An Algorithm for Finding the Veech Group of an Origami

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CONTENTS

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We study the Veech group of an origami, i.e., of a translation surface, tessellated by parallelograms. We show that it is isomorphic to the image of a certain subgroup of $\mathrm{Aut}^+(F_2)$ in $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$. Based on this, we present an algorithm that determines the Veech group.

1. ORIGAMIS AS TEICHMÜLLER CURVES

(Oriented) *origamis* (as defined in Section 2.1) can be described as follows: Take finitely many copies of the unit square in $\mathbb{C}$ and glue them together such that each left edge is glued with a right edge and each upper edge with a lower one (compare e.g., [Lochak 03], [Möller 03]). This defines a compact surface S. We restrict ourselves to the cases where S is connected.

Lifting the structure of $\mathbb{C}$ via the squares defines a translation structure on $S^\star := S - \{P_1, \dots, P_n\}$, where $P_1, \dots, P_n$ are finitely many points on S. One can vary the structure on $S^\star$ as follows: For each $\tau \in \mathbb{H}$, identify the squares on S with the parallelogram $P(\tau)$ in $\mathbb{C}$ defined by the vertices $0, 1, \tau, 1+\tau$. This defines an isometric embedding of the upper half plane $\mathbb{H}$ into the Teichmüller space $T_{g,n}$, where g is the genus of S.

This construction is a special case of the more general concept of Teichmüller geodesic disks: Any holomorphic differential ω on a Riemann surface X defines a translation structure on X. Composing this structure with matrices in $\mathrm{SL}_2(\mathbb{R})$ defines an isometric embedding of $\mathbb{H} = \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$ into the appropriate Teichmüller space $T_{g,n}$ whose image is a complex geodesic called a *Teichmüller disk*.

The image of this disk in the moduli space $M_{g,n}$ under the natural projection $T_{g,n} \to M_{g,n}$ is in some cases an algebraic curve. Then, it is called a *Teichmüller curve*. It is birational to the mirror image of $\mathbb{H}/\Gamma$, where Γ is the Veech group of the surface defined as in Section 2.1.

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(See [Earle and Gardiner 97], [Lochak 03], [McMullen 03] and references therein.)

An origami presents the special case where ω is the pullback $\omega = p^*(\omega_E)$ of the invariant differential ω_E on an elliptic curve E under a finite morphism $p : X \to E$ ramified only over one point. In this case the quotient $\mathbb{H}/\Gamma$ is always an affine algebraic curve, and it is easy to see that it is defined over $\bar{\mathbb{Q}}$ (see Section 3.4). Additionally, the embedded curve C in $M_{g,n}$ is an irreducible component of a Hurwitz space and thus also defined over $\overline{\mathbb{Q}}$ ([Möller 03]). In [Lochak 03], where the name *origami* was introduced, Pierre Lochak suggests to study them in the context of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on combinatorial objects, in some sense as a generalization of the study of dessins d'enfants. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of *origami curves* in $M_{g,n}$ and this action is faithful as shown in [Möller 03]. Recently, it was shown in [Möller 04], using different methods, that arbitrary Teichmüller curves are defined over $\overline{\mathbb{Q}}$.

It would be interesting to know "all" Teichmüller curves. For the case of genus 2 many things are already known. For example, [McMullen 03] and [Calta 02] classify Teichmüller curves in the moduli space M_2 using different methods (Jacobians with real multiplication; and Kenyon-Smillie invariants as defined in [Kenyon and Smillie 00]). [McMullen 03] obtains an infinite family of *primitive Teichmüller curves*, where "primitive" means that the differential ω defining the Teichmüller curve is not the pullback of a holomorphic differential on a surface of lower genus.

In [Hubert and Lelièvre 04a] one can find explicit combinatorial descriptions of the origami curves in genus 2 where the differential ω has only one zero.

Since the Teichmüller curve is birational to $\mathbb{H}/\Gamma$, knowledge of the Veech groups Γ that occur will help to understand the Teichmüller curves. Veech groups are defined in general for translation surfaces. They are discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ ([Veech 89]), but not all discrete subgroups occur as Veech groups ([Hubert and Schmidt 01]). The construction for the Teichmüller geodesic disk described above leads to a Teichmüller curve iff the Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$, i.e., if it has finite covolume.

Therefore, there is a particular interest in translation surfaces whose Veech groups are lattices (*Veech surfaces*). The first examples were given by Veech himself, e.g., the surfaces obtained by gluing parallel sides of two regular n-gons (see [Veech 89]). Their Veech groups are the hyperbolic triangle groups $\Delta(2, n, \infty)$ if n is odd and $\Delta(m, \infty, \infty)$ if $n = 2m$, $n \geq 5$. (Here $\Delta(r, s, t)$ denotes the Fuchsian triangle group with signature r, s, t.) One gets these translation surfaces using the construction in [Katok and Zemlyakov 75] starting from billiard tables in the shape of an isosceles triangle with base angles π/n.

Other examples were found using the same construction starting from rational triangles with angles (q_1, q_2, q_3). The Veech groups associated with isosceles triangles with base angles $q_1 = q_2 = \frac{2k-1}{4k}\pi$ and $q_1 = q_2 = \frac{k}{2k+1}\pi$ ($k \geq 2$) are the triangle groups $\Delta(2k, \infty, \infty)$ and $\Delta(2k+1, \infty, \infty)$, respectively ([Earle and Gardiner 97], [Hubert and Schmidt 01]); those associated with the triangles defined by $q_1 = \frac{\pi}{2n}$, $q_2 = \frac{\pi}{n}$, $q_3 = \frac{2n-3}{2n}\pi$ ($n \geq 4$) are the triangle groups $\Delta(3, n, \infty)$ ([Ward 98]). The three triangles where (q_1, q_2, q_3) equals $(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5}{12}\pi)$, $(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7}{15}\pi)$, and $(\frac{2}{9}\pi, \frac{\pi}{3}, \frac{4}{9}\pi)$ (in [Vorobets 96] and [Kenyon and Smillie 00]) also have Veech groups that are lattices, namely $\Delta(6, \infty, \infty)$, $\Delta(15, \infty, \infty)$, and $\Delta(9, \infty, \infty)$, respectively ([Hubert and Schmidt 01]).[1]

Not all Veech groups are commensurable to a triangle group. Starting with L-shaped billiard tables instead of triangles, McMullen finds in [McMullen 03] an infinite sequence of Veech surfaces of genus 2, among them surfaces whose Veech groups are not commensurable to a triangle group. Their associated Teichmüller curves belong to the infinite family of curves in M_2 mentioned previously.

Nevertheless, being a lattice should—as noted above—be considered to be an exception for a Veech group. For example, the last three triangle-shaped billiard tables given above are the only acute nonisosceles triangles whose associated Veech group is a lattice ([Kenyon and Smillie 00], [Puchta 01]).

The Veech group of an origami, however, is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index and thus a lattice. In fact, Gutkin and Judge obtain the following equivalence in [Gutkin and Judge 00]: A translation surface has a Veech group commensurable to $\mathrm{SL}_2(\mathbb{Z})$ iff it covers a flat torus with at most one branch point. In other words, origamis can be characterized as those Veech surfaces whose Veech groups are arithmetic.

Origamis already occur implicitly in the work of Thurston and Veech and examples have been studied by a number of authors since then—for example, under the name *square tiled surfaces* or *branched coverings of marked flat tori* (see e.g., [Eskin et al. 03], [Eskin and Okounkov 01], [Hubert and Lelièvre 04a] and references therein).

In this article we describe how to calculate the Veech group $\Gamma(O)$ of an arbitrary origami O. We present an algorithm that finds generators and coset representatives of

[1]For a more detailled overview see e.g., [Lelièvre 02].

$\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$ and calculates the genus and the number of points at infinity of $\mathbb{H}/\Gamma(O)$.

In Section 2, we provide a characterization of the Veech groups of origamis in terms of automorphisms of the free group F_2 in two generators. They are the images in $\mathrm{SL}_2(\mathbb{Z})$—the outer automorphism group of F_2—of certain subgroups of the automorhism group, namely those which stabilize a finite index subgroup of F_2 (Proposition 2.1). As a consequence we get a new proof that the Veech groups of origamis are subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index (Corollary 2.9).

Proposition 2.1 is the key result for the algorithm that is described in Section 3. In Section 4, we give examples and state some final remarks.

2. VEECH GROUPS OF ORIGAMIS

The algorithm we want to present is based on Proposition 2.1. We denote by F_2 the free group in two generators and by $\mathrm{Aut}^+(F_2)$ the group of orientation preserving automorphisms of F_2. Furthermore, we use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathrm{Out}^+(F_2)$, the group of outer orientation preserving automorphisms of F_2, and denote by $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$ the canonical projection (see Lemma 2.8). To an origami O we will associate a subgroup H of F_2 (see Notation 2.3).

Proposition 2.1. *Let O be an origami. Let $\mathrm{Aff}^+(H) := \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(H) = H\}$. Then we have*

$$\Gamma(O) = \hat{\beta}(\mathrm{Aff}^+(H)) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

The aim of Section 2. is to explain the notation and prove the statement of Proposition 2.1.

2.1 Origamis, Translation Surfaces and the Veech Group

In the following let E be a fixed torus and

$$E^\star := E - \{\bar{P}\} \quad (\text{for some } \bar{P} \in E)$$

be a once punctured torus.

Definition 2.2. An (oriented) *origami* O (of genus $g \geq 1$) is a (topological) unramified covering $p : X \to E^\star$, where X is obtained by erasing finitely many points of a compact surface $\bar{X}$ of genus g.

Fix a (topological) unramified universal covering $u : \tilde{X} \to X$ of X. Then $v := p \circ u$ is a universal covering of $E^\star$.

Let $\mathrm{Gal}(\tilde{X}/E^\star)$ be the group of deck transformations of v. $\mathrm{Gal}(\tilde{X}/E^\star)$ is naturally isomorphic to the fundamental group $\pi_1(E^\star, \bar{Q})$ of $E^\star$ with an arbitrary base point $\bar{Q} \in E^\star$. Furthermore, $\pi_1(E^\star, \bar{Q})$ is isomorphic to $F_2 := F_2(x, y)$, the free group in the two generators x and y. Fix the isomorphism $\alpha : F_2 \to \pi_1(E^\star, \bar{Q}) \overset{\text{can}}{\cong} \mathrm{Gal}(\tilde{X}/E^\star)$ such that $\alpha(x)$ and $\alpha(y)$ define a canonical marking on $E^\star$.

Then, $H := \mathrm{Gal}(\tilde{X}/X) \subseteq \mathrm{Gal}(\tilde{X}/E^\star)$ is considered (via $(\mathrm{can} \circ \alpha)^{-1}$) as a subgroup of F_2.

Notation 2.3.

$$H := \mathrm{Gal}(\tilde{X}/X) \subseteq \mathrm{Gal}(\tilde{X}/E^\star) = F_2(x, y) =: F_2.$$

We will consider translation structures on X induced by translation structures on $E^\star$. Therefore, we first want to recall some definitions and notation (see e.g., [Gutkin and Judge 00], [Thurston 97]).

An atlas on a surface X such that all transition maps are translations, defines a *translation structure* μ on X. $X_\mu := (X, \mu)$ is called a *translation surface*. We call

$$\mathrm{Aff}^+(X_\mu) := \{f : X_\mu \to X_\mu | f \text{ is an orientation preserving affine diffeomorphism}\}^2$$

the *affine group* of X_μ.

Let $u : \tilde{X} \to X$ be a (topological) universal covering of X. Then $\tilde{X}$ becomes a translation surface $\tilde{X}_\eta$ by lifting the structure μ on X via u to η on $\tilde{X}$. A fixed chart (U, η_U) of $\tilde{X}_\eta$ defines a holomorphic map $\mathbf{dev} : \tilde{X}_\eta \to \mathbb{C}$ (*developing map*) such that

$$\eta_U = dev|_U \quad \text{and} \quad \eta_{U'} = t \circ dev|_{U'}$$

for a translation $t := t(U', \eta_{U'})$ for any other chart $(U', \eta_{U'})$ of $\tilde{X}_\eta$.

For any affine diffeomorphism $\hat{f}$ of $\tilde{X}_\eta$ there is a unique affine diffeomorphism $\mathbf{aff}(\hat{f})$ of $\mathbb{C}$ such that $\mathbf{dev} \circ \hat{f} = \mathbf{aff}(\hat{f}) \circ \mathbf{dev}$. We call $\mathbf{aff}$ the group homomorphism

$$\mathbf{aff} : \mathrm{Aff}^+(\tilde{X}_\eta) \to \mathrm{Aff}^+(\mathbb{C}), \hat{f} \mapsto \mathbf{aff}(\hat{f}).$$

The *holonomy mapping* **hol** is the restriction of **aff** to the subgroup $H = \mathrm{Gal}(\tilde{X}/X)$ of $\mathrm{Aff}^+(\tilde{X})$. If proj is the natural projection $\mathrm{proj} : \mathrm{Aff}^+(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{R})$, then the group homomorphism

$$\mathbf{der} : \mathrm{Aff}^+(X_\mu) \to \mathrm{GL}_2(\mathbb{R}), f \mapsto \mathrm{proj}(\mathbf{aff}(\hat{f}))$$

[2] In the following, all diffeomorphisms are orientation preserving.

where $\hat{f}$ is some lift of f to $\tilde{X}$ is well defined and called a *derived map*.

$\Gamma(X_\mu) := \mathbf{der}(\mathrm{Aff}^+(X_\mu)) \subseteq \mathrm{GL}_2(\mathbb{R})$ is called the *Veech group* of X_μ. It is independent of the choice of the chart (U, μ_U) which we used to define **dev**. If X is precompact, i.e., X is obtained by erasing finitely many points from a compact Riemann surface $\bar{X}$, then every $f \in \mathrm{Aff}^+(X_\mu)$ preserves the volume. Thus, $\Gamma(X_\mu)$ is in $\mathrm{SL}_2(\mathbb{R})$.

Now, given an origami $O = (p : X \to E^\star)$ as above, any matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

defines a translation structure on X as follows:
Take the lattice

$$\Lambda_B :=< \vec{v}_1 := \begin{pmatrix} a \\ c \end{pmatrix}, \vec{v}_2 := \begin{pmatrix} b \\ d \end{pmatrix} > \text{ in } \mathbb{C}.$$

Let $E_B := \mathbb{C}/\Lambda_B$ be the elliptic curve defined by Λ_B and let $E_B^\star$ be the once punctured elliptic curve (obtained by erasing the image of 0 from E_B) with the induced translation structure. Fix some point Q_B in $\mathbb{C} - \Lambda_B$. Let $\bar{Q}_B$ be its image on $E_B^\star$. Furthermore, set as canonical marking the images of the segments from Q_B to $Q_B + \vec{v}_1$ and from Q_B to $Q_B + \vec{v}_2$ on $E_B^\star$. Identify $E_B^\star$ with $E^\star$ via a diffeomorphism respecting the canonical markings. This way, p defines an unramified covering of $E_B^\star$. Let μ_B be the translation structure on X defined by lifting the translation structure on $E_B^\star$ to X via p (Note that μ_B depends also on p). Similarly let η_B be the translation structure on the fixed universal covering $\tilde{X}$ defined via u.

Notation 2.4. Denote by $X_B := X_B(O) := (X, \mu_B)$ the surface X with translation structure μ_B. Furthermore, denote by $\tilde{X}_B$ the translation surface $(\tilde{X}, \eta_B)$.

Then the maps $p_B : X_B \to E_B^\star$, $u_B : \tilde{X}_B \to X_B$, and $v_B : \tilde{X}_B \to E_B^\star$ induced by p, u, and v are translation maps.

Let $\mathbf{dev}_B : \tilde{X}_B \to \mathbb{C}$ be a developing map of $\tilde{X}_B$ (and thus also for X_B and $E_B^\star$) and $\mathbf{der}_B : \mathrm{Aff}^+(\tilde{X}_B) \to \mathrm{GL}_2(\mathbb{R})$ the corresponding derived map.

The proof of Remark 2.5 shows that the affine group of an origami surface X_B does not depend (up to conjugacy) on the choice of the matrix B.

Remark 2.5. Let B, B' be in $\mathrm{SL}_2(\mathbb{R})$. Then

$$\mathrm{Aff}^+(X_B(O)) \cong \mathrm{Aff}^+(X_{B'}(O))$$

and

$$\Gamma(X_{B'}(O)) = B'B^{-1}\Gamma(X_B(O))BB'^{-1}.$$

Proof: The map $\varphi : X_B(O) \to X_{B'}(O)$ that is topologically the identity on X is an affine diffeomorphism and induces the group isomorphism:

$$\mathrm{Aff}^+(X_B(O)) \to \mathrm{Aff}^+(X_{B'}(O)), f \mapsto \varphi \circ f \circ \varphi^{-1}.$$

Since $\mathbf{der}(\varphi) = B'B^{-1}$, we have $\mathbf{der}(\varphi f \varphi^{-1}) = B'B^{-1}\mathbf{der}(f)BB'^{-1}$ □

Since the Veech group depends only up to conjugacy on the choice of B, we will restrict to the case of $B = I$, the identity matrix. If not stated otherwise, we will denote $\tilde{X} := \tilde{X}_I$, $\mathbf{der} := \mathbf{der}_I$, $\mathbf{dev} := \mathbf{dev}_I$, $X := X_I$, $E := E_I$, $\Lambda := \Lambda_I$, $E^\star := E_I^\star$, $\mu := \mu_I$, and $\Gamma(O) := \Gamma(X_I(O))$.

By the uniformization theorem there exists a biholomorphic map $\delta : \mathbb{H} \to \tilde{X} = \tilde{X}_I$, where $\mathbb{H}$ is the complex upper half plane. $\mathbb{H}$ becomes, via δ, a translation surface. We will identify $\mathbb{H}$ with $\tilde{X} = \tilde{X}_I$.

Proposition 2.6. *Let $O = (p : X \to E^\star)$ be an origami and $\mathbb{H}$ be the upper half plane, endowed with the translation structure induced by O as above. Then we have:*

(1) *$\Gamma(O)$ is a subgroup of $\Gamma(\mathbb{H})$.*

(2) $\Gamma(E^\star) = \Gamma(\mathbb{H}) = \mathrm{SL}_2(\mathbb{Z})$.

(3) *Let f be in $\mathrm{Aff}^+(X)$. Then f descends via p to some $\bar{f} \in \mathrm{Aff}^+(E^\star)$ and the diagram in Figure 1 is commutative with $A := \mathbf{der}(f)$, with $\hat{f}$ some lift of f to $\mathbb{H}$ and with some $b \in \mathbb{Z}^2$.*

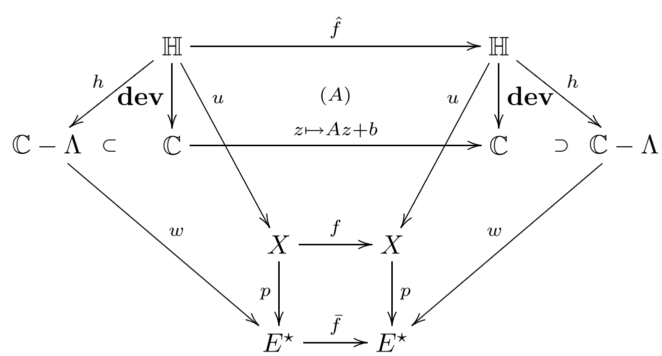


FIGURE 1. Diagram 1.

Proof:
(1). Let f be in $\mathrm{Aff}^+(X)$ and $\hat{f}$ be some lift of f via u. Since the translation structure on $\mathbb{H}$ is lifted via u, $\hat{f}$ is also affine and $\mathbf{der}(\hat{f}) = \mathbf{der}(f)$. Hence, $\Gamma(O) \subseteq \Gamma(\mathbb{H})$.

(2). Let $\mathbb{C} \to E$ be the universal covering and $w : \mathbb{C} - \Lambda \to E^\star$ its restriction to $\mathbb{C} - \Lambda$. Since $v = p \circ u$ is the universal covering of $E^\star$, there is an unramified covering $h : \mathbb{H} \to \mathbb{C} - \Lambda$, such that $w \circ h = v = p \circ u$. But since the structure on $\mathbb{H}$ was obtained by lifting the translation structure on $E^\star$ via v, this map h is locally a chart of $\mathbb{H} = \tilde{X}_I$. Thus, the map h is a developing map and the image of this developing map $\mathbf{dev}$ is $\mathbb{C} - \Lambda$.

Now, let A be in $\Gamma(\mathbb{H})$, hence $A = \mathbf{der}(\hat{f})$ for some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$. By the definition of **der** and **dev**, Part (A) of the diagram in Figure 1 is commutative for some $b \in \mathbb{Z}^2$, i. e.,

$$(z \mapsto Az + b) \circ \mathbf{dev} = \mathbf{dev} \circ \hat{f}.$$

Since the image of **dev** is in $\mathbb{C} - \Lambda$, the map $z \mapsto Az + b$ respects $\Lambda = \mathbb{Z}^2$. Thus, A is in $\mathrm{SL}_2(\mathbb{Z})$. Hence, we have $\Gamma(\mathbb{H}) \subset \mathrm{SL}_2(\mathbb{Z})$.

Conversely, taking a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ the map $z \mapsto Az$ descends to an affine diffeomorphism $\bar{f} \in \mathrm{Aff}^+(E^\star)$. This can be lifted to some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ with $\mathbf{der}(\hat{f}) = A$. Thus, we have $\mathrm{SL}_2(\mathbb{Z}) \subset \Gamma(\mathbb{H})$.

Using the same arguments it follows that also $\Gamma(E^\star) = \mathrm{SL}_2(\mathbb{Z})$.

(3). Let $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ be some lift of f to $\mathbb{H}$. By the proof of (2) it follows that $\hat{f}$ descends via $w \circ h = v$ to some $\bar{f} \in \mathrm{Aff}^+(E^\star)$ and that the diagram in Figure 1 is commutative. □

From (1) and (2) of Proposition 2.6 we see in particular that the Veech group $\Gamma(O)$ of an origami O is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It follows from [Gutkin and Judge 00, Thm. 5.5] that it has finite index in $\mathrm{SL}_2(\mathbb{Z})$. We will obtain this later in Corollary 2.9. This fact will play a crucial role in Section 3.3.

An immediate consequence of Proposition 2.6 is:

Corollary 2.7.

$$\Gamma(O) = \{A \in \mathrm{SL}_2(\mathbb{Z}) | A = \mathbf{der}(\hat{f}) \text{ for some } \hat{f} \in \mathrm{Aff}^+(\mathbb{H}) \text{ that descends to } X \text{ via } u\}.$$

To prove Proposition 2.1 from Corollary 2.7 we have to state a condition for $\hat{f}$ in $\mathrm{Aff}^+(\mathbb{H})$ to descend via u to some $f \in \mathrm{Aff}^+(X)$.

2.2 When Does an Element in $\mathrm{Aff}^+(\mathbb{H})$ Descend to X?

Recall that $H = \mathrm{Gal}(\mathbb{H}/X) \subset F_2 = \mathrm{Gal}(\mathbb{H}/E^\star) \subseteq \mathrm{PSL}_2(\mathbb{R})$ (Notation 2.3). We define the group homomorphism

$$\begin{array}{rccl} \star: & \mathrm{Aff}^+(\mathbb{H}) & \to & \mathrm{Aut}^+(F_2) \\ & \hat{f} & \mapsto & (\hat{f}_\star : \sigma \mapsto \hat{f} \circ \sigma \circ \hat{f}^{-1}). \end{array}$$

Notice that

$$F_2 = \mathrm{Gal}(\mathbb{H}/E^\star) = \{\hat{f} \in \mathrm{Aff}^+(\mathbb{H}) | \mathbf{der}(\hat{f}) = I\}. \quad (2–1)$$

The map $\star$ is well defined, since $\hat{f} \circ \sigma \circ \hat{f}^{-1}$ is again affine with the derivative $\mathbf{der}(\hat{f}) \cdot I \cdot \mathbf{der}(\hat{f})^{-1} = I$ and thus in F_2.

Lemma 2.8. *We have the following properties of* $\star$*:*

(1) *The following two sequences are exact and the diagram is commutative:*

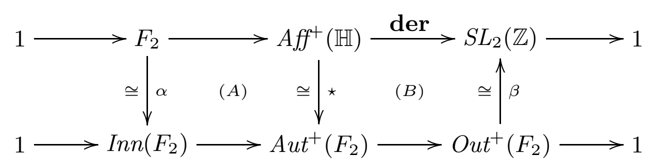


FIGURE 2. Diagram 2.

Here, $\mathrm{Inn}(F_2)$ *is the group of inner automorphisms of* F_2*,* α *is the natural isomorphism* $F_2 \to \mathrm{Inn}(F_2)$, $x \mapsto (y \mapsto xyx^{-1})$, $\beta : \mathrm{Out}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ *is the group isomorphism induced by the natural homomorphism:*

$$\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z}),\ \varphi \mapsto A := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a *is the number of* x *appearing in* $\varphi(x)$*,* b *the number of* x *appearing in* $\varphi(y)$*,* c *the number of* y *in* $\varphi(x)$*, and* d *the number of* y *in* $\varphi(y)$ *(see* [Lyndon and Schupp 77, I 4.5, p.25]*). Recall that for the canonical projection* proj : $F_2 \to \mathbb{Z}^2$ *sending* x *to* $(1,0)^t$ *and* y *to* $(0,1)^t$ *one has* $\forall \varphi \in \mathrm{Aut}^+(F_2)$,

$$A := \hat{\beta}(\varphi), \quad \mathrm{proj} \circ \varphi = (z \mapsto A \cdot z) \circ \mathrm{proj}. \quad (2–2)$$

(2) *An element* $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ *descends to* X *via* u *iff* $\hat{f}_\star(H) = H$.

Proof:
(1). The exactness of the first sequence follows from Equation 2–1 and from Proposition 2.6. The exactness of the second sequence is true by the definition of $\mathrm{Out}^+(F_2)$.

The commutativity of Part (A) of Diagram 2 is true by definition of $\star$. We prove now the commutativity of Part (B):

We have chosen the isomorphism $F_2 = F_2(x, y) \cong \mathrm{Gal}(\mathbb{H}/E^\star)$ and the translation structure on $E^\star = E_I^\star$ in such a way that

$$\mathbf{aff}(x) = (z \mapsto z + \begin{pmatrix}1\\0\end{pmatrix}) \text{ and } \mathbf{aff}(y) = (z \mapsto z + \begin{pmatrix}0\\1\end{pmatrix}).$$

Thus, $\mathbf{aff}|_{F_2}(= \mathbf{hol})$ is the natural projection $\mathrm{proj} : F_2 \to \mathbb{Z}^2$. Here we identify the group of translations of $\mathbb{C}$ along some vector in $\mathbb{Z}^2$ canonically with $\mathbb{Z}^2$.

Consider the commutative diagram in Figure 3.

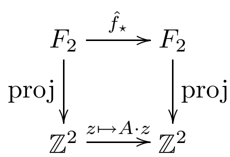


FIGURE 3. Diagram 3.

We will show that Diagram 3 is commutative with $A := \mathbf{der}(\hat{f})$.

Let σ be in $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$. We have to show that $\mathrm{proj}(\hat{f}_\star(\sigma)) = A \cdot \mathrm{proj}(\sigma)$. We have $\mathbf{aff}(\sigma) = (z \mapsto z + c)$ and $\mathbf{aff}(\hat{f}) = (z \mapsto Az + b)$ for some $b, c \in \mathbb{Z}^2$. Thus we get

$$\begin{aligned}\mathrm{proj}(\hat{f}_\star(\sigma)) &= \mathbf{aff}(\hat{f}_\star(\sigma)) = \mathbf{aff}(\hat{f})\mathbf{aff}(\sigma)\mathbf{aff}(\hat{f}^{-1})\\ &= (z \mapsto z + Ac).\end{aligned}$$

Hence, Diagram 3 is commutative with $A = \mathbf{der}(\hat{f})$.

To conclude we use the fact that Diagram 3 is also commutative with $A = \hat{\beta}(\hat{f}_\star)$ (see Equation (2–2)). Thus, $\mathbf{der}(\hat{f}) = \hat{\beta}(\hat{f}_\star)$ and (B) is commutative.

Finally, α and β are both isomorphisms, thus $\star$ is also an isomorphism.

(2). $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ descends to X via $u \Leftrightarrow$ for all $z \in \mathbb{H}, \sigma \in H = \mathrm{Gal}(\mathbb{H}/X)$ there is some $\tilde{\sigma}_{z,\sigma} \in H$ such that $\tilde{\sigma}_{z,\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$.

For $\tilde{\sigma} := \hat{f}_\star(\sigma)$ we have by definition of $\hat{f}_\star$: $\tilde{\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$ for all $z \in \mathbb{H}$. Since F_2 operates fix point free on $\mathbb{H}$ it follows from the last equation that $\tilde{\sigma}_{z,\sigma}$ has to be equal to $\tilde{\sigma} = \hat{f}_\star(\sigma)$. On the other hand, $\tilde{\sigma}_{z,\sigma}$ has to be in H. This proves (2). □

Now Proposition 2.1 follows from Corollary 2.7 and Lemma 2.8.

From Proposition 2.1 we immediately obtain the following:

Corollary 2.9. (to Proposition 2.1)
$\Gamma(O)$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Proof: Let H be defined as above and $d := [F_2 : H]$.

We have a natural action of $\mathrm{Aut}^+(F_2)$ on the subgroups of F_2 of index d and $\mathrm{Aff}^+(H) = \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(H) = H\}$ is the stabilizer of H under this action. Since there are only finitely many subgroups of index d in F_2 the orbit of H under $\mathrm{Aut}^+(F_2)$ is finite and therefore we have $[\mathrm{Aut}^+(F_2) : \mathrm{Aff}^+(H)] < \infty$.

From Proposition 2.1, it follows that $\Gamma(O) = \hat{\beta}(\mathrm{Aff}^+(H))$ also has finite index in $\mathrm{SL}_2(\mathbb{Z}) = \hat{\beta}(\mathrm{Aut}^+(F_2))$. □

As an application of Proposition 2.1 we get the following: In order to check whether $A \in \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, we have to check if there exists a lift $\gamma_A \in \mathrm{Aut}^+(F_2)$ of A (i.e., a preimage of A under $\hat{\beta}$) that fixes H. The following corollary translates this into a finite problem that can be left to a computer.

Corollary 2.10. (to Proposition 2.1)
Let $O = (p : X \to E^\star)$ be an origami of degree d, $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$, $H = \mathrm{Gal}(\mathbb{H}/X)$ as above. Let $h_1, \ldots, h_k$ be generators of H and $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of $H \backslash F_2$ (denote the right coset $H \cdot \sigma_i$ by $\bar{\sigma}_i$).

Further let $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ be some fixed lift of $A \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\begin{aligned}A \in \Gamma(O) \Leftrightarrow \exists\, i \in \{1, \ldots, d\} &\text{ such that}\\ &\bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i \text{ for all } j \in \{1, \ldots, k\}.\end{aligned}$$

Proof: Let γ_A be another lift of A. Thus $\gamma_A^0 = \sigma^{-1} \cdot \gamma_A \cdot \sigma$ for some $\sigma \in F_2$ and we have for all h in H:

$$\begin{aligned}\gamma_A(h) \in H \Leftrightarrow \sigma \cdot \gamma_A^0(h) \cdot \sigma^{-1} \in H \Leftrightarrow H \cdot \sigma \cdot \gamma_A^0(h)\\ = H \cdot \sigma \Leftrightarrow \bar{\sigma} \cdot \gamma_A^0(h) = \bar{\sigma}.\end{aligned}$$

Hence, the claim follows from Proposition 2.1. □

3. THE ALGORITHM

Let $O = (p : X \to E^\star)$ be a given origami of degree d. In this section we present our algorithm that determines the Veech group $\Gamma(O)$. We have subdivided this description into four parts: In Section 3.1 we describe how to find a lift $\gamma_A \in \mathrm{Aut}^+(F_2)$ for any matrix A in $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$, in Section 3.2 we show how to decide whether a given matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, in

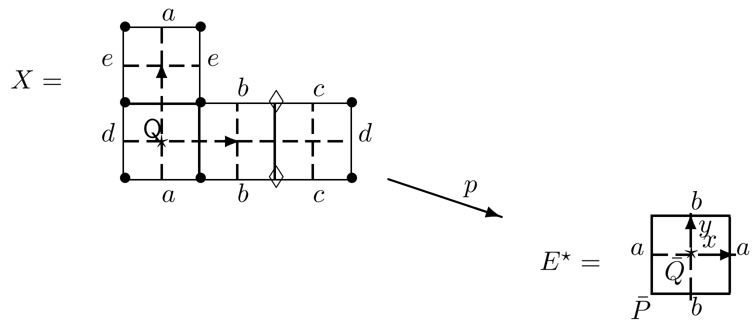


FIGURE 4. Example 3.1.

Section 3.3 we give an algorithm that determines generators and a system of coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$, and finally in Section 3.4 we state how to calculate the genus and the points at infinity of the corresponding *Veech curve* $\mathbb{H}/\Gamma(O)$.

In order to illustrate the algorithm we will use the example $O = L(2,3)$.

Example 3.1. (The Origami $O = L(2,3)$)
Example 3.1 is illustrated in Figure 4.

In Example 3.1 the edges labelled with the same letters are glued together. This way X becomes a surface of genus 2. The squares describe the covering map to $E^\star$. The point $\bar{P} \in E$ (at infinity) has two preimages on the surface X (the points $\bullet$ and $\Diamond$), the degree d of p is 4.

We identify $F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$ with the fundamental group of $E^\star$ (with base point $\bar{Q}$) and $H = \mathrm{Gal}(\mathbb{H}/X)$ with the fundamental group of X (with base point Q). The projection of the closed paths on X to $E^\star$ defines the embedding of H into F_2, x and y are the fixed generators of F_2 on $E^\star$. Since the $L(2,3)$-shape is simply connected, the generators of H are obtained by the identifications of the edges. Thus, $H = < x^3, x^2yx^{-2}, xyx^{-1}, yxy^{-1}, y^2 >$. The index $[F_2 : H]$ is equal to $d = 4$.

3.1 Lifts from $\mathrm{SL}_2(\mathbb{Z})$ to the Automorphism Group of F_2

Let

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T and that $S^{-1} = -S$ and $T^{-1} = -STSTS$. Thus, every $A \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $A = W(S,T)$ or $A = -W(S,T)$, where W is a word in the letters S and T.

The homomorphisms

$$\begin{aligned}
\gamma_S : \quad & F_2 \to F_2 \text{ defined by } \gamma_S(x) = y \text{ and} \\
& \qquad\qquad\qquad\quad \gamma_S(y) = x^{-1}, \\
\gamma_T : \quad & F_2 \to F_2 \text{ defined by } \gamma_T(x) = x \text{ and} \\
& \qquad\qquad\qquad\quad \gamma_T(y) = xy, \\
\gamma_{-I} : \quad & F_2 \to F_2 \text{ defined by } \gamma_{-I}(x) = x^{-1} \text{ and} \\
& \qquad\qquad\qquad\quad \gamma_{-I}(y) = y^{-1}
\end{aligned}$$

are in $\mathrm{Aut}^+(F_2)$ with $\hat{\beta}(\gamma_S) = S$, $\hat{\beta}(\gamma_T) = T$, and $\hat{\beta}(\gamma_{-I}) = -I$, where the morphism $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is the projection defined in Section 2.2 (Lemma 2.8).

Hence, for $A = \pm W(S,T)$ the automorphism $\gamma_A := \pm W(\gamma_S, \gamma_T) \in \mathrm{Aut}^+(F_2)$ is a lift of A. We denote $-W(\gamma_S, \gamma_T) := \gamma_{-I} \circ W(\gamma_S, \gamma_T)$.

In order to find a word W such that $A = W(S,T)$ or $A = -W(S,T)$ we will define a sequence $A_1 := A$, A_2, $\ldots$, A_N such that (for $1 \le n < N$)

$$\begin{aligned}
A_{n+1} &= A_n \cdot T^{-k_n} \cdot S \text{ (with } k_n \in \mathbb{Z}\text{) and} \\
A_N &= \pm T^{\pm b_N} \text{ (with } b_N \in \mathbb{Z}\text{).}
\end{aligned}$$

From this we get that $A = \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1}$. We will conclude using the fact that $T^{-1} = -STSTS$.

These considerations give rise to Algorithm 1, in which we denote

$$A_n =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \text{ with } a_n, b_n, c_n, d_n \in \mathbb{Z}.$$

Algorithm 1. Finding a lift in $\mathrm{Aut}^+(F_2)$.

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

$n := 1; A_1 := A$.

1. If $c_n \neq 0$ find $k_n \in \mathbb{Z}$, such that
$$A_{n+1} := A_n T^{-k_n} S \text{ fulfills } |c_{n+1}| < |c_n|.$$
$k_n := d_n \operatorname{div} c_n$ does this job: $d_n = k_n c_n + r_n$ with $r_n \in \{0, 1, \ldots, |c_n| - 1\}$
$$\Rightarrow A_{n+1} = \begin{pmatrix} -a_n k_n + b_n & -a_n \\ r_n & -c_n \end{pmatrix}.$$
Increase n by 1.
2. Iterate Step (1) until $c_n = 0$. Thus
$$\begin{aligned} A_n &= \begin{pmatrix} \pm 1 & b_n \\ 0 & \pm 1 \end{pmatrix} = \pm T^{\pm b_n} \text{ and} \\ A &= \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1} \\ &=: \pm \tilde{W}(S, T, T^{-1}). \end{aligned}$$
3. Replace in $\tilde{W}$ each T^{-1} by $-STSTS$
$\Rightarrow$ Word W in S and T with $A = W(S,T)$ or $A = -W(S,T)$.
4. Compute $\gamma_A := W(\gamma_S, \gamma_T)$ or $\gamma_A := -W(\gamma_S, \gamma_T)$.

Result: $\gamma_A \in \mathrm{Aut}^+(F_2)$ with $\hat{\beta}(\gamma_A) = A$.

Example 3.2.

$$\begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix} = -T^2 S T^3 STS \Rightarrow \gamma_A^0 = \gamma_{-I} \gamma_T^2 \gamma_S \gamma_T^3 \gamma_S \gamma_T \gamma_S$$
$$\Rightarrow \gamma_A^0 : \; x \mapsto x^{-2} y^{-1} x^{-2} y^{-1} x^{-2} y^{-1} x y x^2,$$
$$y \mapsto x^{-1} y x^2 y x^2 y x^2.$$

3.2 Decide Whether A is in the Veech Group $\Gamma(O)$

Let A be in $\mathrm{SL}_2(\mathbb{Z})$. We want to decide whether A is in $\Gamma(O)$ or not. As in Corollary 2.10 let $h_1, \ldots, h_k$ be generators of $H = \mathrm{Gal}(\mathbb{H}/X) \subseteq F_2 = \mathrm{Gal}(\mathbb{H}/E^\star)$, $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of H in F_2 ($\bar{\sigma}_i := H \cdot \sigma_i$), and γ_A^0 some fixed lift of A in $\mathrm{Aut}^+(F_2)$.

Corollary 2.10 suggests how to build the algorithm:

$$A \in \Gamma(O) \Leftrightarrow \exists i \in \{1, \ldots, d\} \text{ such that } \forall j \in \{1, \ldots, k\} \quad \bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i.$$

Hence, the *main step* will be to decide for some $\tau \in F_2$ whether
$$\bar{\sigma}_i \cdot \tau = \bar{\sigma}_i.$$

In order to do this we present the origami O as directed graph G with edges labelled by x and y (see Figure 5). The cosets $\bar{\sigma}_1, \ldots, \bar{\sigma}_d$ are the vertices of G. Each vertex $\bar{\sigma}_i$ is the start point of one x-edge and one y-edge. The endpoint is $\overline{\sigma_i \cdot x}$ and $\overline{\sigma_i \cdot y}$, respectively.

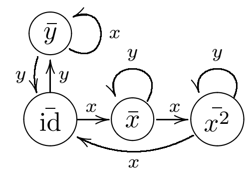


FIGURE 5. Graph for $O = L(2,3)$.

Writing $\tau \in F_2$ as a word in x, y, x^{-1}, and y^{-1} defines a not necessarily oriented path in G starting at the vertex $\bar{\sigma}_i$ with end point $\bar{\sigma}_i \cdot \tau$. We have
$$\bar{\sigma}_i \cdot \tau = \bar{\sigma}_i \Leftrightarrow \text{ this path is closed.}$$

Thus we get the following algorithm.

Algorithm 2. Deciding whether A is in $\Gamma(O)$.

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

Calculate some lift $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ of A (see Section 3.1).
For $j = 1$ to k do: $\tilde{h}_j := \gamma_A^0(h_j)$.
result := false.
for $i = 1$ to d do
 help := true.
 for $j = 1$ to k do:
  if $\bar{\sigma}_i \cdot \tilde{h}_j \neq \bar{\sigma}_i$ (main step, see above)
   then help := false.
 if help = true then result := true.

Result: If the variable "result" is true, then $A \in \Gamma(O)$, else $A \notin \Gamma(O)$.

Example 3.3. (for $O = L(2,3)$)

Let $A := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Take the lift:

$$\gamma_A^0 : \; x \mapsto xyxyx^{-1} =: u \quad y \mapsto xyxyx^{-1}y^{-1}x^{-1} =: v.$$

Generators of H (see Example 3.1) are

$$h_1 := x^3, \; h_2 := xyx^{-1}, \; h_3 := x^2 y x^{-2},$$
$$h_4 := yxy^{-1}, \; h_5 := y^2.$$

For example, $\bar{\mathrm{id}}\cdot\gamma_A^0(h_2) = \bar{\mathrm{id}}\cdot uvu^{-1} = \bar{x}vu^{-1} = \bar{x^2}u^{-1} = \bar{x^2} \Rightarrow \gamma_A^0(H) \neq H$.

But one has $\bar{x}\cdot\gamma_A^0(h_i) = \bar{x}\ \forall i \in \{1,\ldots,5\}; \Rightarrow \gamma_A(H) = H$ for $\gamma_A = x\cdot\gamma_A^0\cdot x^{-1}$ and $A \in \Gamma(O)$.

3.3 Generators and Coset Representatives of $\Gamma(O)$

Let $\bar{\Gamma}(O)$ be the *projective Veech group*, i.e., the image of $\Gamma(O)$ under the projection of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{PSL}_2(\mathbb{Z})$. We first give an algorithm that calculates a list **Gen** of generators and a list **Rep** of right coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$; then we determine $\Gamma(O)$. The way we proceed is based on the Reidemeister-Schreier method ([Lyndon and Schupp 77, II.4]).

We denote by $\bar{A}$ the image of an element $A \in \mathrm{SL}_2(\mathbb{Z})$ under the projection of $\mathrm{PSL}_2(\mathbb{Z})$ and conversely, denote for $\bar{A}$ in $\mathrm{PSL}_2(\mathbb{Z})$ by A some lift of $\bar{A}$. Moreover, we write $A \sim B$ (respectively, $\bar{A} \sim \bar{B}$) if they are in the same coset, i.e., $\Gamma(O)\cdot A = \Gamma(O)\cdot B$ (respectively $\bar{\Gamma}(O)\cdot\bar{A} = \bar{\Gamma}(O)\cdot\bar{B}$).

Each element of $\mathrm{PSL}_2(\mathbb{Z})$ can be presented as a word in $\bar{S}$ and $\bar{T}$. We use the directed infinite tree shown in Figure 6. The vertices $v_0, v_1, v_2, \ldots$ of the tree are labelled by elements of $\mathrm{PSL}_2(\mathbb{Z})$. The root v_0 is labelled by $\bar{I}$, the image of the identity matrix. Each vertex is the starting point of two edges, one labelled by $\bar{S}$, one labelled by $\bar{T}$.

Each element of $\mathrm{PSL}_2(\mathbb{Z})$ occurs as the label of at least one vertex. Starting with v_0, we will visit each vertex v (with label $\bar{B}$) and check if it is not yet represented by the list **Rep**. In this case we will add it to **Rep**. Otherwise, for each $\bar{D}$ in **Rep** that is in the same coset as $\bar{B}$, we add $\bar{B}\cdot\bar{D}^{-1}$ to the list **Gen** of generators.

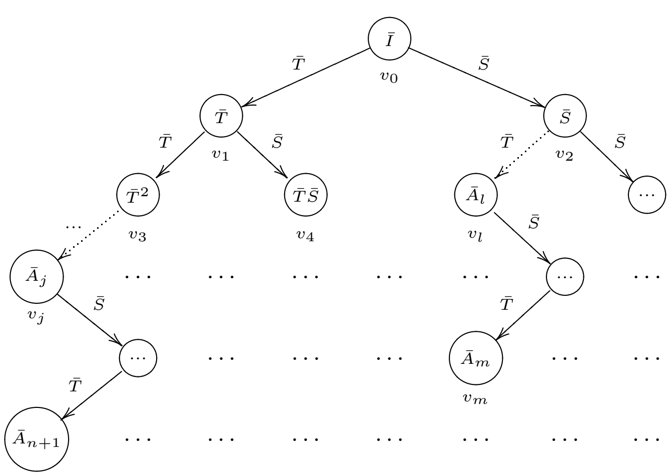


FIGURE 6. Tree labelled by the elements of $PSL_2(\mathbb{Z})$.

We will first give the algorithm and then prove that the lists **Gen** and **Rep** that are calculated are what they should be.

Algorithm 3. Calculating $\bar{\Gamma}(O)$.

Given: Origami O.

Let **Rep** and **Gen** be empty lists.
Add $\bar{I}$ to **Rep**. $\bar{A} := \bar{I}$.
Loop:
$B := A\cdot T$, $C := A\cdot S$
Check whether $\bar{B}$ is already represented by **Rep**:
 For each $\bar{D}$ in **Rep**, check whether $B\cdot D^{-1}$ is in $\Gamma(O)$ or $-B\cdot D^{-1}$ is in $\Gamma(O)$.
 If so, add $\bar{B}\cdot\bar{D}^{-1}$ to **Gen**.
 If none is found, add $\bar{B}$ to **Rep**.
Do the same for C instead of B.
If there exists a successor of $\bar{A}$ in **Rep**, let $\bar{A}$ be now this successor and go to the beginning of the loop. If not, finish the loop.

Result: **Gen**: list of generators of $\bar{\Gamma}(O)$, **Rep**: list of coset representatives in $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 3.4.

(1) Any two elements of **Rep** belong to different cosets.

(2) The algorithm stops after finitely many steps.

(3) In the end, each coset is represented by a member of **Rep**.

(4) In the end, $\bar{\Gamma}(O)$ is generated by the elements of **Gen**.

Proof:
(1). The statement follows by induction. It is true initially, since **Rep** contains only $\bar{I}$. After passing through the loop it is still true, since $\bar{B}$ (respectively, $\bar{C}$) is only added if $\bar{B}\cdot\bar{D}^{-1}$ (respectivley, $\bar{C}\cdot\bar{D}^{-1}$) is not in $\bar{\Gamma}(O)$ for all $\bar{D}$ in **Rep**.

(2). Follows from (1), since $\bar{\Gamma}(O)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ (Corollary 2.9).

(3). Let $\bar{A}$ be an arbitrary element of $\mathrm{PSL}_2(\mathbb{Z})$. There is at least one vertex in the tree that is labelled by $\bar{A}$. Denote the vertices by v_0, v_1, v_2, … as in Figure 6 and their labels by $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, …, respectively. We do induction by the numeration n of the vertices: $\bar{A}_0 = \bar{I}$ is in **Rep**.

Suppose for a certain $n \in \mathbb{N}$ all $\bar{A}_k$ with $k \le n$ are represented by **Rep**. If A_{n+1} is not itself in **Rep** then consider the path ω from v_0 to v_{n+1} and let v_j be the first vertex on ω that is not in **Rep**. Hence, its predecessor is in **Rep** and $\bar{A}_j$ was checked but not added. Thus, there is some $\bar{A}_l$ ($l < j$) in **Rep** such that $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in $\bar{\Gamma}(O)$, i.e., $\bar{A}_j \sim \bar{A}_l$.

Let $\hat{\omega}$ be the path from v_j to v_{n+1} and $\bar{D}$ the product of the labels of the edges on $\hat{\omega}$. Then $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$.

Walking the "same path" as $\hat{\omega}$ starting at v_l (i.e., a path described by the same sequence of $\bar{S}$ and $\bar{T}$) leads to some vertex v_m with $m < n+1$ and label $\bar{A}_m = \bar{A}_l \cdot \bar{D}$.

We have $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D} \sim \bar{A}_l \cdot \bar{D} = \bar{A}_m$ and by the assumption, $\bar{A}_m$ is represented by **Rep**, hence $\bar{A}_{n+1}$ also is.

(4). Let G be the group generated by the elements of **Gen**. We have by construction of the list **Gen** that $G \subseteq \bar{\Gamma}(O)$.

We show again by induction that each label $\bar{A}_n$ in the tree that is in $\bar{\Gamma}(O)$ is also in G. This is true for $n = 0$. Suppose it is true for all $k \le n$ with a certain $n \in \mathbb{N}$.

If $\bar{A}_{n+1}$ is in $\bar{\Gamma}(O)$, we proceed as in (3) and find some $\bar{A}_j$, $\bar{A}_l$, $\bar{A}_m$, and $\bar{D}$ ($j, l, m < n+1$) such that $\bar{A}_j$ and $\bar{A}_l$ are in the same coset, $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in the list **Gen** (hence, $\bar{A}_j \cdot \bar{A}_l^{-1} \in G$), $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$ and $\bar{A}_m = \bar{A}_l \cdot \bar{D}$. $\bar{A}_m$ is in the same coset as $\bar{A}_{n+1}$, thus it is an element of $\bar{\Gamma}(O)$. By the assumption, $\bar{A}_m$ is then also in G. Hence, we have

$$\bar{A}_{n+1} = \bar{A}_j \cdot \bar{A}_l^{-1} \cdot \bar{A}_l \cdot \bar{D} = (\bar{A}_j \cdot \bar{A}_l^{-1}) \cdot \bar{A}_m \in G. \qquad \square$$

Now—knowing $\bar{\Gamma}(O)$—it is easy to determine $\Gamma(O)$. We just have to distinguish the two cases, whether $-I$ is in $\Gamma(O)$ or not.

Algorithm 4. Calculation of $\Gamma(O)$.

Given: Origami O.

Calculate **Gen** and **Rep**.
Let **Gen′** and **Rep′** be empty lists.
Check, whether $-I \in \Gamma(O)$.
If yes: For each $\bar{A} \in$ **Gen** add A to **Gen′**.
Add $-I$ to **Gen′**.
For each $\bar{A} \in$ **Rep** add A to **Rep′**.
If no: For each $\bar{A} \in$ **Gen**, check whether $A \in \Gamma(O)$.
If it is, add A to **Gen′**; if it is not, add $-A$ to **Gen′**.
For each $\bar{A} \in$ **Rep** add A and $-A$ to **Rep′**.

Result: **Gen′**: list of generators of $\Gamma(O)$,
Rep′: list of right coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Example 3.5. (for $O = L(2,3)$)
(1) Result of calculating $\bar{\Gamma}(O)$:

Gen:

$$\begin{aligned}
\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} &= \bar{T}^3, \\
\begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix} &= \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-1}\bar{T}^{-1}, \\
\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} &= \bar{T}\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}, \\
\begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} &= \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}^{-1}\bar{T}^{-2}
\end{aligned}$$

is a list of generators of $\bar{\Gamma}(O)$.
Rep :

$$\bar{I},\ \bar{T},\ \bar{S},\ \bar{T}^2,\ \bar{T}\bar{S},\ \bar{S}\bar{T},\ \bar{T}^2\bar{S},\ \bar{T}\bar{S}\bar{T},\ \bar{T}^2\bar{S}\bar{T}$$

is a system of coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{SL}_2(\mathbb{Z})$. (The algorithm produces more generators (compare Example 3.7). We eliminated redundant ones.)

(2) Result of calculating $\Gamma(O)$: $(-I \in \Gamma(O))$

$$\begin{aligned}
\mathbf{Gen}' &= \mathbf{Gen} \cup \{-I\}. \\
\mathbf{Rep}' : &= I,\ T,\ S,\ T^2,\ TS,\ ST,\ T^2S,\ TST,\ T^2ST.
\end{aligned}$$

Hence, $\Gamma(O)$ is a subgroup of index 9 in $\mathrm{SL}_2(\mathbb{Z})$.

3.4 Geometrical Type of $\mathbb{H}/\bar{\Gamma}(O)$

The group $\bar{\Gamma}(O)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and of finite index (Corollary 2.9); thus it operates as a Fuchsian group (via Möbius transformations) on $\mathbb{H}$, and $V := \mathbb{H}/\bar{\Gamma}(O)$ is an affine algebraic curve. This curve is defined over $\bar{\mathbb{Q}}$ by the Theorem of Belyi: we have a covering from $\mathbb{H}/\bar{\Gamma}(O)$ to $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{A}^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) - \{\infty\}$ that is ramified, at most, over the images of i and $\rho = \frac{1}{2} + (\frac{1}{2}\sqrt{3})i$. Thus, by Belyi's theorem, the projective curve $\overline{\mathbb{H}/\bar{\Gamma}(O)}$ and hence also the origami curve C (defined in Section 1) is defined over $\overline{\mathbb{Q}}$. In the following, we want to determine the genus and the number of points at infinity of $V = \mathbb{H}/\bar{\Gamma}(O)$.

Let $\Delta := \Delta(P_0, P_1, P_\infty)$ be the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, i.e., the hyperbolic pseudo-triangle with vertices $P_0 := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $P_1 := \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $P_\infty := i\infty$.

We denote by $\bar{A}$ the Möbius transformation defined by the matrix A. Then $\bar{T}$ and $\bar{S}$ (as Möbius transformations) send P_0P_∞ to P_1P_∞, and respectively P_0P_1 to itself (fixing i).

Let **Rep** $= \{\bar{A}_1, \ldots, \bar{A}_k\}$ be the system of right coset representatives we calculated in Section 3.3. Then

$$F := \bigcup_{i=1}^{k} \bar{A}_i(\Delta)$$

is a simply connected fundamental domain of $\bar{\Gamma}(O)$.

The list **Gen** of generators defines how to glue the edges of F to obtain $\mathbb{H}/\bar{\Gamma}(O)$. We get a triangulation of $\mathbb{H}/\bar{\Gamma}(O)$ (compare Figure 7). We calculate the numbers t, e, v of the triangles, the edges and the vertices of this triangulation as described in the following algorithm. Furthermore, the vertices defined by translates of P_∞ are exactly the cusps of $\mathbb{H}/\bar{\Gamma}(O)$. We denote their number by $\hat{v}$. Thus (using the formula of Euler for calculating the genus) we get the result in Remark 3.6.

Algorithm 5. Determining the geometrical type of $\mathbb{H}/\bar{\Gamma}(O)$.

Generate a list of triangles $L := \{\bar{A}_1(\Delta), \ldots, \bar{A}_k(\Delta)\}$. In the triangle $\bar{A}_i(\Delta)$ we call $\bar{A}_i(P_0)\bar{A}_i(P_1)$ (the image of the edge P_0P_1) the "S-edge." Similarly, we call $\bar{A}_i(P_1)\bar{A}_i(P_\infty)$ the "T-edge" and $\bar{A}_i(P_0)\bar{A}_i(P_\infty)$ the "T^{-1}-edge."

For each $i, j \in \{1, \ldots, k\}$ identify

- the T-edge of $\bar{A}_j(\Delta)$ with the T^{-1}-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}$, i.e., if $(\bar{A}_j\bar{T})\bar{A}_i^{-1} \in \bar{\Gamma}(O)$;
- the T^{-1}-edge of $\bar{A}_j(\Delta)$ with the T-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}^{-1}$; and
- the S-edge of $\bar{A}_j(\Delta)$ with the S-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{S}$.
  If an S-edge of some triangle $\bar{A}_j(\Delta)$ is identified with itself (i.e., $i = j$) create an additional triangle. Add a vertex in the middle of this S-edge and add an edge from this new vertex to the opposite vertex in the triangle $\bar{A}_j(\Delta)$. (Compare triangle $\bar{T}^2\bar{S}\bar{T}$ in Figure 7). This is done to get, in the end, a triangulation of the surface.

$t :=$ number of triangles. $e :=$ number of edges.
$v :=$ number of vertices. $\hat{v} :=$ number of vertices that are endpoints of T-edges.
$g := \frac{2-(v-e+t)}{2}$.

Result: g: genus of $\mathbb{H}/\bar{\Gamma}(O)$, $\hat{v}$: number of vertices at infinity of $\mathbb{H}/\bar{\Gamma}(O)$.

Remark 3.6. Let t, e, v, and $\hat{v}$ be the numbers of triangles, edges, vertices, and marked vertices as calculated in Algorithm 5. Then $\mathbb{H}/\bar{\Gamma}(O)$ is an affine curve of genus $g = \frac{2-(v-e+t)}{2}$ with $\hat{v}$ cusps.

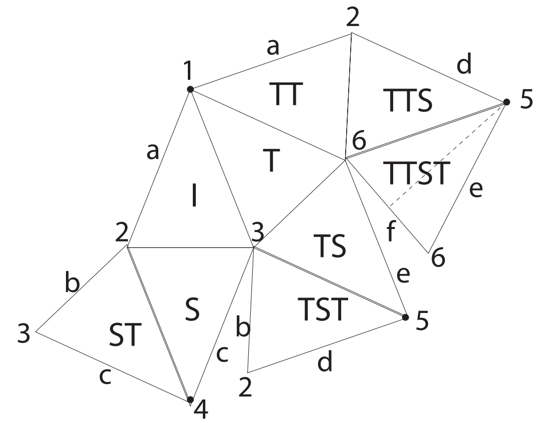


FIGURE 7. Fundamental domain of $\bar{\Gamma}(L(2,3))$.

Example 3.7. (for $O = L(2,3)$)
Rep: $\bar{I}, \bar{T}, \bar{T}^2, \bar{T}^2\bar{S}, \bar{T}^2\bar{S}\bar{T}, \bar{T}\bar{S}, \bar{T}\bar{S}\bar{T}, \bar{S}, \bar{S}\bar{T}$.
Gen: $a := \bar{T}^3$, $b := \bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-1}$,
$c := \bar{S}\bar{T}^2\bar{S}$, $d := \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-2}$,
$e := \bar{T}\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^{-2}$, $f := \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-2}$.

In Figure 7, edges with the same letters are glued. In triangle $\bar{T}^2\bar{S}\bar{T}(\Delta)$ an edge and a vertex were added, since the "S-edge" is glued to itself. Vertices with the same numbers are identified. Vertices at infinity are marked by a filled circle.

Thus, $t = 9+1, e = 14+1, v = 6+1, \hat{v} = 3$.

Result: $g = 0, \hat{v} = 3$. Hence,

$$\mathbb{H}/\bar{\Gamma}(L(2,3)) \cong \mathbb{P}^1 - \{0, 1, \infty\}.$$

Proposition 3.8. $\Gamma(L(2,3))$ *is not a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$.

Recall that a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence group* if it contains a principal congruence group

$$\Gamma(n) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv 1, b \equiv 0, c \equiv 0, d \equiv 1 \bmod n\}$$

for some $n \in \mathbb{N}$.

For the proof of Proposition 3.8 we will use the results of Wohlfahrt. Define the *general level* of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ to be the least common multiple of the amplitudes of the cusps. Then Theorem 2 in [Wohlfahrt 64] states that if Γ is a congruence group of general level m then $\Gamma(m)$ is contained in Γ.

Proof: (of Proposition 3.8) We have from Example 3.5 that

$$\Gamma(L(2,3)) = \left\langle \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}-1 & 3\\ -2 & 5\end{pmatrix}, \begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \right\rangle.$$

$\mathbb{H}/\bar{\Gamma}(L(2,3))$ has three cusps represented in Figure 7 by the vertices 1, 4, and 5. T^3, ST^2S^{-1}, and $TST^4S^{-1}T^{-1}$ are their respective parabolic elements, and 3, 2, and 4 their respective amplitudes. Hence, the general level m of $\Gamma(L(2,3))$ is $\mathrm{lcm}(3,2,4) = 12$.

Suppose that $\Gamma(L(2,3))$ is a congruence subgroup. By Wohlfahrt's theorem we would have:

$$\Gamma(12) \subseteq \Gamma(L(2,3)). \qquad (3\text{–}1)$$

Let $p : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ be the natural projection. Then we have

$$p(\bar{\Gamma}(L(2,3))) =< \begin{pmatrix}\bar{1} & \bar{0}\\ \bar{2} & \bar{1}\end{pmatrix}, \begin{pmatrix}\bar{0} & \bar{1}\\ \bar{2} & \bar{0}\end{pmatrix} >= \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}).$$

Hence the diagram in Figure 8 is commutative with $N := \bar{\Gamma}(L(2,3)) \cap \bar{\Gamma}(3)$.

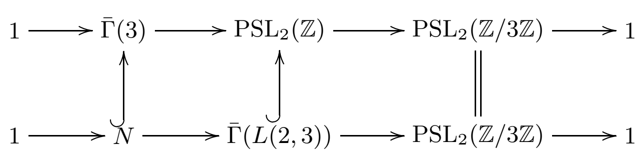


FIGURE 8. Diagram 4.

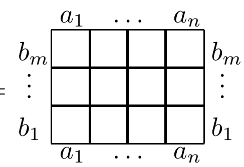
Since the index $[\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}(L(2,3))]$ of $\bar{\Gamma}(L(2,3))$ in $\mathrm{PSL}_2(\mathbb{Z})$ is 9, it follows from Diagram 4 in Figure 8 that $[\bar{\Gamma}(3) : N] = 9$.

By (3–1), $\bar{\Gamma}(12) \subseteq N \subseteq \bar{\Gamma}(3)$. But $[\bar{\Gamma}(3) : \bar{\Gamma}(12)] = 2^4 \cdot 3$ (using [Shimura 71, 1.6.20]). Thus $[\bar{\Gamma}(3) : N] = 9$ would have to be a factor of $2^4 \cdot 3$. Contradiction! □

Hubert and Lelièvre have recently found a generalization of this result to infinitely many origamis of genus 2 ([Hubert and Lelièvre 04b]).

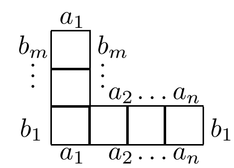
4. SOME EXAMPLES

4.1 Trivial Origamis

$O =$ 

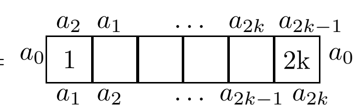
$$\Gamma(O) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | b \equiv 0 \bmod n', c \equiv 0 \bmod m'\}$$
where $t := \gcd(m,n), n' := n/t, m' := m/t$.

4.2 *L*-Sequence

$L(n,m) =$ 

Origami	Index	Genus	♯ Cusps
$L(2,2)$	3	0	2
$L(2,3)$	9	0	3
$L(2,4)$	18	0	5
$L(2,5)$	36	0	8
$L(2,6)$	54	0	10
$L(2,7)$	108	1	17
$L(3,3)$	9	0	3
$L(4,4)$	54	0	10

4.3 Cross-Sequence

$O_{2k} =$ 

Origami	Index	Genus	♯ Cusps
O_2	3	0	2
O_4	6	0	3
O_6	12	0	4
O_8	24	0	6
O_{10}	36	0	8
O_{12}	48	0	10
O_{14}	72	1	12
O_{16}	96	2	14

4.4 Remarks

As in Example 3.1 edges labelled with the same letters are glued. The tables in Sections 4.2 and 4.3 itemize, for an origami O, the index of the projective Veech group $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$ and the genus and number of cusps of $\mathbb{H}/\bar{\Gamma}(O)$.

For the example in Section 4.1, $\Gamma(O)$ can be determined using Proposition 2.1.

The sequence in Section 4.2 was introduced to me by Pierre Lochak. The Veech group example of $L(2,2)$ is also given in [Möller 03]. This sequence is also studied in detail in [Hubert and Lelièvre 04a] and example estimates for the growth of the genus and the number of cusps are obtained. The Veech groups in this sequence are in general not congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (see Proposition 3.8 and the comment following it).

On the contrary one can show—again using Proposition 2.1—that the Veech groups $\Gamma(O_{2k})$ in Section 4.3 are congruence subgroups for all $k \in \mathbb{N}$. Furthermore the genus of the curve $\mathbb{H}/\Gamma(O_{2k})$ is not bounded.[3]

Only a few general statements about Veech groups of origamis are known. There seems to be no obvious relation between the index d of the origami $O = (p : X \to E^\star)$ and the index of its Veech group. In particular, it follows from Proposition 2.1 that each characteristic subgroup of F_2 defines an origami with Veech group $\mathrm{SL}_2(\mathbb{Z})$. (Two nontrivial examples defined by coverings $p : X \to E^\star$ of degree 8 and 108 are calculated explicitly by Frank Herrlich.) Hence, there is a cofinal system of origamis having the full group $\mathrm{SL}_2(\mathbb{Z})$ as Veech group.

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