Gamma) = 2(g(\Gamma)-1)+c+\sum_{i=1}^{r}(1-1/d_i),$$

c is the number of cusps of Γ, r is the number of conjugacy classes of elliptic subgroups of Γ, and $d_1, d_2, \cdots, d_r$ are their orders.

Corollary 3.2. *Let $g_0\in\mathbb{N}$ and let $E_{g_0} = \{n : g(\Gamma_0^+(n)) \le g_0\}$. Then E_{g_0} is finite. Further, if $n\in E_{g_0}$. Then*

$$128(g_0+1) \ge n\prod_{p|n}(p+1)/2p\,.$$

3.1 The Sets E and Ω Are Finite

The main purpose of this section is to determine the sets $E = \{n : g(\Gamma_0^+(n)) = 0\}$ and Ω. Suppose that $g(\Gamma_0^+(f)) = 0$, where f is square-free. Let $\Gamma = G = \Gamma_0^+(f)$; by Theorem 3.1, we have

$$128 > \prod_{p|f}(p+1)/2.$$

Hence the possible prime divisors of f are $2, 3, 5, 7, \cdots, 251$. Direct calculation shows that a complete list of the f such that $g(\Gamma_0^+(f)) = 0$ is given by the following: $1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 38, 39, 41, 42, 46, 47, 51, 55, 59, 62, 66, 69, 70, 71, 78, 87, 94, 95, 105, 110, 119$. Denote the above set by F.

Remark 3.3. Let $f\in F$. An easy observation shows that the possible prime divisors of f are members of $\mathbb{M}_p = \{2,3,5,7,11,13,17,19,23,29,31,41,47,59,71\}$. Note that $\mathbb{M}_p$ is the set of prime divisors of the order of the Monster simple group.

Lemma 3.4. *Let $E = \{n : g(\Gamma_0^+(n)) = 0\}$. Then*

$$E = F\cup\{4,8,9,12,16,18,20,24,25,27,32,36,44,45,49,50,54,56,60,92\}.$$

Proof: Suppose that $n\in E$. Let $n = fB^2$ (f is square-free). Then

$$\Gamma_0^+(n) \le \Gamma_0^+(fB|B) = \begin{pmatrix} B & 0\\ 0 & 1\end{pmatrix}^{-1}\Gamma_0^+(f)\begin{pmatrix} B & 0\\ 0 & 1\end{pmatrix}.$$

Since $\Gamma_0^+(n)$ is of genus 0 and $\Gamma_0^+(n)$ is a subgroup of $\Gamma_0^+(fB|B))$, we have $g(\Gamma_0^+(f)) = g(\Gamma_0^+(fB|B)) = 0$. Hence $f\in F$. Let $\Gamma = \Gamma_0^+(fB|B))$, $G = \Gamma_0^+(n)$. By Theorem 3.1,

$$128 > [\Gamma_0^+(fB|B):\Gamma_0^+(n)]\prod_{p|f}(p+1)/2.$$

It is clear that the choices of B are finite. Since the signature of $\Gamma_0^+(m)$ can be determined for any m (see Section 2), the set E can be determined by direct calculation. □

Corollary 3.5. *Let $m = nh^2$, where h is the largest divisor of 24 such that $h^2|m$. Then $m\in\Omega$ if and only if $n\in E$. In particular, Ω is finite.*

Proof: Suppose $m\in\Omega$. Let $m = nh^2$, where h is the largest divisor of 24 such that $h^2|m$. Since $g(\Gamma_0^+(nh|h)) =$

0, we have $g(\Gamma_0^+(n)) = 0$. Hence $n \in E$. By Lemma 3.4, the choices of n are finite. Since h is a divisor of 24 and $m = nh^2$, the set Ω is finite. □

3.2 The Set Δ Is Finite

$G \in \Delta$ if and only if $\Gamma_0(m) \le G \le N(\Gamma_0(m))$, $m \in \Omega$. Since both Ω and $N(\Gamma_0(m))/\Gamma_0(m)$ are finite, Δ is finite. Further, the following holds.

Proposition 3.6. *Suppose that $G \in \Delta$. Then $\Gamma_0(m) \le G \le N(\Gamma_0(m))$ for some $m \in \Omega$. Let $m = nh^2$, where h is the largest divisor of m such that $h^2|m$. Then $N(\Gamma_0(m)) = \Gamma_0^+(nh|h)$, $n \in E$. Further, $64/3\chi(\Gamma_0^+(nh|h)) > [\Gamma_0^+(nh|h) : G]$. If*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generates the stabiliser of the infinite cusp of G, then

$$\begin{aligned} \frac{64}{3\chi(\Gamma_0^+(nh|h))} &> [\Gamma_0^+(nh|h) : G] \\ &\ge [\Gamma_0^+(nh|h)_\infty : G_\infty] = h. \end{aligned}$$

Proof: Since $g(G) = 0$, we have that $\Gamma_0^+(nh|h)$ is of genus 0. It follows that $\Gamma_0^+(n)$ is of genus zero. By Lemma 3.4, $n \in E$. Applying Theorem 3.1, $64/3\chi(\Gamma_0^+(nh|h)) > [\Gamma_0^+(nh|h) : G]$. The rest follows easily. □

Proposition 3.6 is useful if one wants to investigate groups G that satisfy

$$G_\infty = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Example 3.7. Suppose that $m = 119 \cdot 24^2 \in \Omega$, $\Gamma_0(m) \le G \le N(\Gamma_0(m))$, $G \in \Delta$. Then $32/9 = 64/3\chi(\Gamma_0^+(119 \cdot 24|24)) > [\Gamma_0^+(119 \cdot 24|24)) : G]$. As a consequence, G is a maximal subgroup of $\Gamma_0^+(119 \cdot 24|24)$. Applying Proposition 3.6, G_∞ cannot be

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In summary, $\Gamma_0(119 \cdot 24^2) \le N(\Gamma_0(119 \cdot 24^2))$ possesses no subgroups G such that $g(G) = 0$ and

$$G_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

4. THE DETERMINATION OF Δ

Let $m = nh^2$, where h is the largest divisor of 24 such that $h^2|m$. By Corollary 3.5, $m \in \Omega$ if and only if $n \in E$, where E is the collection of the following 64 numbers: 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 38, 39,41, 42, 46, 47, 51, 55, 59, 62, 66, 69, 70, 71, 78, 87, 94, 95, 105, 110, 119, 4, 8, 9, 12, 16, 18, 20, 24, 25, 27, 32, 36, 44, 45, 49, 50, 54, 56, 60, 92. For each $m \in \Omega$, we may now apply Theorem 2.5 to all the intermediate groups of $\Gamma_0(m) \le N(\Gamma_0(m))$ to determine the set $\{G : g(G) = 0, \Gamma_0(m) \le G \le N(\Gamma_0(m))\}$. This is achieved by a computer program written in GAP 4.3. This program, which is available upon request, computes the genera of maximal subgroups recursively until all subgroups of genus 0 are obtained. A branch is cut off when a subgroup is of positive genus or the index of the subgroup exceeds the bound (see Remark 4.1). The last step of the program checks the conjugation in $N(\Gamma_0(m))$ and the output is the list of all proper subgroups of genus 0. A few useful remarks can be found in the following:

(i) a complete set of representatives of cusps of $\Gamma_0(nh^2)$ can be found in Section 6.

(ii) denote the set in (i) by T. Let

$$x = \begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 & 0 \\ nh & 1 \end{pmatrix}.$$

By the results of Akbas and Singerman [Akbas and Singerman 90], $\Gamma_0^+(nh|h)/\Gamma_0(nh^2)$ is generated by x, y and w_e, $e||n$. By Lemma 2.3, $\Gamma_0^+(nh|h)/\Gamma_0(nh^2) \cong \rho(\Gamma_0^+(nh|h))$, where $\rho(\Gamma_0^+(nh|h))$ is a subgroup of the symmetric group S_T.

(iii) a complete list of nonconjugate elliptic subgroups of order 2, 3, 4, 6 of $\Gamma_0^+(nh|h)$, where $n \in E$, can be found in Section 5. Their permutation representations can be determined by our results in Section 6.

Remark 4.1. Take $m = 119 \cdot 24^2$, for example, $\Gamma_0^+(119 \cdot 24|24)$ is of order 4608. Proposition 3.6 implies that if G is a subgroup of $N(\Gamma_0(119 \cdot 24^2))$ of genus 0, then the index $[N(\Gamma_0(119 \cdot 24^2)) : G]$ is at most 32/9.

4.1 The Output

For each $m = nh^2 \in \Omega$, a complete list of intermediate groups of $\Gamma_0(m) \le N(\Gamma_0(m))$ of genus 0 can be found on the following web site: www.math.nus.edu.sg/~matlml/.

Group	Index	Width of ∞	$[c, v_2, v_3, v_4, v_6]$	Generators
1.4.8.1	6	1	$[2, 2, 0, 0, 0]$	$[3, -1/2, -4, 1], [1, 0, -8, 1]$
1.4.8.2	6	1	$[2, 2, 0, 0, 0]$	$[2, -1/4, -12, 2], [5, -1/2, -8, 1]$
1.4.8.3	6	1/2	$[2, 2, 0, 0, 0]$	$[-1, 1/4, -8, 1], [1, -1/2, 0, 1]$

TABLE 1.

Group	Index	Width of ∞	$[c, v_2, v_3, v_4, v_6]$	Generator
1.4.9.1	12	1	$[3, 2, 0, 0, 0]$	$[3, -1/2, -4, 1]$
1.4.9.2	12	1	$[3, 2, 0, 0, 0]$	$[-1, -1/2, 4, 1]$
1.4.9.3	12	1	$[3, 2, 0, 0, 0]$	$[-1, 1/4, -8, 1]$
1.4.9.4	12	1	$[3, 2, 0, 0, 0]$	$[3, -1/4, -8, 1]$
1.4.9.5	12	1	$[3, 2, 0, 0, 0]$	$[2, -1/4, -12, 2]$
1.4.9.6	12	1	$[3, 2, 0, 0, 0]$	$[0, 1/4, -4, 0]$

TABLE 2.

All the intermediate groups of width 1 (at ∞) are also listed on this web site. Notations used in our lists can be found in the following:

(i) intermediate groups of genus 0 of $\Gamma_0(nh^2) \leq N(\Gamma_0(nh^2)) = \Gamma_0^+(nh|h)$ form t conjugacy classes. They are listed as $n.h.1$, $n.h.2$, $\cdots$, $n.h.t$.

(ii) suppose that the rth class has k groups. Then these groups are listed as $n.h.r.1$, $n.h.r.2$, $\cdots$, $n.h.r.k$. The groups are listed according to their width at ∞.

(iii) class with $*$ $(n.h.r^*)$ in $\Gamma_0^+(nh|h)$ possesses a single member only. As a consequence, this group is normal in $\Gamma_0^+(nh|h)$. Take $1.2.1^*$, for example, the $*$ means that the class 1.2.1 possesses a single member 1.2.1.1 only and the group 1.2.1.1 is a normal subgroup of $\Gamma_0^+(2|2)$.

See the following for two easy examples.

Example 4.2. Table 1 gives all the intermediate groups of genus 0 of the 8th conjugacy class of $\Gamma_0(16) \leq N(\Gamma_0(16)) = \Gamma_0^+(4|4)$.

Remark 4.3.

(i) The c in the forth column are the number of cusps of the group $1.4.8.x$.

(ii) The last column is a set of generators of $1.4.8.x$ modulo $\Gamma_0(16)$. The tuple $[a, b, c, d]$ represents the matrix

$$\frac{1}{\sqrt{ad-bc}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(iii) The last column gives a set of generators modulo $\Gamma_0(16)$.

(iv) Width is not invariant under conjugation.

Example 4.4. Table 2 gives all the groups of genus 0, width 1 (at ∞) of the 9th conjugacy class of the intermediate groups of $\Gamma_0(16) \leq N(\Gamma_0(16)) = \Gamma_0^+(4|4)$.

Remark 4.5. The above actually gives all the groups in the 9th class.

4.2 The Table

We shall now give the number of intermediate groups of genus zero of $\Gamma_0(nh^2) \leq N(\Gamma_0(nh^2))$ in Table 3. The entry 52(177,25) for $n = 2$, $h = 12\,(2.12^2 = 288)$ means that

(i) $m = nh^2 = 288$, there are 52 conjugacy classes of intermediate groups of $\Gamma_0(288) < N(\Gamma_0(288))$ of genus zero,

(ii) there are altogether 177 intermediate groups of $\Gamma_0(288) < N(\Gamma_0(288))$ of genus zero; 25 of them have width 1 at infinity.

The remaining entries can be read similarly. An entry $**$ denotes that h is not the largest divisor of 24 such that h^2 divides nh^2. Note that there are 419 m and 64 n.

(n,h)	1	2	3	4	6	8	12	24
1	1(1,1)	4(6,4)	5(10,8)	11(30,21)	20(84,56)	17(70,32)	34(166,42)	41(209,2)
2	2(2,2)	8(10,7)	11(30,26)	20(42,27)	34(119,67)	25(60,16)	52(177,25)	57(195,0)
3	2(2,2)	10(16,12)	7(12,9)	21(53,31)	24(55,22)	24(68,10)	37(99,6)	40(114,0)
4	**	**	**	**	**	18(30,10)	**	32(65,1)
5	2(2,2)	6(12,8)	6(13,10)	11(29,14)	12(30,6)	12(32,2)	17(47,0)	18(50,0)
6	5(5,5)	19(27,19)	12(30,21)	32(59,27)	32(66,14)	35(69,10)	48(111,13)	51(121,0)
7	2(2,2)	7(9,6)	4(6,4)	8(12,2)	11(18,4)	8(12,0)	12(21,0)	12(21,0)
8	**	**	**	**	**	7(9,1)	**	8(13,0)
9	**	**	5(7,5)	**	10(18,1)	**	11(21,0)	11(21,0)
10	5(5,5)	14(16,10)	8(12,7)	19(24,8)	20(30,7)	19(24,0)	25(38,0)	25(38,0)
11	1(1,1)	3(5,3)	2(5,3)	5(12,6)	4(9,0)	5(12,0)	6(16,0)	6(16,0)
12	**	**	**	**	**	17(21,0)	**	20(25,0)
13	2(2,2)	3(3,1)	3(3,1)	3(3,0)	4(4,0)	3(3,0)	4(4,0)	4(4,0)
14	3(3,3)	8(10,6)	6(10,7)	11(15,5)	11(17,0)	11(15,0)	14(22,0)	14(22,0)
15	3(3,3)	9(15,10)	5(7,4)	10(18,2)	11(19,0)	10(18,0)	12(22,0)	12(22,0)
16	**	**	**	**	**	3(3,0)	**	3(3,0)
17	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
18	**	**	7(9,4)	**	10(14,0)	**	10(14,0)	10(14,0)
19	1(1,1)	2(2,1)	2(2,1)	2(2,0)	3(3,0)	2(2,0)	3(3,0)	3(3,0)
20	**	**	**	**	**	7(7,0)	**	7(7,0)
21	3(3,3)	6(6,3)	4(4,1)	6(6,0)	8(8,1)	6(6,0)	8(8,0)	8(8,0)
22	2(2,2)	5(5,3)	2(2,0)	5(5,0)	5(5,0)	5(5,0)	5(5,0)	5(5,0)
23	1(1,1)	2(4,2)	1(1,0)	2(4,0)	2(4,0)	2(4,0)	2(4,0)	2(4,0)
24	**	**	**	**	**	4(4,0)	**	4(4,0)
25	2(2,2)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
26	2(2,2)	5(5,3)	2(5,0)	6(6,1)	5(5,0)	6(6,0)	6(6,0)	6(6,0)
27	**	**	1(1,0)	**	1(1,0)	**	1(1,0)	1(1,0)
29	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
30	6(6,6)	11(13,6)	8(10,4)	11(13,0)	13(17,0)	11(13,0)	13(17,0)	13(17,0)
31	1(1,1)	1(1,0)	2(2,1)	1(1,0)	2(2,0)	1(1,0)	2(2,0)	2(2,0)
32	**	**	**	**	**	1(1,0)	**	1(1,0)
33	2(2,2)	3(3,1)	2(2,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)
34	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
35	2(2,2)	3(3,1)	2(2,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)
36	**	**	**	**	**	**	**	4(4,0)
38	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
39	2(2,2)	2(2,0)	3(3,1)	2(2,0)	3(3,0)	2(2,0)	3(3,0)	3(3,0)
41	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
42	3(3,3)	6(6,3)	3(3,0)	6(6,0)	6(6,0)	6(6,0)	6(6,0)	6(6,0)
44	**	**	**	**	**	2(2,0)	**	2(2,0)
45	**	**	1(1,0)	**	1(1,0)	**	1(1,0)	1(1,0)
46	2(2,2)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
47	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
49	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
50	2(2,2)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
51	1(1,1)	2(2,1)	1(1,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
54	**	**	1(1,0)	**	1(1,0)	**	1(1,0)	1(1,0)
55	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
56	**	**	**	**	**	1(1,0)	**	1(1,0)
59	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
60	**	**	**	**	**	4(4,0)	**	4(4,0)
62	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
66	2(2,2)	3(3,1)	2(2,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)
69	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
70	2(2,2)	3(3,1)	2(2,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)	3(3,0)
71	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
78	2(2,2)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)	2(2,0)
87	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
92	**	**	**	**	**	1(1,0)	**	1(1,0)
94	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
95	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
105	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
110	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)
119	1(1,1)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)	1(1,0)

TABLE 3.

n	v_2	v_3	v_4	v_6	c	g	n	v_2	v_3	v_4	v_6	c	g
1	1	1	0	0	1	0	34	5	0	1	0	1	0
2	1	0	1	0	1	0	35	6	0	0	0	1	0
3	1	0	0	1	1	0	36	2	0	0	0	4	0
4	1	0	0	0	2	0	38	7	0	0	0	1	0
5	3	0	0	0	1	0	39	5	0	0	1	1	0
6	3	0	0	0	1	0	41	9	0	0	0	1	0
7	2	1	0	0	1	0	42	6	0	0	0	1	0
8	2	0	0	0	2	0	44	6	0	0	0	2	0
9	2	0	0	0	2	0	45	6	0	0	0	2	0
10	2	0	1	0	1	0	46	8	0	0	0	1	0
11	4	0	0	0	1	0	47	10	0	0	0	1	0
12	2	0	0	0	2	0	49	4	1	0	0	4	0
13	3	1	0	0	1	0	50	4	0	1	0	3	0
14	4	0	0	0	1	0	51	8	0	0	0	1	0
15	4	0	0	0	1	0	54	7	0	0	0	3	0
16	2	0	0	0	3	0	55	8	0	0	0	1	0
17	5	0	0	0	1	0	56	8	0	0	0	2	0
18	3	0	0	0	2	0	59	12	0	0	0	1	0
19	4	1	0	0	1	0	60	6	0	0	0	2	0
20	3	0	0	0	2	0	62	10	0	0	0	1	0
21	3	0	0	1	1	0	66	8	0	0	0	1	0
22	5	0	0	0	1	0	69	10	0	0	0	1	0
23	6	0	0	0	1	0	70	8	0	0	0	1	0
24	4	0	0	0	2	0	71	14	0	0	0	1	0
25	3	0	0	0	3	0	78	9	0	0	0	1	0
26	4	0	1	0	1	0	87	12	0	0	0	1	0
27	4	0	0	0	3	0	92	12	0	0	0	2	0
29	7	0	0	0	1	0	94	14	0	0	0	1	0
30	5	0	0	0	1	0	95	12	0	0	0	1	0
31	6	1	0	0	1	0	105	10	0	0	0	1	0
32	4	0	0	0	4	0	110	11	0	0	0	1	0
33	6	0	0	0	1	0	119	14	0	0	0	1	0

TABLE 4. Signature of $\Gamma_0^+(n), n \in E$.

Recall that E is the set of all n such that $g(\Gamma_0^+(n)) = 0$.

(i) The signature of the groups in Table 4 determined by Riemann-Hurwitz formula and results of [Akbas and Singerman 92], [Conway 79], and [Maclachlan 81].

(ii) $\Gamma_0^+(25)$, $\Gamma_0^+(49)$, and $\Gamma_0^+(50)$ are known as the *ghost* classes of the Monster. (see [Conway 79] for more detail).

For Table 5,

(i) $n+e+f+\cdots$ is the group $\Gamma_0(n)+w_e+w_f+\cdots = \langle \Gamma_0(n), w_e, w_f, \cdots \rangle$ and $n+$ is the group generated by $\Gamma_0(n)$ and all the Atkin-Lehner involutions of $\Gamma_0(n)$. Table 5 has 123 groups.

(ii) The completeness of Table 5 was first proved by P. G. Kluit in his thesis [Kluit 79].

1	$34+$
$2, 2+$	$35+35, 35+$
$3, 3+$	$36+4, 36+36, 36+$
$4, 4+$	$38+$
$5, 5+$	$39+39, 39+$
$6, 6+2, 6+3, 6+6, 6+$	$41+$
$7, 7+$	$42+3+14, 42+6+14, 42+$
$8, 8+$	$44+$
$9, 9+$	$45+$
$10, 10+2, 10+5, 10+10, 10+$	$46+23, 46+$
$11+$	$47+$
$12, 12+3, 12+4, 12+12, 12+$	$49+$
$13, 13+$	$50+50, 50+$
$14+7, 14+14, 14+$	$51+$
$15+5, 15+15, 15+$	$54+$
$16, 16+$	$55+$
$17+$	$56+$
$18, 18+2, 18+9, 18+18, 18+$	$59+$
$19+19$	$60+4+15, 60+15+20, 60+$
$20+4, 20+20, 20+$	$62+$
$21+3, 21+21, 21+$	$66+6+11, 66+$
$22+11, 22+$	$69+$
$23+$	$70+10+14, 70+$
$24+8, 24+24, 24+$	$71+$
$25, 25+$	$78+26+39, 78+$
$26+26, 26+$	$87+$
$27+$	$92+$
$28+7, 28+$	$94+$
$29+$	$95+$
$30+15, 30+5, 6, 30+3+5, 30+2+15, 30+6+10, 30+$	$105+$
$31+$	$110+$
$32+$	$119+$
$33+11, 33+$	

TABLE 5. G such that $g(G) = 0$, $\Gamma_0(n) \leq G \leq \Gamma_0^+(n)$.

5. ELLIPTIC PERIODS

Let $\sigma \in \Gamma_0^+(n)$ be an element of order k. $\langle\sigma\rangle$ is an elliptic subgroup of order k of $\Gamma_0^+(n)$ if and only if $\langle\sigma\rangle$ is a maximal cyclic subgroup of $\Gamma_0^+(n)$. Since the entries of members of $\Gamma_0^+(n)$ are elements of $\mathbb{Q}(\sqrt{a}, a \in \mathbb{N})$, one can show easily that k is either 2, 3, 4, or 6. By results of Maclachlan and Akbas-Singerman ([Maclachlan 81], [Akbas and Singerman 92]), the number of nonconjugate elliptic subgroups of order k ($k = 3, 4$, or 6) is either 0 or 1.

Further,

(i) $\Gamma_0^+(n)$ has a unique conjugacy class of elliptic subgroups of order 3 if and only if all the prime divisors of n are of the form $3k + 1$,

(ii) $\Gamma_0^+(n)$ has a unique conjugacy class of elliptic subgroups of order 4 if and only if n is even and all the prime divisors of $n/2$ are of the form $4k + 1$,

(iii) $\Gamma_0^+(n)$ has a unique conjugacy class of elliptic subgroups of order 6 if and only if n is a multiple of

3 and all the prime divisors of $n/3$ are of the form $3k+1$.

Since an element is of order 3 (4, 6) if and only if the trace is ± 1 ($\pm\sqrt{2}$, $\pm\sqrt{3}$), a representative of an elliptic element of order 3 (4, 6) of $\Gamma_0^+(n)$ can be obtained easily (see (i), (ii) and (iii) of Remark 5.4). It is clear that an element g is of order 2 if and only if the trace of g is zero. Let

$$\sigma = \begin{pmatrix} a\sqrt{e} & -b/\sqrt{e} \\ cn/\sqrt{e} & -a\sqrt{e} \end{pmatrix}, \tau = \begin{pmatrix} x\sqrt{f} & y/\sqrt{f} \\ zn/\sqrt{f} & w\sqrt{f} \end{pmatrix}$$

$\in \Gamma_0^+(n)$. Then the $(2,1)$ and $(1,2)$ entries of $\tau\sigma\tau^{-1}$ are given by the following:

$$(cfw^2 + 2azwe + bnz^2/f)n/\sqrt{e},$$
$$-(bfx^2 + 2axye + cny^2/f)/\sqrt{e}.$$

Direct calculation shows that

$$cfw^2 + 2azwe + bnz^2/f = cf(w + aez/cf)^2 + ez^2/cf,$$
$$bfx^2 + 2axye + cny^2/f = bf(x + aey/bf)^2 + ey^2/bf.$$

Since $f, e > 0$ (replace b, c by $-b$, $-c$ if necessary), we may assume that the quadratic forms

$$cfw^2 + 2azwe + bnz^2/f, \; bfx^2 + 2axye + cny^2/f \quad (5\text{–}1)$$

are positive definite. As a consequence, whether $\sigma_1, \sigma_2 \in w_e\Gamma_0(n)$ ($o(\sigma_1) = o(\sigma_2) = 2$) are conjugate to each other in $\Gamma_0^+(n)$ can be determined by direct calculation. Applying the results of Maclachlan [Maclachlan 81], the number of nonconjugate elliptic subgroups of order 2 in $w_e\Gamma_0(n)$ (where n is square-free) is known. Hence representatives of nonconjugate elliptic subgroups of order 2 in $w_e\Gamma_0(n)$ can be determined. In summary, a complete set of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(n)$, where n is square-free, can be obtained.

Remark 5.1. In the case n is not square-free, let $n = fB^2$, where f is square-free, then $f \in F$ (see Section 3.1) and $\Gamma_0^+(n) \le \Gamma_0^+(fB|B)$. The set of elliptic subgroups of order 2 of $\Gamma_0^+(fB|B)$ can be determined by the above and the number of nonconjugating elliptic subgroups of order 2 of $\Gamma_0^+(n)$ can be determined by applying (ii)–(v) of Section 2.2.

Example 5.2. An element in $\Gamma_0^+(11)$ is of the form σ or σw_{11}, where $\sigma \in \Gamma_0(11)$,

$$w_{11} = \begin{pmatrix} \sqrt{11} & -4/\sqrt{11} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix}.$$

Direct calculation shows that

$$\tau \begin{pmatrix} \sqrt{11} & -4/\sqrt{11} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix} \tau^{-1} = \begin{pmatrix} * & -3/\sqrt{11} \\ 4\sqrt{11} & * \end{pmatrix}$$

if and only if

$$\tau = \pm \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}$$

or

$$\tau = \pm \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \begin{pmatrix} \sqrt{11} & -4/\sqrt{11} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix}.$$

Further,

$$\begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix} \begin{pmatrix} \sqrt{11} & -4/\sqrt{11} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix} \times \begin{pmatrix} 3 & -1 \\ -11 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -\sqrt{11} & -3/\sqrt{11} \\ 4\sqrt{11} & \sqrt{11} \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} \sqrt{11} & -4/\sqrt{11} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix} \text{ and } \begin{pmatrix} \sqrt{11} & -3/\sqrt{11} \\ 4\sqrt{11} & -\sqrt{11} \end{pmatrix}$$

are not conjugate to each other in $\Gamma_0^+(11)$.

Example 5.3. The following is a complete set of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(38)$:

$$\begin{pmatrix} 3\sqrt{2} & -1/\sqrt{2} \\ 19\sqrt{2} & -3\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{19} & -10/\sqrt{19} \\ 2\sqrt{19} & -\sqrt{19} \end{pmatrix},$$
$$\begin{pmatrix} \sqrt{19} & -5/\sqrt{19} \\ 4\sqrt{19} & -\sqrt{19} \end{pmatrix}, \begin{pmatrix} \sqrt{19} & -2/\sqrt{19} \\ 10\sqrt{19} & -\sqrt{19} \end{pmatrix},$$
$$\begin{pmatrix} 0 & -1/\sqrt{38} \\ \sqrt{38} & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{38} & -13/\sqrt{38} \\ 3\sqrt{38} & -\sqrt{38} \end{pmatrix},$$
$$\begin{pmatrix} \sqrt{38} & -3/\sqrt{38} \\ 13\sqrt{38} & -\sqrt{38} \end{pmatrix}.$$

Proof: It is well known that $\Gamma_0(38)$ has no elements of order 2. Further, by results of Maclachlan [Maclachlan 81], $w_2\Gamma_0(38), w_{19}\Gamma_0(38)$, and $w_{38}\Gamma_0(38)$ admit 1, 3, and 3 nonconjugate elliptic subgroups of order 2, respectively. Direct calculation shows that the above is a complete set of representatives of nonconjugate elliptic subgroups of order 2. Note that our calculation is *finite* since the forms in (5–1) are positive definite. □

Remark 5.4.

(i) Suppose that $n \in E$. Then $v_6 = 1$ if and only if $n = 3, 21, 39$. It follows that an involution $\sigma \in w_3\Gamma_0(n)$ is an elliptic element of $\Gamma_0^+(n)$ order 2 if and only if it is not conjugate to the cubic of $u \in w_3\Gamma_0(n)$, where u is elliptic of order 6 as follows:

$$\begin{pmatrix} 0 & -1/\sqrt{3} \\ \sqrt{3} & -\sqrt{3} \end{pmatrix} \in w_3\Gamma_0(3),$$
$$\begin{pmatrix} \sqrt{3} & -1/\sqrt{3} \\ 7\sqrt{3} & -2\sqrt{3} \end{pmatrix} \in w_3\Gamma_0(21),$$
$$\begin{pmatrix} 5\sqrt{3} & -1/\sqrt{3} \\ 7.13\sqrt{3} & -6\sqrt{3} \end{pmatrix} \in w_3\Gamma_0(39).$$

Note that the above can be checked easily in $\rho(\Gamma_0^+(n))$ by GAP.

(ii) Suppose that $n \in E$. Then $v_4 = 1$ if and only if $n = 2, 10, 26, 34$, and 50. By results of Maclachlan [Maclachlan 81], every $\sigma \in \Gamma_0(n)$ of order 2 is a conjugate of the square of $u \in w_2\Gamma_0(n)$, where u is elliptic of order 4. As a consequence, elements of order 2 of $\Gamma_0(n)$ are not elliptic elements. Further, elliptic elements of order 4 of the above groups are given by

$$\begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \in w_2\Gamma_0(2),$$
$$\begin{pmatrix} 3\sqrt{2} & -1/\sqrt{2} \\ 25\sqrt{2} & -4\sqrt{2} \end{pmatrix} \in w_2\Gamma_0(10),$$
$$\begin{pmatrix} 3\sqrt{2} & -1/\sqrt{2} \\ 13\sqrt{2} & -2\sqrt{2} \end{pmatrix} \in w_2\Gamma_0(26),$$
$$\begin{pmatrix} 6\sqrt{2} & -1/\sqrt{2} \\ 85\sqrt{2} & -7\sqrt{2} \end{pmatrix} \in w_2\Gamma_0(34),$$
$$\begin{pmatrix} 3\sqrt{2} & -1/\sqrt{2} \\ 25\sqrt{2} & -4\sqrt{2} \end{pmatrix} \in w_2\Gamma_0(50).$$

(iii) Suppose that $n \in E$. Then $v_3 = 1$ if and only if $n = 1, 7, 13, 19, 31$, and 49. Elliptic elements of order 3 of the above groups are given by

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 7 & 3 \end{pmatrix},$$
$$\begin{pmatrix} -3 & -1 \\ 13 & 4 \end{pmatrix}, \begin{pmatrix} -7 & -3 \\ 19 & 8 \end{pmatrix},$$
$$\begin{pmatrix} -5 & -1 \\ 31 & 6 \end{pmatrix}, \begin{pmatrix} -18 & -7 \\ 49 & 19 \end{pmatrix}.$$

Denote the matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

by $(x, y : z, w)$. Table 6 gives a complete set of representatives of nonconjugate elliptic subgroups of order 2 of $\Gamma_0^+(n)$, where $n \in E$ and $g(\Gamma_0^+(n)) = 0$.

6. THE CUSPS OF $\Gamma_0(N)$ AND $\Gamma(N)$

The purpose of this section is to study the cusps of $\Gamma_0(N)$ and $\Gamma(N)$. Let $N \in \mathbb{N}$ and let h be the largest divisor of 24 such that $N = nh^2$. The following gives a complete description of the set of cusps of $\Gamma_0(nh^2)$ and a systematic description of $\rho(\tau)$, where $\tau \in \Gamma_0(nh|h)$.

Let d be a positive divisor of N and let $e = \gcd(d, N/d)$. Denote by S_d the following set:

$$S_d = \{x_i/d : \gcd(x_i, d) = 1, 0 \le x_i \le d-1, \quad x_i \not\equiv x_j \pmod{e}\}.$$

Then $\Delta = \cup_{d|N} S_d$ is a complete set of inequivalent cusps of $\Gamma_0(N)$. Note that ∞ is equivalent to $1/N$.

Remark 6.1. Let d be a positive divisor of N and let $e = \gcd(d, N/d)$. Suppose that $\gcd(x, d) = 1 = \gcd(y, d)$. Then x/d and y/d are equivalent to each other if and only if $x \equiv y \pmod{e}$.

Let $\tau \in \Gamma_0^+(nh|h)$. The permutation $\rho(\tau)$ can be determined easily by the following lemma.

Lemma 6.2. *Let x/n_0y be a cusp of $\Gamma_0(n)$, where n_0 is a divisor of n and $gcd(n, n_0y) = n_0$, $gcd(x, n_0y) = 1$. Let r be chosen such that $1 \le r \le n_0$, $r \equiv xy \pmod{(n_0, n/n_0)}$, $gcd(r, n_0) = 1$. Then x/n_0y is equivalent to r/n_0.*

Proof: Let

$$\sigma = \begin{pmatrix} a & b \\ cn & d \end{pmatrix} \in \Gamma_0(n)$$

be chosen such that $cnx + dn_0y = n_0$. This implies that $\sigma(x/n_0y) = \frac{ax+bn_0y}{n_0}$. Let

$$\tau = \begin{pmatrix} 1 & -by \\ 0 & 1 \end{pmatrix} \in \Gamma_0(n).$$

It follows that $\tau\sigma(x/n_0y) = ax/n_0$, $\gcd(ax, n_0) = 1$. Direct calculation shows that $cnx + dn_0y = n_0$, $ad - bcn = 1$. Hence $a \equiv y \pmod{n/n_0}$. This implies that $ax/n_0 = (xy + nk/n_0)/n_0$, for some $k \in \mathbb{Z}$. Since $(xy + nk/n_0)/n_0$ is equivalent to s/n_0 if and only if $xy + nk/n_0 \equiv s \pmod{(n_0, n/n_0)}$ (see Remark 6.1), it follows that x/n_0y is equivalent to r/n_0. □

n	v_2	elliptic elements of order 2
1	1	$(0, -1 : 1, 0)$
2	1	$(0, -1/\sqrt{2} : \sqrt{2}, 0)$
3	1	$(0, -1/\sqrt{3} : \sqrt{3}, 0)$
4	1	$(0, -1/2 : 2, 0)$
5	3	$(3, -2 : 5, -3), (0, -1/\sqrt{5} : \sqrt{5}, 0), (\sqrt{5}, -3/\sqrt{5} : 2\sqrt{5}, -\sqrt{5})$
6	3	$(\sqrt{2}, -1/\sqrt{2} : 3\sqrt{2}, -\sqrt{2}), (\sqrt{3}, -1/\sqrt{3} : 4\sqrt{3}, -\sqrt{3}), (0, -1/\sqrt{6} : \sqrt{6}, 0)$
7	2	$(0, -1/\sqrt{7} : \sqrt{7}, 0), (\sqrt{7}, -2/\sqrt{7} : 4\sqrt{7}, -\sqrt{7})$
8	2	$(0, -1/\sqrt{8} : \sqrt{8}, 0), (\sqrt{8}, -3/\sqrt{8} : 3\sqrt{8}, -\sqrt{8})$
9	2	$(0, -1/\sqrt{9} : \sqrt{9}, 0), (\sqrt{9}, -5/\sqrt{9} : 2\sqrt{9}, -\sqrt{9})$
10	2	$(\sqrt{5}, -3/\sqrt{5} : 2\sqrt{5}, -\sqrt{5}), (0, -1/\sqrt{10} : \sqrt{10}, 0)$
11	4	$(0, -1/\sqrt{11} : \sqrt{11}, 0), (\sqrt{11}, -2/\sqrt{11} : 6\sqrt{11}, -\sqrt{11}), (\sqrt{11}, -4/\sqrt{11} : 3\sqrt{11}, -\sqrt{11}),$ $(\sqrt{11}, -3/\sqrt{11} : 4\sqrt{11}, -\sqrt{11})$
12	2	$(0, -1/\sqrt{12} : \sqrt{12}, 0), (\sqrt{3}, -1/\sqrt{3} : 4\sqrt{3}, -\sqrt{3})$
13	3	$(5, -2 : 13, -5), (0, -1/\sqrt{13} : \sqrt{13}, 0), (\sqrt{13}, -2/\sqrt{13} : 7\sqrt{13}, -\sqrt{13})$
14	4	$(\sqrt{7}, -1/\sqrt{7} : 8\sqrt{7}, -\sqrt{7}), (\sqrt{7}, -2/\sqrt{7} : 4\sqrt{7}, -\sqrt{7}), (0, -1/\sqrt{14} : \sqrt{14}, 0),$ $(\sqrt{14}, -3/\sqrt{14} : 5\sqrt{14}, -\sqrt{14})$
15	4	$(\sqrt{5}, -2/\sqrt{5} : 3\sqrt{5}, -\sqrt{5}), (\sqrt{5}, -1/\sqrt{5} : 6\sqrt{5}, -\sqrt{5}), (0, -1/\sqrt{15} : \sqrt{15}, 0),$ $(\sqrt{15}, -2/\sqrt{15} : 8\sqrt{15}, -\sqrt{15})$
16	2	$(0, -1/\sqrt{16} : \sqrt{16}, 0), (2\sqrt{16}, -13/\sqrt{16} : 5\sqrt{16}, -2\sqrt{16})$
17	5	$(4, -1 : 17, -4), (0, -1/\sqrt{17} : \sqrt{17}, 0), (\sqrt{17}, -2/\sqrt{17} : 9\sqrt{17}, -\sqrt{17}),$ $(\sqrt{17}, -6/\sqrt{17} : 3\sqrt{17}, -\sqrt{17}), (\sqrt{17}, -3/\sqrt{17} : 6\sqrt{17}, -\sqrt{17})$
18	3	$(0, -1/\sqrt{18} : \sqrt{18}, 0), (\sqrt{9}, -5/\sqrt{9} : 2\sqrt{9}, -\sqrt{9}), (2\sqrt{2}, -1/\sqrt{2} : 9\sqrt{2}, -2\sqrt{2})$
19	4	$(0, -1/\sqrt{19} : \sqrt{19}, 0), (\sqrt{19}, -2/\sqrt{19} : 10\sqrt{19}, -\sqrt{19}), (\sqrt{19}, -5/\sqrt{19} : 4\sqrt{19}, -\sqrt{19}),$ $(\sqrt{19}, -4/\sqrt{19} : 5\sqrt{19}, -\sqrt{19})$
21	3	$(4\sqrt{3}, -7/\sqrt{3} : 7\sqrt{3}, -4\sqrt{3}), (0, -1/\sqrt{21} : \sqrt{21}, 0), (\sqrt{21}, -2/\sqrt{21} : 11\sqrt{21}, -\sqrt{21})$
22	5	$(4\sqrt{2}, -3/\sqrt{2} : 11\sqrt{2}, -4\sqrt{2}), (\sqrt{11}, -1/\sqrt{11} : 12\sqrt{11}, -\sqrt{11}), (\sqrt{11}, -3/\sqrt{11} : 4\sqrt{11}, -\sqrt{11}),$ $(\sqrt{11}, -2/\sqrt{11} : 6\sqrt{11}, -\sqrt{11}), (0, -1/\sqrt{22} : \sqrt{22}, 0)$
23	6	$(0, -1/\sqrt{23} : \sqrt{23}, 0), (\sqrt{23}, -2/\sqrt{23} : 12\sqrt{23}, -\sqrt{23}), (\sqrt{23}, -3/\sqrt{23} : 8\sqrt{23}, -\sqrt{23}),$ $\sqrt{23}, -4/\sqrt{23} : 6\sqrt{23}, -\sqrt{23}), (\sqrt{23}, -6/\sqrt{23} : 4\sqrt{23}, -\sqrt{23}), (\sqrt{23}, -8/\sqrt{23} : 3\sqrt{23}, -\sqrt{23})$
24	4	$([0, -1/\sqrt{24} : \sqrt{24}, 0), (\sqrt{8}, -3/\sqrt{8} : 3\sqrt{8}, -\sqrt{8}), (\sqrt{8}, -1/\sqrt{8}; 9\sqrt{8}, -\sqrt{8}),$ $(\sqrt{24}, -5/\sqrt{24}; 5\sqrt{24}, -\sqrt{24})$
25	3	$(0, -1/\sqrt{25} : \sqrt{25}, 0)(\sqrt{25}, -13/\sqrt{25} : 2\sqrt{25}, -\sqrt{25}), (7, -2 : 25, -7)$
26	4	$(\sqrt{13}, -7/\sqrt{13} : 2\sqrt{13}, -\sqrt{13}), (0, -1/\sqrt{26} : \sqrt{26}, 0), (\sqrt{26}, -3/\sqrt{26} : 9\sqrt{26}, -\sqrt{26}),$ $(\sqrt{26}, -9/\sqrt{26} : 3\sqrt{26}, -\sqrt{26})$
27	4	$(0, -1/\sqrt{27} : \sqrt{27}, 0), (\sqrt{27}, -14/\sqrt{27} : 2\sqrt{27}, -\sqrt{27}), (\sqrt{27}, -7/\sqrt{27} : 4\sqrt{27}, -\sqrt{27}),$ $(\sqrt{27}, -4/\sqrt{27} : 7\sqrt{27}, -\sqrt{27})$
29	7	$(12, -5 : 29, -12), (0, -1/\sqrt{29} : \sqrt{29}, 0), (\sqrt{29}, -2/\sqrt{29} : 15\sqrt{29}, -\sqrt{29}),$ $(\sqrt{29}, -3/\sqrt{29} : 10\sqrt{29}, -\sqrt{29}), (\sqrt{29}, -5/\sqrt{29} : 6\sqrt{29}, -\sqrt{29}), (\sqrt{29}, -6/\sqrt{29} : 5\sqrt{29}, -\sqrt{29}),$ $(\sqrt{29}, -10/\sqrt{29} : 3\sqrt{29}, -\sqrt{29})$
30	5	$(\sqrt{5}, -1/\sqrt{5} : 6\sqrt{5}, -\sqrt{5}), (0, -1/\sqrt{30} : \sqrt{30}, 0), (\sqrt{15}, -1/\sqrt{15} : 16\sqrt{15}, -\sqrt{15}),$ $(\sqrt{15}, -2/\sqrt{15} : 8\sqrt{15}, -\sqrt{15}), (2\sqrt{6}, -5/\sqrt{6} : 5\sqrt{6}, -2\sqrt{6})$

TABLE 6.

31	6	$(\sqrt{31},-2/\sqrt{31}:16\sqrt{31},-\sqrt{31}),(\sqrt{31},-8/\sqrt{31}:4\sqrt{31},-\sqrt{31}),(\sqrt{31},-4/\sqrt{31}:8\sqrt{31},-\sqrt{31}),$ $(0,-1/\sqrt{31}:\sqrt{31},0),(2\sqrt{31},-25/\sqrt{31}:5\sqrt{31},-2\sqrt{31}),(2\sqrt{31},-5/\sqrt{31}:25\sqrt{31},-2\sqrt{31})$
32	4	$(0,-1/\sqrt{32}:\sqrt{32},0),(\sqrt{32},-11/\sqrt{32}:3\sqrt{32},-\sqrt{32}),(\sqrt{32},-3/\sqrt{32}:11\sqrt{32},-\sqrt{32}),$ $(3\sqrt{32},-17/\sqrt{32}:17\sqrt{32},-3\sqrt{32})$
33	6	$(\sqrt{11},-4/\sqrt{11}:3\sqrt{11},-\sqrt{11}),(\sqrt{11},-2/\sqrt{11}:6\sqrt{11},-\sqrt{11}),(\sqrt{11},-1/\sqrt{11}:12\sqrt{11},-\sqrt{11}),$ $(2\sqrt{11},-3/\sqrt{11}:15\sqrt{11},-2\sqrt{11}),(0,-1/\sqrt{33}:\sqrt{33},0),(\sqrt{33},-2/\sqrt{33}:17\sqrt{33},-\sqrt{33}),$
34	5	$(5\sqrt{2},-3/\sqrt{2}:17\sqrt{2},-5\sqrt{2}),(0,-1/\sqrt{34}:\sqrt{34},0),(\sqrt{34},-5/\sqrt{34}:7\sqrt{34},-\sqrt{34}),$ $(\sqrt{17},-1/\sqrt{17}:18\sqrt{17},-\sqrt{17}),(\sqrt{17},-3/\sqrt{17}:6\sqrt{17},-\sqrt{17})$
35	6	$(2\sqrt{5},-3/\sqrt{5}:7\sqrt{5},-2\sqrt{5}),(2\sqrt{5},-1/\sqrt{5}:21\sqrt{5},-2\sqrt{5}),(0,-1/\sqrt{35}:\sqrt{35},0),$ $(\sqrt{35},-2/\sqrt{35}:18\sqrt{35},-\sqrt{35}),(\sqrt{35},-3/\sqrt{35}:12\sqrt{35},-\sqrt{35}),(\sqrt{35},-9/\sqrt{35}:4\sqrt{35},-\sqrt{35})$
36	2	$(0,-1/\sqrt{36}:\sqrt{36},0),(2\sqrt{36},-29/\sqrt{36}:5\sqrt{36},-2\sqrt{36})$
38	7	$(3\sqrt{2},-1/\sqrt{2}:19\sqrt{2},-3\sqrt{2}),(\sqrt{19},-10/\sqrt{19}:2\sqrt{19},-\sqrt{19}),(\sqrt{19},-5/\sqrt{19}:4\sqrt{19},-\sqrt{19}),$ $(\sqrt{19},-2/\sqrt{19}:10\sqrt{19},-\sqrt{19}),(0,-1/\sqrt{38}:\sqrt{38},0),(\sqrt{38},-13/\sqrt{38}:3\sqrt{38},-\sqrt{38}),$ $(\sqrt{38},-3/\sqrt{38}:13\sqrt{38},-\sqrt{38})$
39	5	$(2\sqrt{3},-1/\sqrt{3}:13\sqrt{3},-2\sqrt{3}),(0,-1/\sqrt{39}:\sqrt{39},0),(\sqrt{39},-20/\sqrt{39}:2\sqrt{39},-\sqrt{39}),$ $(\sqrt{39},-10/\sqrt{39}:4\sqrt{39},-\sqrt{39}),(\sqrt{39},-8/\sqrt{39}:5\sqrt{39},-\sqrt{39})$
41	9	$(0,-1/\sqrt{41}:\sqrt{41},0),(\sqrt{41},-2/\sqrt{41}:21\sqrt{41},-\sqrt{41}),(\sqrt{41},-14/\sqrt{41}:3\sqrt{41},-\sqrt{41}),$ $(\sqrt{41},-7/\sqrt{41}:6\sqrt{41},-\sqrt{41}),(\sqrt{41},-6/\sqrt{41}:7\sqrt{41},-\sqrt{41}),(\sqrt{41},-3/\sqrt{41}:14\sqrt{41},-\sqrt{41}),$ $(9,-2:41,-9),(2\sqrt{41},-33/\sqrt{41}:5\sqrt{41},-2\sqrt{41}),(2\sqrt{41},-5/\sqrt{41}:33\sqrt{41},-2\sqrt{41})$
42	6	$(3\sqrt{3},-2/\sqrt{3}:14\sqrt{3},-3\sqrt{3}),(\sqrt{6},-1/\sqrt{6}:7\sqrt{6},-\sqrt{6}),(\sqrt{14},-5/\sqrt{14}:3\sqrt{14},-\sqrt{14}),$ $(\sqrt{14},-1/\sqrt{14}:15\sqrt{14},-\sqrt{14}),(\sqrt{21},-11/\sqrt{21}:2\sqrt{21},-\sqrt{21}),(0,-1/\sqrt{42}:\sqrt{42},0)$
44	6	$(0,-1/\sqrt{44}:\sqrt{44},0),(\sqrt{11},-3/\sqrt{11}:4\sqrt{11},-\sqrt{11}),(\sqrt{11},-1/\sqrt{11}:12\sqrt{11},-\sqrt{11}),$ $(\sqrt{44},-15/\sqrt{44}:3\sqrt{44},-\sqrt{44}),(\sqrt{44},-5/\sqrt{44}:9\sqrt{44},-\sqrt{44}),(3\sqrt{11},-5/\sqrt{11}:20\sqrt{11},-3\sqrt{11})$
45	6	$(0,-1/\sqrt{45}:\sqrt{45},0),(\sqrt{9},-2/\sqrt{9}:5\sqrt{9},-\sqrt{9}),(\sqrt{9},-1/\sqrt{9}:10\sqrt{9},-\sqrt{9}),$ $(\sqrt{45},-23/\sqrt{45}:2\sqrt{45},-\sqrt{45}),(4\sqrt{5},-9/\sqrt{5}:9\sqrt{5},-4\sqrt{5}),(4\sqrt{5},-3/\sqrt{5}:27\sqrt{5},-4\sqrt{5})$
46	8	$(\sqrt{23},-1/\sqrt{23}:24\sqrt{23},-\sqrt{23}),(\sqrt{23},-6/\sqrt{23}:4\sqrt{23},-\sqrt{23}),(\sqrt{23},-4/\sqrt{23}:6\sqrt{23},-\sqrt{23}),$ $(\sqrt{23},-3/\sqrt{23}:8\sqrt{23},-\sqrt{23}),(\sqrt{23},-2/\sqrt{23}:12\sqrt{23},-\sqrt{23}),(3\sqrt{23},-52/\sqrt{23}:4\sqrt{23},-3\sqrt{23}),$ $(0,-1/\sqrt{46}:\sqrt{46},0),(2\sqrt{46},-5/\sqrt{46}:37\sqrt{46},-2\sqrt{46})$
47	10	$(0,-1/\sqrt{47}:\sqrt{47},0),(\sqrt{47},-x/\sqrt{47}:y\sqrt{47},-\sqrt{47}),2\le y\le 16,xy=48.$ $(2\sqrt{47},-21/\sqrt{47}:9\sqrt{47},-2\sqrt{47}),(2\sqrt{47},-7/\sqrt{47}:27\sqrt{47},-2\sqrt{47})$
49	4	$(0,-1/\sqrt{49}:\sqrt{49},0),(\sqrt{49},-25/\sqrt{49}:2\sqrt{49},-\sqrt{49}),$ $(\sqrt{49},-10/\sqrt{49}:5\sqrt{49},-\sqrt{49}),(\sqrt{49},-5/\sqrt{49}:10\sqrt{49},-\sqrt{49})$
50	4	$(0,-1/\sqrt{50}:\sqrt{50},0),(\sqrt{25},-13/\sqrt{25}:2\sqrt{25},-\sqrt{25}),$ $(\sqrt{50},-17/\sqrt{50}:3\sqrt{50},-\sqrt{50}),(\sqrt{50},-3/\sqrt{50}:17\sqrt{50},-\sqrt{50})$
51	8	$(\sqrt{17},-x/\sqrt{17}:3y\sqrt{17},-\sqrt{17}),x=1,2,3,6,xy=6.$ $(0,-1/\sqrt{51}:\sqrt{51},0),(\sqrt{51},-x/\sqrt{51}:y\sqrt{51},-\sqrt{51}),x=2,4,13,xy=52.$
54	7	$(0,-1/\sqrt{54}:\sqrt{54},0),(\sqrt{27},-14/\sqrt{27}:2\sqrt{27},-\sqrt{27}),(\sqrt{27},-7/\sqrt{27}:4\sqrt{27},-\sqrt{27}),$ $(\sqrt{27},-2/\sqrt{27}:14\sqrt{27},-\sqrt{27}),(\sqrt{54},-11/\sqrt{54}:5\sqrt{54},-\sqrt{54}),(\sqrt{54},-5/\sqrt{54}:11\sqrt{54},-\sqrt{54}),$ $(11\sqrt{2},-9/\sqrt{2}:27\sqrt{2},-11\sqrt{2})$
55	8	$(2\sqrt{11},-x/\sqrt{11}:5y\sqrt{11},-2\sqrt{11}),x=1,2,9,xy=9,(3\sqrt{11},-2/\sqrt{11}:50\sqrt{11},-3\sqrt{11}).$ $(0,-1/\sqrt{55}:\sqrt{55},0),(\sqrt{55},-x/\sqrt{55}:y\sqrt{55},-\sqrt{55}),y=2,4,7,xy=56.$
59	12	$(0,-1/\sqrt{59}:\sqrt{59},0),(\sqrt{59},-x/\sqrt{59}:y\sqrt{59},-\sqrt{59}),2\le x\le 20,xy=60.$ $(3\sqrt{59},-7/\sqrt{59}:76\sqrt{59},-3\sqrt{59}),(3\sqrt{59},-76/\sqrt{59}:7\sqrt{59},-3\sqrt{59}).$

TABLE 6. Continued.

60	6	$(0,-1/\sqrt{60}:\sqrt{60},0),(\sqrt{15},-4/\sqrt{15}:4\sqrt{15},-\sqrt{15}),(\sqrt{15},-2/\sqrt{15}:8\sqrt{15},-\sqrt{15}),$ $(\sqrt{15},-1/\sqrt{15}:16\sqrt{15},-\sqrt{15},(\sqrt{20},-7/\sqrt{20}:3\sqrt{20},-\sqrt{20}),(\sqrt{20},-1/\sqrt{20}:21\sqrt{20},-\sqrt{20})$
62	10	$(\sqrt{31},-x/\sqrt{31}:2y\sqrt{31},-\sqrt{31}),x=1,2,4,8,xy=16.$ $(3\sqrt{31},-4/\sqrt{31}:70\sqrt{31},-3\sqrt{31}),(3\sqrt{31},-20/\sqrt{31}:14\sqrt{31},-3\sqrt{31}),$ $(0,-1/\sqrt{62}:\sqrt{62},0),(\sqrt{62},-x/\sqrt{62}:y\sqrt{62},-\sqrt{62}),x=3,7,21,xy=63.$
66	8	$(4\sqrt{2},-1/\sqrt{2}:33\sqrt{2},-4\sqrt{2}),(3\sqrt{6},-5/\sqrt{6}:11\sqrt{6},-3\sqrt{6}),(\sqrt{33},-17/\sqrt{33}:2\sqrt{33},-\sqrt{33}),$ $(0,-1/\sqrt{66}:\sqrt{66},0),(2\sqrt{66},-5/\sqrt{66}:53\sqrt{66},-2\sqrt{66}),(\sqrt{11},-2/\sqrt{11}:6\sqrt{11},-\sqrt{11}),$ $(\sqrt{11},-1/\sqrt{11}:12\sqrt{11},-\sqrt{11}),(5\sqrt{11},-2/\sqrt{11}:23.6\sqrt{11},-5\sqrt{11})$
69	10	$(\sqrt{23},-x/\sqrt{23}:3y\sqrt{23},-\sqrt{23}),x=1,2,4,8,xy=8.$ $(4\sqrt{23},-3/\sqrt{23}:123\sqrt{23},-4\sqrt{23}),(5\sqrt{23},-4/\sqrt{23}:144\sqrt{23},-5\sqrt{23})$ $(0,-1/\sqrt{69}:\sqrt{69},0),(\sqrt{69},-x/\sqrt{69}:y\sqrt{69},-\sqrt{69}),x=2,5,10,xy=70.$
70	8	$(5\sqrt{5},-9/\sqrt{5}:14\sqrt{5},-5\sqrt{5}),(3\sqrt{10},-13/\sqrt{10}:7\sqrt{10},-3\sqrt{10}),(\sqrt{14},-3/\sqrt{14}:5\sqrt{14},-\sqrt{14}),$ $(\sqrt{14},-1/\sqrt{14}:15\sqrt{14},-\sqrt{14}),(\sqrt{35},-1/\sqrt{35}:36\sqrt{35},-\sqrt{35}),(\sqrt{35},-2/\sqrt{35}:18\sqrt{35},-\sqrt{35}),$ $(\sqrt{35},-6/\sqrt{35}:6\sqrt{35},-\sqrt{35}),(0,-1/\sqrt{70}:\sqrt{70},0)$
71	14	$(0,-1/\sqrt{71}:\sqrt{71},0),(\sqrt{71},-x/\sqrt{71}:y\sqrt{71},-\sqrt{71}),2\le y\le 24,xy=72.$ $(2\sqrt{71},-x/\sqrt{71}:y\sqrt{71},-2\sqrt{71}),x=5,57,xy=285.$ $(3\sqrt{71},-x/\sqrt{71}:y\sqrt{71},-3\sqrt{71}),x=8,64,xy=640.$
78	9	$(11\sqrt{3},-1/\sqrt{26}:14.29\sqrt{3},-11\sqrt{3}),(\sqrt{26},-x/\sqrt{26}:3y\sqrt{26},-\sqrt{26}),x=1,3,9,xy=9.$ $(\sqrt{39},-x/\sqrt{39}:2y\sqrt{39},-\sqrt{39}),x=1,2,4,10,xy=20.$ $(0,-1/\sqrt{78}:\sqrt{78},0)$
87	12	$(\sqrt{29},-x/\sqrt{29}:3y\sqrt{29},-\sqrt{29}),x=1,2,5,10,xy=10.$ $(2\sqrt{29},-13/\sqrt{29}:9\sqrt{29},-2\sqrt{29}),(2\sqrt{29},-3/\sqrt{29}:39\sqrt{29},-2\sqrt{29}),$ $(0,-1/\sqrt{87}:\sqrt{87},0),(\sqrt{87},-x/\sqrt{87}:y\sqrt{87},-\sqrt{87}),y=2,4,8,11,22,xy=88.$
92	12	$(0,-1/\sqrt{92}:\sqrt{92},0),(\sqrt{23},-6/\sqrt{23}:4\sqrt{23},-\sqrt{23}),(\sqrt{23},-3/\sqrt{23}:8\sqrt{23},-\sqrt{23}),$ $(\sqrt{23},-2/\sqrt{23}:12\sqrt{23},-\sqrt{23}),(\sqrt{23},-1/\sqrt{23}:24\sqrt{23},-\sqrt{23}),(\sqrt{92},-31/\sqrt{92}:3\sqrt{92},-\sqrt{92}]),$ $(\sqrt{92},-3/\sqrt{92}:31\sqrt{92},-\sqrt{92}),(3\sqrt{23},-26/\sqrt{23}:8\sqrt{23},-3\sqrt{23}),(3\sqrt{23},-13/\sqrt{23}:16\sqrt{23},-3\sqrt{23}),$ $(3\sqrt{23},-4/\sqrt{23}:52\sqrt{23},-3\sqrt{23}),(3\sqrt{23},-2/\sqrt{23}:104\sqrt{23},-3\sqrt{23}),(5\sqrt{23},-18/\sqrt{23}:32\sqrt{23},-5\sqrt{23})$
94	14	$(\sqrt{47},-x/\sqrt{47}:2y\sqrt{47},-\sqrt{47}),x=1,2,3,4,6,8,12,xy=24.$ $(5\sqrt{47},-6/\sqrt{47}:196\sqrt{47},-5\sqrt{47}),(3\sqrt{47},-x/\sqrt{47}:2y\sqrt{47},-3\sqrt{47}),x=4,53,xy=212.$ $(0,-1/\sqrt{94}:\sqrt{94},0),(\sqrt{94},-x/\sqrt{94}:y\sqrt{94},-\sqrt{94}),x=5,19,xy=95.$ $(3\sqrt{94},-7/\sqrt{94}:121\sqrt{94},-3\sqrt{94})$
95	12	$(9\sqrt{19},-4/\sqrt{19}:385\sqrt{19},-9\sqrt{19}),(\sqrt{19},-x/\sqrt{19}:5y\sqrt{19},-\sqrt{19}),x=1,2,4,xy=4.$ $(0,-1/\sqrt{95}:\sqrt{95},0),(4\sqrt{95},-9/\sqrt{95}:169\sqrt{95},-4\sqrt{95}),$ $(\sqrt{95},-x/\sqrt{95}:y\sqrt{95},-\sqrt{95}),x=2,3,4,8,16,32,xy=96.$
105	10	$(2\sqrt{5},-1/\sqrt{5}:21\sqrt{5},-2\sqrt{5}),(5\sqrt{5},-1/\sqrt{5}:121\sqrt{5},-5\sqrt{5}),$ $(2\sqrt{21},-17/\sqrt{21}:5\sqrt{21},-2\sqrt{21}),(2\sqrt{21},-1/\sqrt{21}:85\sqrt{21},-2\sqrt{21}),$ $(0,-1/\sqrt{105}:\sqrt{105},0),(\sqrt{105},-53/\sqrt{105}:2\sqrt{105},-\sqrt{105}),$ $(\sqrt{35},-x/\sqrt{35}:3y\sqrt{35},-\sqrt{35}),x=1,2,3,12,xy=12.$
110	11	$(\sqrt{10},-1/\sqrt{10}:11\sqrt{10},-\sqrt{10}),(3\sqrt{11},-x/\sqrt{11}:10y\sqrt{11},-3\sqrt{11}),x=1,2,10,xy=10.$ $(\sqrt{55},-x/\sqrt{55}:2y\sqrt{55},-\sqrt{55}),x=1,2,4,14,xy=28.$ $(0,-1/\sqrt{110}:\sqrt{110},0),(\sqrt{110},-x/\sqrt{110}:y\sqrt{110},-\sqrt{110}),x=3,37,xy=111.$
119	14	$(3\sqrt{17},-x/\sqrt{17}:7y\sqrt{17},-3\sqrt{17}),x=1,2,11,22,xy=22.\ (0,-1/\sqrt{119}:\sqrt{119},0),$ $(\sqrt{119},-x/\sqrt{119}:y\sqrt{119},-\sqrt{119}),x=2,3,4,5,10,12,15,20,24,xy=120.$

TABLE 6. Continued.

Example 6.3. The set of cusps of $\Gamma_0(144)$ is

$$\Delta = \{0, 1/2, 1/3, 2/3, 1/4, 3/4, 1/6, 5/6, 1/8, 1/9, 1/12, 5/12, 7/12, 11/12, 1/16, 1/18, 1/24, 5/24, 1/36, 7/36, 1/48, 5/48, 1/72, \infty\}.$$

Let

$$\tau = \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} \in \Gamma_0(n).$$

Then $\tau(5/48)$ is equivalent to $7/36$.

Proof: Direct calculation shows that $\tau(5/48) = 5/36.3$. By Lemma 6.2, $5/48$ is equivalent to $(5.3 + 4k)/36$ for some k. Since $15 \equiv 7 \pmod 4$ and $\gcd(36, 7) = 1$, we conclude that $\tau(5/48)$ is equivalent to $7/36$. □

Remark 6.4. Note that since $15 \equiv 11 \pmod 4$ and $\gcd(36, 11) = 1$, we have $\tau(5/48)$ is equivalent to $11/36$ as well.

The following result is taken from [Shimura 71].

Lemma 6.5. *Let a/b, $c/d \in \mathbb{Q} \cup \{\infty\}$. Suppose that $gcd(a, b) = gcd(c, d) = 1$. Then a/b and c/d are equivalent to each other in $\Gamma(n)$ if and only if*

$(*)$ $a \equiv c$, $b \equiv d \pmod n$, *or* $a \equiv -c$, $b \equiv -d \pmod n$.

Further, $\Gamma(n)$ has $[PSL_2(\mathbb{Z}) : \Gamma(n)]/n$ cusps.

Applying Lemma 6.5, one may write down a set of representatives of inequivalent cusps as follows.

1. **Cusps of $\boldsymbol{\Gamma(2m)}$.** For each k $(1 \le k \le m - 1)$, define A_k to be the set $A_k = \{\hat{x} : 1 \le \hat{x} \le 2m, \gcd(\gcd(k, \hat{x}), 2m)) = 1\}$. For each $\hat{x} \in A_k$, let $x = \hat{x} + 2my$ be the smallest positive integer such that $\gcd(x, k) = 1$. Define S_k to be the set $S_k = \{x/k : \hat{x} \in A_k\}$. Let

   $$A_m = \{\hat{x} : 1 \le \hat{x} \le m, \gcd(\gcd(m, \hat{x}), 2m)) = 1\},$$

   $$A_{2m} = \{\hat{x} : 1 \le \hat{x} \le m, \gcd(\gcd(2m, \hat{x}), 2m)) = 1\}.$$

   Define

   (i) $S_m = \{x/m : \hat{x} \in A_m\}$, where $x = \hat{x} + 2my$ is the smallest positive integer such that $\gcd(x, m) = 1$,

   (ii) $S_{2m} = \{x/2m : \hat{x} \in A_{2m}\}$, where $x = \hat{x} + 2my$ is the smallest positive integer such that $\gcd(x, 2m) = 1$.

Corollary 6.6. *$S_1 \cup S_2 \cup \cdots \cup S_{m-1} \cup S_m \cup S_{2m}$ is a set of cusps of $\Gamma(2m)$.*

Remark 6.7. In $\Gamma(2m)$, $1/2m$ is equivalent to ∞ and 0 is equivalent to $2m$.

Example 6.8. A set of inequivalent cusps of $\Gamma(6)$ is given by

$$\{\infty = 1/0, 0, 1/1, 2/1, 3/1, 4/1, 5/1, 1/2, 3/2, 5/2, 1/3, 2/3\}.$$

2. **Cusps of $\boldsymbol{\Gamma(2m+1)}$.** For each k $(1 \le k \le m)$, define A_k to be the set $A_k = \{\hat{x} : 1 \le \hat{x} \le 2m + 1, \gcd(\gcd(k, \hat{x}), 2m+1)) = 1\}$. For each $\hat{x} \in A_k$, let $x = \hat{x} + (2m + 1)y$ be the smallest positive integer such that $\gcd(x, k) = 1$. Define S_k to be the set $S_k = \{x/k : \hat{x} \in A_k\}$. Let

   $$A_{2m+1} = \{\hat{x} : 1 \le \hat{x} \le m, \gcd(\gcd(k, \hat{x}), 2m+1)) = 1\}.$$

   Define $S_{2m+1} = \{x/(2m + 1) : \hat{x} \in A_{2m+1}\}$, where $x = \hat{x} + (2m + 1)y$ is the smallest positive integer such that $\gcd(x, 2m + 1) = 1$.

Corollary 6.9. *$S_1 \cup S_2 \cup \cdots \cup S_m \cup S_{2m+1}$ is a set of cusps of $\Gamma(2m + 1)$.*

Remark 6.10. In $\Gamma(2m+1)$, $1/(2m+1)$ is equivalent to ∞ and 0 is equivalent to $2m + 1$.

Example 6.11.

(i) A set of inequivalent cusps of $\Gamma(5)$ is given by

$$\{\infty, 0, 1/1, 2/1, 3/1, 4/1, 1/2, 3/2, 5/2, 7/2, 9/2, 2/5\}.$$

(ii) A set of inequivalent cusps of $\Gamma(7)$ is given by $\{\infty, 0, 1/1, 2/1, 3/1, 4/1, 5/1, 6/1, 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2, 1/3, 2/3, 10/3, 4/3, 5/3, 13/3, 7/3, 2/7, 3/7\}$.

Remark 6.12. By Lemma 6.5, the permutation representation of $PSL_2(\mathbb{Z})$ on the set of cusps of $\Gamma(n)$ can be determined easily. Recall that

$$PSL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

7. PROOF

The main purpose of this section is to give a proof of (i) of Section 2.2. Our proof can be found in Section 7.2.

7.1 Some Basic Results

Let M be a maximal discrete subgroup of $PSL_2(\mathbb{R})$ commensurable with $PSL_2(\mathbb{Z})$. By results of Helling [Helling 66], M is a conjugate of $\Gamma_0^+(f)$, where f is square-free.

Definition 7.1. Let ν be an element of order 6 of M and let G be a subgroup of M. Suppose that $M = \cup_{i=1}^n x_i G$. We define the following:

(i) $\Delta_\nu(G) = \{x_i : x_i^{-1}\nu x_i \in G\}$,

(ii) for each $x \in \Delta_{\nu^2}(G) - \Delta_\nu(G)$, define $E_x = \{x_i \in \Delta_{\nu^2}(G) - \Delta_\nu(G) : x_i^{-1}\nu^2 x_i$ and $x^{-1}\nu^2 x$ are conjugate to each other in $G\}$,

(iii) for each $x \in \Delta_{\nu^3}(G) - \Delta_\nu(G)$, define $F_x = \{x_i \in \Delta_{\nu^3}(G) - \Delta_\nu(G) : x_i^{-1}\nu^3 x_i$ and $x^{-1}\nu^3 x$ are conjugate to each other in $G\}$.

Remark 7.2. Note that $x^{-1}\nu^2 x$ ($x \in \Delta_{\nu^2}(G) - \Delta_\nu(G)$) is an elliptic element of order 3 of G (see [Shimura 71, 1.15 and 1.16]). Similarly, $x^{-1}\nu^3 x$ ($x \in \Delta_{\nu^3}(G) - \Delta_\nu(G)$) is an elliptic element of order 2 of G.

The following lemma is taken from Section 4 of [Lang 01]. For readers' convenience, the proof is included.

Lemma 7.3. *Let ν be an elliptic element of order 6 of $M = \cup_{i=1}^n x_i G$. Then*

(i) for any $x, y \in \Delta_\nu(G)$, $x^{-1}\nu x$ and $y^{-1}\nu y$ are not conjugate to each other in G if and only if $x \neq y$,

(ii) $|E_x| = 2$ for every $x \in \Delta_\nu(G)$,

(iii) $|F_x| = 3$ for every $x \in \Delta_\nu(G)$.

Proof: (i) Suppose that $o(\nu) = 6$ and that $\tau_1 = x_1^{-1}\nu x_1$, $\tau_2 = x_2^{-1}\nu x_2 \in G$ are conjugate to each other in G. It follows that there exists some $z \in G$ such that $z\tau_1 z^{-1} = \tau_2$. This implies that $x_2 z x_1^{-1}$ commutes with ν. By a standard result of discrete groups (see 1.15 and 1.16 of [Sh]), $x_2 z x_1^{-1} = \nu^k$ for some k. Hence $z = x_2^{-1}\nu^k x_1 = x_2^{-1}x_1 x_1^{-1}\nu^k x_1$. Since $z, x_1^{-1}\nu^k x_1 \in G$, we conclude that $x_2^{-1}x_1 \in G$. This implies that $x_1 = x_2$. The rest follows similarly.

(iia) For each $x \in \Delta_{\nu^2}(G) - \Delta_\nu(G)$, there exists a unique x_j such that $\nu x G = x_j G$. This implies that $x_j = \nu x z$ for some $z \in G$. Consequently,

(i) $x_j^{-1}\nu^2 x_j \in G$,

(ii) $x_j^{-1}\nu x_j \notin G$ ($x \notin \Delta_\nu(G)$, $x^{-1}\nu x \notin G$),

(iii) $x_j^{-1}\nu^2 x_j$ and $x^{-1}\nu^2 x$ are conjugate to each other in G.

This implies that $x_j \in E_x$. Since

(i) $(x_j^{-1}x)x^{-1}\nu x = x_j^{-1}\nu x = z^{-1} \in G$,

(ii) $x^{-1}\nu x \notin G$,

we conclude that $x_j \neq x$. Hence $|E_x| \geq 2$.
(iib) Suppose that $x_j \in E_x - \{x\}$. It follows that $x^{-1}\nu^2 x = z^{-1}x_j^{-1}\nu^2 x_j z$ for some z in G. Hence $x_j z x^{-1}$ commutes with ν^2. Consequently, $x_j z x^{-1} = \nu^k$, $0 \leq k \leq 5$ (see [Shimura 71, 1.15 and 1.16]). This gives

$$z = x_j^{-1}\nu^k x = (x_j^{-1}x)(x^{-1}\nu^k x) \in G\,.$$

Since

$$x_j^{-1}x \notin G\ ,\ x^{-1}\nu^2 x \in G\,,$$

k cannot be 0, 2, or 4. It follows that $x_j G = x_j z G = \nu^k x G$, $k = 1$ or 5. Since $x^{-1}\nu^2 x \in G$, we have

$$x_j G = \nu^k x G = \nu x G\,.$$

It is clear that the choice for such x_j is unique. As a consequence, $|E_x| \leq 2$. This completes the proof of (ii). (iii) can be proved similarly. □

Proof of (ii) of Section 2.2:

(a) $v_6(G) = r$. Suppose that $\langle g_i^{-1}ug_i\rangle$, $\langle g_j^{-1}ug_j\rangle \in G$ are conjugate to each other in G. Then there exists some $z \in G$ such that $g_i^{-1}ug_i = z^{-1}g_j^{-1}u^k g_j z$, where $k = 1$ or -1. It follows that

$$g_j z g_i^{-1} u g_i z^{-1} g_j^{-1} = u^k.$$

Since u and u^{-1} are not conjugate to each other (see [Shimura 71, 1.22]), we conclude that

$$g_j z g_i^{-1} u g_i z^{-1} g_j^{-1} = u.$$

It follows that $g_i^{-1}ug_i$ and $g_j^{-1}ug_j$ are conjugate to each other in G. Applying (i) of Lemma 7.3, $i = j$. Hence $v_6(G) = r$.

(b) $v_3(G) = (k-r)/2$. Note first that $\langle g_i^{-1}u^2g_i\rangle$ is an elliptic subgroup of order 3 of G if and only if $g_i \in \Delta_{u^2}(G) - \Delta_u(G)$. Suppose that $\langle g_i^{-1}u^2g_i\rangle$ and $\langle g_j^{-1}u^2g_j\rangle$ are conjugate to each other in G. Similar to the proof of (a), we conclude that $g_i^{-1}u^2g_i$ and $g_j^{-1}u^2g_j$ are conjugate to each other in G. As a consequence, $g_j \in E_{g_i}$. Applying (ii) of Lemma 7.3, we have $v_3(G) = (k-r)/2$.

□

Similar to the above, we may show that $v_2(G) = e_1 + e_2 + \cdots + e_s + (e-r)/3$.

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