The Residually Weakly Primitive Geometries of J_3

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CONTENTS

We summarize the classification of all firm and residually connected geometries satisfying the conditions $(IP)_2$ and $(2T)_1$ and on which the Janko group J_3 acts flag-transitively and residually weakly primitively. We state some facts regarding the results. The complete list of geometries is available as a supplement to this note [Leemans 03c].

1. INTRODUCTION

The basic concepts about geometries constructed from a group and some of its subgroups are due to Tits [Tits 62] (see also [Buekenhout 95, Chapter 3]).

Let $I = \{1, \ldots, n\}$ and let $\Gamma(G; (G_i)_{i \in I})$ be a coset geometry constructed from a group G and some subgroups G_i of G (where $i \in I$). The geometry Γ is

- *flag-transitive* (FT) if G acts transitively on all chambers of Γ, hence also on all flags of any given type J, where J is a subset of I;
- *firm* (F) if every flag of rank $\mid I \mid -1$ is contained in at least two chambers;
- *residually connected* (RC) if the incidence graph of each residue of rank ≥ 2 is a connected graph;
- *residually weakly primitive* (RWPRI) if each residue Γ_F of a flag F is primitive on the set of elements of type i for at least one $i \in t(\Gamma_F)$;
- $(IP)_2$ if every rank 2 residue of Γ is either a partial linear space or a generalized digon [Buekenhout 95];
- $(2T)_1$ if the stabilizer G_F of any given flag F of Γ of corank 1 acts two-transitively on the residue Γ_F.

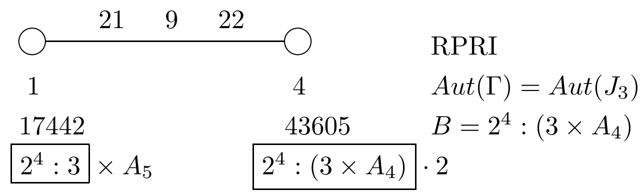
The Janko group J_3 is the eighth smallest sporadic group. It has 50,232,960 elements. Very few geometries are known on which J_3 acts as a flag-transitive automorphism group. We found only three such geometries in the literature. We give their diagrams in Figure 1.

2000 AMS Subject Classification: Primary 51E24; 20D08;

Keywords: Sproadic groups, incidence geometry

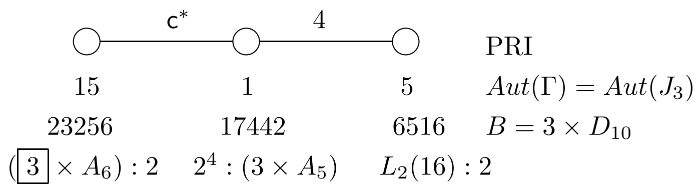
1058-6458/2004$ 0.50 per page

2.1



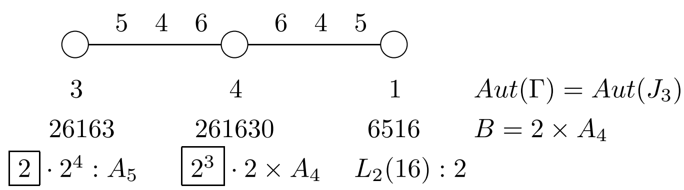
Due to Richard Weiss [Weiss 82]. This geometry was used by Weiss to give a computer-free existence proof of J_3. See also [Buekenhout 85, Geometry (54)].

3.2



Due to Francis Buekenhout (unpublished work, 1978). See also [Baumeister 97]. This geometry was used in the latter paper to give a computer-free existence proof of J_3.

3.3



Due to Buekenhout (see [Buekenhout 85, Geometry (106)]).

FIGURE 1. The geometries of J_3 available in the literature.

Using a series of Magma [Bosma et al. 97] programs described in [Dehon and Leemans 99], we classified all firm and residually connected geometries satisfying the conditions $(IP)_2$ and $(2T)_1$ and on which the Janko group J_3 acts flag-transitively and residually weakly primitively. If a geometry satisfies these conditions we say it is a $RWPRI + (2T)_1$ geometry. Such a classification has already been obtained for the seven smallest sporadic groups. We refer to [Dehon et al. 96] (respectively, [Dehon and Leemans 99], [Gottschalk and Leemans 96], [Leemans 02], [Leemans 03a], [Leemans 04c], [Leemans 04a]) for the geometries of M_{11}, M_{12}, J_1, J_2, M_{22}, M_{23}, and HS. This work is part of a project initiated by Buekenhout in the 1980s. It aims to find a unified geometric interpretation of all the finite simple groups. We refer to [Buekenhout et al. 03] for a survey of this project.

The list of geometries we obtained for J_3 is extremely important in the sense that it will help to understand the third group of Janko in a geometrical way. The programs used to obtain the classification as well as the lists of subgroups forming the geometries we obtained are available from the author upon request.

2. THE RESULTS

The Janko group J_3 is a group of order 50,232,960 and its smallest permutation representation is on 6,156 points. This makes it by far the biggest group we were able to fully analyze in this project.

To obtain our classification, we needed first to compute one representative of each conjugacy class of subgroups of J_3. This has been done using a custom program which is based on ideas described in [Pfeiffer 97]. The main idea is to first compute the maximal subgroups of J_3 and then their maximal subgroups and so on until we arrive at the identity subgroup.

The computer we used runs a Pentium III Xeon processor at 2.0 GHz with 3 gigabytes of memory under Linux. It took 190 seconds to obtain the subgroup lattice of J_3 with this computer.

It is obvious that J_3 cannot have rank-1 $RWPRI + (2T)_1$ geometries since it does not have a 2-transitive permutation representation. Using our set of programs, we obtained 7, 17, 0 $RWPRI + (2T)_1$ geometries of rank 2, 3, and ≥ 4.

We used what is described as the bottom-up approach in [Dehon and Leemans 99] to obtain the complete classification of geometries presented in this paper. In this method, if we try to construct a geometry $\Gamma := \Gamma(J_3, (G_i)_{i\in\{1,\dots,n\}})$, we start from the Borel subgroup $B = \bigcap_{i\in\{1,\dots,n\}} G_i$ to build the geometry Γ. Since the subgroup lattice of J_3 gives us 137 conjugacy classes of subgroups, the task of classifying all geometries for J_3 may be divided into 137 subtasks of classifying geometries with B being one representative of each conjugacy class of subgroups of J_3.

Almost all representatives of conjugacy classes of subgroups could be analyzed using our programs. It took the computer roughly two months of computing time to obtain all these results. For two conjugacy classes of subgroups, namely one class of subgroups of order 3 and the identity, we had to work slightly differently.

In the case of the identity, we found out using our program that no maximal subgroup of J_3 has a geometry with Borel equal to the identity. Hence, J_3 cannot have a geometry with Borel equal to the identity.

In the case of the subgroups of order 3 (which correspond to the class number 134 in the subgroup lattice available in the supplement [Leemans 03b]), we found that some maximal subgroups of J_3 have geometries with Borel subgroup in this conjugacy class of subgroups.

Then, we took these geometries one by one. Say $\Gamma := \Gamma(M, (M_i)_{i\in\{1,\dots n-1\}})$ with M a maximal subgroup of J_3 and $B := \cap_{i\in\{1,\dots,n-1\}} M_i$ is a cyclic group of order 3 conjugate to the subgroups of class 134. We checked that for each such Γ, it was not possible to find a subgroup H of J_3, containing B as a maximal subgroup and such that $\Gamma(J_3, \{M\} \cup (< M_i, H >)_{i\in\{1,\dots,n-1\}})$ is a geometry satisfying our axioms. This check was made interactively with Magma.

The complete list is available online as a supplement to this note (see [Leemans 03b]). For every geometry Γ, we give its diagram, the automorphism and correlation groups, the Borel subgroup, and the kernels. We mention when a geometry is a truncation of a higher rank geometry. We also say when a geometry can be constructed from another using a construction described in Theorem 4.1 of [Lefèvre-Percsy et al. 97] that Pasini calls doubling [Pasini 95].

In the supplement, we also give the complete subgroup lattice of J_3. This subgroup lattice has already been computed by Goetz Pfeiffer [Pfeiffer 97].

Two rank-2 geometries are truncations of geometries of higher rank. This yields 5, 17, 0 geometries of rank 2, 3, and ≥ 4 that are not truncations of a geometry of higher rank. Out of these remaining geometries, 1 (respectively, 2) geometry of rank 2 (respectively, 3) is a doubling.

3. OBSERVATIONS

We collect here a list of observations we made while looking at the list of geometries we obtained. For a group G, we call the (RWPRI+$(2T)_1$)-rank of G the highest rank of the geometries satisfying our axioms.

- The Janko group J_3 has the smallest (RWPRI+$(2T)_1$)-rank of all the eight smallest sporadic groups investigated so far. This rank is equal to 3 for J_3; 4 for M_{11}, J_1, and J_2; 5 for M_{12}, M_{22}, and M_{23}; and 6 for HS (see [Leemans 04b], [Leemans 04a]).
- There are no thin geometries. Out of the eight sporadic groups investigated so far, only J_2 has thin geometries satisfying our axioms.
- There are no Petersen graphs arising as rank-2 residues.
- In the rank-3 geometries, the rank-2 residue which appears most often is a 14-9-14. We quote Vanmeerbeek [Vanmeerbeek 99]: "This geometry is a graph first found by Norman Biggs and Derek Smith [Biggs and Smith 71]. As mentioned in [Vanmeerbeek 99], a mistake in the observation was found by Arjeh Cohen [Cohen 80] and later confirmed by Richard Weiss (personal communication of Francis Buekenhout). This graph is 3-Moufang."
- The subgroup of J_3 which appears most often as a maximal parabolic subgroup of a geometry is $PSL(2,17)$. It appears in 11 of the 17 rank-3 geometries. Moreover, all the $RWPRI + (2T)_1$ geometries of rank 2 of $PSL(2,17)$ have a rank-3 $RWPRI + (2T)_1$ extension in J_3.

4. THE EIGHT SPORADIC GROUPS ANALYSED SO FAR

As we mentioned in the introduction, all $RWPRI+(2T)_1$ geometries are known for the eight smallest sporadic

Name	Order	#Conj.Classes	#Subgroups	Rk 2	Rk 3	Rk 4	Rk 5	Rk 6
M_{11}	7920	39	8651	0	9	5	0	0
M_{12}	95040	147	214871	6	11	17	1	0
J_1	175560	40	158485	8	1	2	0	0
M_{22}	443520	156	941627	1	19	12	9	0
J_2	604800	146	1104344	5	16	3	0	0
M_{23}	10200960	204	17318406	0	3	16	10	0
HS	44352000	589	149985646	4	20	41	8	1
J_3	50232960	137	71564248	5	17	0	0	0

TABLE 1. The eight sporadic groups explored so far.

groups. No geometry of rank at least 7 exists for any of these groups.

Table 1 gives for each of the eight sporadic groups explored so far its order, the number of conjugacy classes of subgroups, the number of subgroups, and the number of geometries obtained in rank 2, 3, 4, 5, and 6 without counting those that appear as truncations of higher-rank geometries for the same group.

ACKNOWLEDGMENTS

The author gratefully acknowledges financial support from the Belgian National Fund for Scientific Research for this project.

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Received January 27, 2004; accepted May 6, 2004.