On the Number of Perfect Binary Quadratic Forms

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CONTENTS

A *perfect form* is a form $f = mx^2 + ny^2 + kxy$ with integral coefficients (m, n, k) such that $f(\mathbb{Z}^2)$ is a multiplicative semigroup. The growth rate of the number of perfect forms in cubes of increasing side L in the space of the coefficients is known for small cubes, where all perfect forms are known. A form is perfect if its coefficients belong to the image of a map, Q, from $\mathbb{Z}^4$ to $\mathbb{Z}^3$. This property of perfect forms allows us to estimate from below the growth rate of their number for larger values of L. The conjecture that all perfect forms are generated by Q allows us to reformulate results and conjectures on the numbers of the images $Q(\mathbb{Z}^4)$ in cubes of side L in terms of the numbers of perfect forms. In particular, the proportion of perfect elliptic forms in a ball of radius R should decrease faster than $R^{-3/4}$ and the proportion of all perfect forms in a ball of radius R should decrease faster than $2/\sqrt{R}$.

1. INTRODUCTION

Arnold found some remarkable properties of binary quadratic forms with integer coefficients [Arnold 03],

$$f = mx^2 + ny^2 + kxy, \tag{1–1}$$

where (m, n, k) are three fixed integers and (x, y) varies over the integer points plane lattice.

In general, the set of values of a binary quadratic form on integers is a multiplicative trigroup (the product of three values attained by f is still a value attained by f). Sometimes, i.e., for special triples (m, n, k), the set of values form a semigroup. Arnold called such triples *perfect* (the corresponding forms are also called perfect). This is the case, for example, for the Gauss form $x^2 + y^2$, whose semigroup property follows from the multiplication rule of the complex numbers $z = x + iy$.

One of the problems singled out in [Arnold 03] is to estimate the number of perfect forms, i.e., their proportion with respect to the total number of forms inside a ball in the space (m, n, k) with increasing radius.

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2. THE EMPIRICAL DATA

A general criterion for establishing whether a form is perfect is still unknown. However, several theorems in [Arnold 03] offer criteria for doing this in many cases, especially when $k = 0$. Using these theorems, Arnold proved that in the square $[k = 0, -16 \le m \le 16, -16 \le n \le 16]$ 266 of the 1,089 triples are perfect ([Arnold 03]). Using Arnold's results and the $\mathrm{SL}(2,\mathbb{Z})$ invariance (the set of values attained by a form is an invariant of its orbit under the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$), we found that in the cube $[-5 \le m \le 5, -5 \le n \le 5, -5 \le k \le -5]$, 691 of the 1,331 triples are perfect.

We denote the cube of integer side $L = 2i + 1$ by

$$C_i \equiv \{(m,n,k) \in \mathbb{Z}^3 : |m| \le i, |n| \le i, |k| \le i\}.$$

The number of perfect forms and the number of those forms that are elliptic and hyperbolic, in cubes C_i for i up to 5 are shown in Table 1. (The corresponding perfect triples are listed in [Aicardi 03] and in Appendix 1 at http://www.expmath.org/expmath/volumes/13/13.4/Aicardi/appendices.pdf/.)

i	1	2	3	4	5
# of perfect forms	22	81	195	408	691
# of elliptic forms	3	14	36	81	126
# of hyperbolic forms	16	62	154	316	554

TABLE 1. Forms inside C_i.

These sequences of numbers of perfect forms, as functions of side L of the cube, are plotted in bilogarithmic scale in Figure 1. They grow more slowly than the third power of L (the dotted line corresponds to a power law with exponent equal to 3). The straight lines fitting the data, obtained by the least square regression method, are shown as solid lines. The values of the exponents of the corresponding empirical laws are $\alpha = 2.7$ (for the total number), $\alpha_E = 2.9$ (for the elliptic forms), and $\alpha_H = 2.7$ (for the hyperbolic forms).

2.1 Case $k = 0$

Using the table, in [Arnold 03], of all the perfect triples of type $(m, n, 0)$ in the square $|m| \le 16, |n| \le 16$, we considered the sequences of numbers of perfect forms in squares of increasing side:

$$Sq_i \equiv \{(m,n) \in \mathbb{Z}^2 : |m| \le i, |n| \le i\}.$$

These numbers are plotted in bilogarithmic scale in Figure 2 versus the square's side $L = 2i+1$, for $i = 4, \ldots, 16$.

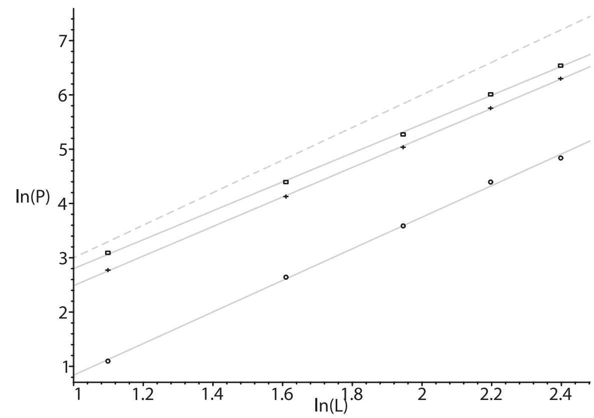

FIGURE 1. P denotes the number of perfect forms in a cube of side L. From bottom to top: elliptic forms, hyperbolic forms, all forms; the dotted line corresponds to a power law with exponent equal to 3.

First of all, we observe that in this case the data fitting straight lines for the numbers of elliptic and of hyperbolic forms are evidently nonparallel to each other. The empirical exponents are $\alpha_\square = 1.5$ for all forms, $\alpha_{\square E} = 1.4$ for the elliptic forms, and $\alpha_{\square H} = 1.65$ for the hyperbolic forms. Note that all these exponents are less than 2.

Definition 2.1. Let $X_i \subset \mathbb{Z}^3$ be the intersection of the cone $4mn - k^2 = 0$ with C_i. Let E_i^+ be the region in the cube C_i inside the upper cone $\{(m,n,k) \in C_i : 4mn - k^2 > 0, \;\; m > 0, n > 0\}$. Let H_i be the hyperbolic region $\{(m,n,k) \in C_i : 4mn - k^2 < 0\}$ inside the same cube. Let A_i be the union $E_i^+ \cup H_i \cup X_i$, i.e., the subset of C_i where perfect triples lie.

Definition 2.2. Let U be a finite subset of $\mathbb{Z}^3$. We denote by $Vol(U)$ the number of points in U.

Remark 2.3. The proportions of all elliptic positive forms and of all hyperbolic forms with respect to the total number of forms in C_i satisfy

$$Vol(E_i^+) \approx 18.5\% Vol(C_i); \qquad Vol(H_i) \approx 63\% Vol(C_i),$$

whereas for the squares Sq_i, one has

$$Vol(E_i^+ \cap Sq_i) \approx 25\% Vol(Sq_i);$$
$$Vol(H_i \cap Sq_i) \approx 50\% Vol(Sq_i).$$

In order to compare the empirical data concerning the cubes C_i and the squares Sq_i we introduce the relative proportion of perfect forms (P. F.) in a subset $U \in \mathbb{Z}^3$

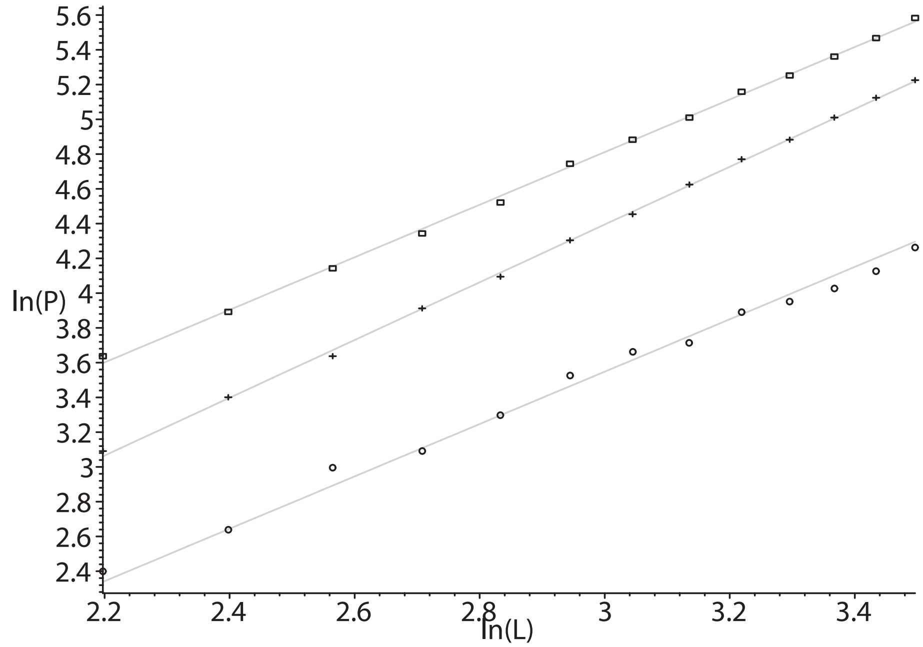

FIGURE 2. P denotes the number of perfect forms in a square of side L. Circles are used for elliptic forms, crosses for hyperbolic forms, and squares for all other forms.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P_{E\square}$	1.	1.	.67	.69	.56	.56	.45	.42	.42	.39	.34	.34	.31	.29	.28	.28
$P_{H\square}$	1.	.75	.67	.69	.60	.53	.51	.47	.46	.43	.42	.41	.39	.38	.37	.36
$P_{A\square}$	.75	.62	.53	.58	.51	.47	.44	.41	.41	.39	.37	.36	.34	.33	.32	.32

TABLE 2.

with respect to the number of points in U:

$$P_U = \frac{\#(P.F. \in U)}{Vol(U)}.$$

So, we denote, for instance:

$$P_E(i) = \frac{\#(P.F. \in E_i^+)}{Vol(E_i^+)},$$

$$P_{E\square}(i) = \frac{\#(P.F. \in Sq_i \cap E_i^+)}{Vol(Sq_i \cap E_i^+)}.$$

Similar definitions hold for the relative proportions of hyperbolic (P_H) and of all forms (P_A).

Table 2 shows the relative proportions of perfect forms in squares Sq_i (for i up to 16) and Table 3 shows the relative proportions of perfect forms in C_i (for i up to 5).

Comparing these tables, we see that the relative proportions of elliptic and of hyperbolic perfect forms are similar in cubes C_i and in squares Sq_i ($i \leq 5$). Hence the section $k = 0$ seems to be a nonspecial section with respect to the statistics of perfect forms. (The fact that, for small i, the relative proportions of all perfect forms in the squares are lower than in the cubes is due to the fact that the contribution to the $Vol(A_i)$ of the parabolic region X_i is higher in square Sq_i than in cube C_i.)

i	1	2	3	4	5
P_E	1.	.78	.65	.68	.56
P_H	1.	.82	.71	.69	.64
P_A	.92	.76	.68	.67	.63

TABLE 3.

3. A "PERFECT" MAPPING

In [Aicardi 03], I have defined a map Q from $\mathbb{Z}^4$ to $\mathbb{Z}^3$, providing, for any quadruple (a, b, c, d), a perfect triple (m, n, k):

$$Q(a, b, c, d) \rightarrow ((a^2 - dc), (c^2 - ab), (ac - bd)).$$

Conjecture 3.1. *A triple (m, n, k) is perfect only if it belongs to the image under Q of $\mathbb{Z}^4$.*

Remark 3.2. In the case $k = 0$, the map Q allows us to find all the perfect forms in the series found by Arnold (see Appendix 3 at http://www.expmath.org/expmath/volumes/13/13.4/Aicardi/appendices.pdf/). Moreover, we found quadruples a, b, c, d for all the perfect forms in the above mentioned cube (see Appendix 2 at http://www.expmath.org/expmath/volumes/13/13.4/Aicardi/appendices.pdf/).

If it is true that all the perfect forms are generated by quadruples (a, b, c, d), the problem of counting the perfect forms inside a region of the space (m, n, k) is reduced to studying the image of the mapping Q inside that region.

Remark 3.3. The map Q evidently possesses the symmetry $Q(a, b, c, d) = Q(-a, -b, -c, -d)$.

Suppose that a, b, c, d vary inside a volume $V = (|a| \leq A,\ |b| \leq B,\ |c| \leq C,\ |d| \leq D)$. Then the image of V is contained in the region

$$
\begin{aligned}
&- DC \leq m \leq A^2 + DC,\\
&- AB \leq n \leq C^2 + AB,\\
&- (AC + BD) \leq k \leq (AC + BD).
\end{aligned}
$$

In particular, if V is a cube centered at the origin of the space (a, b, c, d) with side $2L + 1$, then

$$|m| \leq 2L^2, \quad |n| \leq 2L^2, \quad |k| \leq 2L^2.$$

This means that no triples outside a cube of side of order ℓ can be attained by quadruples in a cube with side of order less than $\sqrt{\ell/2}$.

Definition 3.4. We denote by K_i the cube, in the space (a, b, c, d),

$$K_i \equiv \{(a, b, c, d) \in \mathbb{Z}^4 : |a| \leq i, |b| \leq i, |c| \leq i, |d| \leq i\}.$$

Our approach to the problem of counting perfect forms consists of the following steps:

1. Estimate the upper and lower bounds of volume V in the space (a, b, c, d) containing all the preimages under Q of the points in a volume U of the space (m, n, k);

2. Estimate from above the volume V in the space (a, b, c, d), containing at least one preimage of each point, attained by Q, in a volume U of the space (m, n, k).

Step 2 is particularly difficult. To find quadruples (a, b, c, d) for all the perfect forms whose perfectness we already knew in cube C_5, we had to search for values of the parameter b, as far as 244 (see Appendix 4 in [Aicardi 03] and Appendix 2 in http://www.expmath.org/expmath/volumes/13/13.4/Aicardi/appendices.pdf/).

We remarked that, on the contrary, all the positive elliptic forms, as well as the parabolic ones, are attained from points in the cube K_5. We were unable to prove this without the knowledge of the perfect forms list. However, it is generally not true that all elliptic perfect triples in cube C_i are generated by quadruples in cube K_i.

Notation For any subset $V \subset \mathbb{Z}^4$ we denote by $Q_i(V)$ the subset of C_i $Q(V) \cap C_i$, by $Q_i^H(V)$ the subset $Q(V) \cap H_i$, by $Q_i^E(V)$ the subset $Q(V) \cap E_i^+$ (there are no images under Q in E_i^-), and by $Q_i^X(V)$ the subset $Q(V) \cap X_i$.

Numerical computations show that, for i up to 30, the number of different images $Q_i^E(K_j)$ grows for $j \leq i$, and afterwards increases (slowly) while j is growing to a value $\bar{j} < 4i$ at which point it stops growing (I verified its constancy up to $j = 200$). The graph of $\bar{j}(i)$ is shown in Figure 3. The likely numbers of all images of $Q(\mathbb{Z}^4)$ in E_i^+, for i up to 30, are shown in Table 4. These numbers, answering the problem posed in Step 2 in the elliptical case, would also give the proportions of positive perfect forms in E_i^+, provided that Conjecture 3.1 is true. They give, however, a lower estimate of them, independently of the conjecture validity.

Remark 3.5. In Table 4 we denote by $Pc_E(i)$ the proportion of the number of images of $Q(K_{200})$ in E_i^+ with

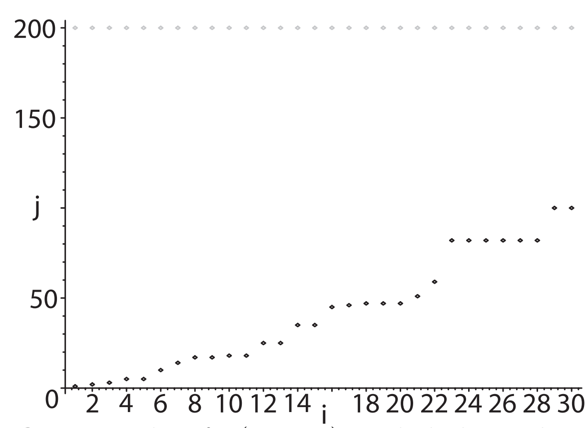

FIGURE 3. Value of j ($j \leq 200$) at which the number of images $Q(K_j)$ in E_i^+ stops growing, for $i \leq 30$.

i	1	2	3	4	5	6	7	8	9	10
$\#(Q_i^E(K_{200}))$	3	14	36	81	126	188	266	337	484	615
Pc_E	100	77.8	65.5	67.5	55.5	49.2	44.7	38.6	39.4	36.9
i	11	12	13	14	15	16	17	18	19	20
$\#(Q_i^E(K_{200}))$	723	871	1054	1202	1368	1703	1896	2099	2377	2628
Pc_E	32.8	30.8	29.4	27.0	25.2	25.9	24.1	22.6	21.8	20.7
i	21	22	23	24	25	26	27	28	29	30
$\#(Q_i^E(K_{200}))$	2944	3242	3454	3810	4395	4672	4950	5430	5827	6237
Pc_E	20.1	19.3	18.0	17.6	18.0	17.6	16.1	15.9	15.3	14.9

TABLE 4.

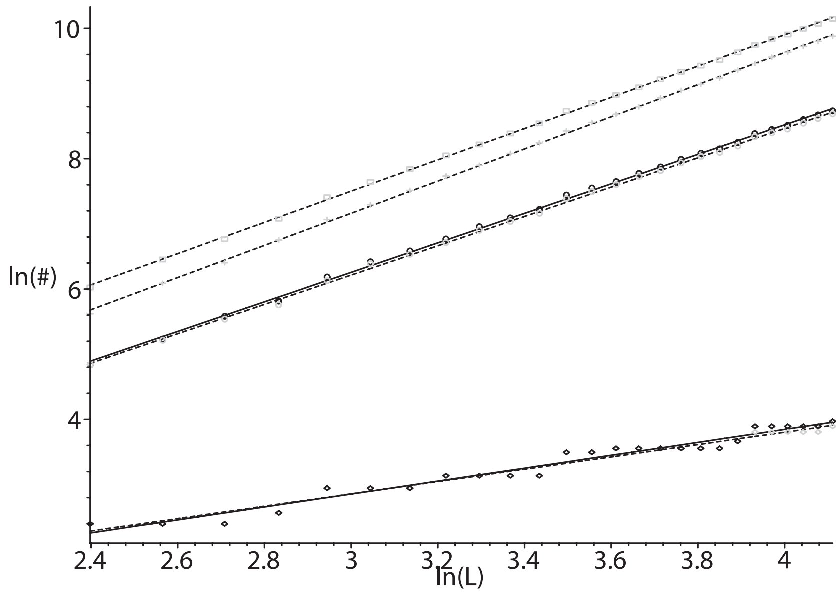

FIGURE 4. Gray colour: bilogarithmic plots of the number (#) of images of $Q(K_i)$ in C_i versus $L = 2i + 1$; from top to bottom: all, hyperbolic, elliptic, parabolic. Black symbols near the gray symbols in the elliptic and parabolic cases indicate the conjectured numbers of perfect forms. Dotted lines: best fits.

respect to the volume of E_i^+. (Pc means "conjectured proportion" of perfect forms). Note that these proportions are similar to those of Table 3, for $i \leq 16$.

The numbers of Table 4 (for $5 \leq i \leq 30$) are shown by black circles in Figure 4 using a bilogarithmic scale. The empirical law obtained for these numbers is

$$\#(Q_i^E(K_{200})) \approx \frac{3}{5} L^{\frac{9}{4}}.$$

Conjecture 3.6. *The asymptotic growth rate of the number of elliptic perfect forms in a cube of side L is no greater than $\frac{9}{4}$.*

This conjecture follows from Conjecture 3.1 and from the observation that the exponent of L of the empirical law decreases slightly for the increasing intervals of L considered.

Remark 3.7. Note that the proportion of the perfect elliptic forms with respect to the entire volume of C_i should decrease from the 11% in cube C_5 to the 2.7% in cube C_{30}.

3.1 Parabolic Forms

For the parabolic forms, it happens that the number of images $Q_i^X(K_j)$, when $i < 25$, is growing only for $j <= i$; afterwards it is constant (We verified this up to $j = 200$). For $25 \leq i \leq 30$, the number of images $Q_i^X(K_j)$ increases by 4 at $j = 50$, and later it is constant.

The likely numbers of parabolic triples attained by $Q(\mathbb{Z}^4)$ are shown in Table 5. The empirical law obtained

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\#(Q_i^X(K_{200}))$	3	5	5	11	11	11	11	13	19	19	19	23	23	23	23
i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\#(Q_i^X(K_{200}))$	33	33	35	35	35	35	35	35	39	49	49	49	49	49	53

TABLE 5.

for these numbers versus $L = 2i + 1$ is

$$\#(Q_i^X(K_{200})) \approx .9L.$$

3.2 Conjectured Upper Bounds for Perfect Forms

In Figure 4, gray circles, very close to the black ones, show the numbers of images $Q_i^E(K_i)$. In fact, the number of the images in E_i^+ not reached by $Q(K_i)$ compared to the number of the images $Q(K_i)$ is approximately constant and equal to the 5%.

In Figure 4, gray squares (upper line) and gray crosses show the numbers of images of K_i, respectively inside C_i and H_i.

Unfortunately, the computational time limits do not allow us to count all the images of $Q(\mathbb{Z}^4)$ inside H_i (and hence inside C_i) for $i > 5$. The numerical computations show that the number of different images $Q(K_j)$ of the cube K_j inside H_i (for i up to 5) stop growing (reaching, in fact, the number of perfect forms) at a value of $j = \bar{j}$, rapidly growing with respect to i, namely $\bar{j}(i) = 2, 9, 28, 74, 244$ for $i = 1, 2, 3, 4, 5$.

The empirical laws for the numbers of images $Q_i^H(K_i)$ and $Q_i(K_i)$ are, in terms of $L = 2i + 1$,

$$\#(Q_i^H(K_i)) \approx .8L^{2.47}, \quad \#(Q_i(K_i)) \approx 1.4L^{2.4}. \quad (3\text{–}1)$$

These laws provide an estimate from below of the number of perfect hyperbolic forms and of all perfect forms in C_i, for $i \leq 30$.

We observe that, for i up to 5, the amount of different images in H_i of points of K_i is approximately one-half the total number $\#Q_i^H(\mathbb{Z}^4)$. We have (compare with Table 1): $\#(Q_i^H(K_i)) = 12, 34, 82, 180, 276$, for $i = 1, 2, 3, 4, 5$.

Moreover, we observe that the growth rate for $\#(Q_i^H(K_i))$ is equal to $2.47 = \frac{3}{2}1.65$, i.e., it is in agreement with the growth rate of hyperbolic perfect forms calculated in the squares Sq_i.

Hence Conjecture 3.1 and the above observations lead to the conjecture that *the number of different images* $Q(\mathbb{Z}^4)$ *in* H_i, *denoted by* $\#Q_i^H(\mathbb{Z}^4)$, *is proportional to the number of the images in* H_i *of* $Q(K_i)$ (as in the case of the images inside E_i^+).

If we suppose, moreover, that $\#Q_i^H(\mathbb{Z}^4) \approx 2\#Q_i^H(K_i)$, as for $i < 6$, then we may formulate the following conjecture:

Conjecture 3.8. *The number of all different images under* Q *of* $\mathbb{Z}^4$ *in* C_i *are bounded, for every* i, *by*

$$\#(Q_i(\mathbb{Z}^4)) \leq 2L^{5/2}.$$

The same upper bound should hold for the number of perfect forms, if the statement of Conjecture 3.1 is true.

Remark 3.9. The exponents of the empirical laws (3–1) for hyperbolic forms and for all forms decrease slightly when the interval chosen for fitting is increased. A similar phenomenon was observed in the elliptic case.

We conjectured that these exponents coincide with those for the whole image $Q(\mathbb{Z}^4)$. However, if they converge, when L grows to infinity, to some asymptotical values, then these values will coincide. Indeed, we can formulate two hypotheses:

1. The exponent for the hyperbolic images converges to the growth rate of the number of elliptic images in C_i. Thus, the exponent for all forms will coincide with them, making the proportion of parabolic forms negligible;

2. The exponent for the hyperbolic images converges to a number greater than the growth rate of the number of elliptic images in C_i. Thus the proportion of the number of images $Q_i^E(\mathbb{Z}^4)$ compared to the total number of images will vanish, and the asymptotical exponent of the total number of images will coincide with that of the hyperbolic ones.

The empirical data we have indicates that the second hypothesis is more likely.

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