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Stability of Equilibrium Points In The Photogravitational Two-Body Problem

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Abstract

This paper deals with the two-body problem in a photogravitational field, using a model introduced by Constantin Popovici (1923). Using Valeev and Scheglov's (1965) transformations, the differential equations of motion are obtained under an nonlinear autonomous form. The equilibrium solutions of this system of equations are determined and their stability is investigated. Numerical simulations are also considered.

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Key words: two-body problem, photogravitational field, equilibrium, stability

1 Introduction

It is known that sunlight exerts a force on illuminated particles; this explains why the comets' tails are oriented in an opposite direction to the Sun. This idea has been known since J.Kepler. In the early twenties, P.N. Lebedev, E.F.Nichols and A.W.Hull have measured the pressure of the light on a reflecting surface. Although this force is very small, it is unlimited and it acts continuously. For this reason, its perturbation on the movement of a small body in a gravitational field might be significant.

2 Basic equations

According to the model introduced by C. Popovici (1923), A. Pal (1992), the magnitude of the photogravitational force is:

$$F = -\frac{A}{r^2} + \frac{R}{r^2} - R\frac{\dot{r}}{cr^2}, \tag{1}$$

where $-A/r^2$ represents the newtonian force (with A - the newtonian force at the unit of distance $r = 1$), R/r^2 the repulsion force of the light at the unit distance and $-R\dot{r}/(cr^2)$ a corrective term due to the finite speed of light.

Equation (1) can be written as:

$$F = -\frac{k}{r^2}(1 + \varepsilon\dot{r}) \tag{2}$$

where

$$k = A - R, \; \varepsilon = \frac{R}{ck}. \tag{3}$$

Anisiu (1995) has used for (2) the following expression:

$$F = -\frac{1}{r^2}(k + l\dot{r}), \tag{4}$$

where

$$l = \frac{R}{c} \tag{5}$$

Note that (4) is not singular at $k = 0$ (when repulsion equals the attraction).

In polar coordinates (r, θ), the equations of motion take the form:

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -(k + l\dot{r})\frac{1}{r^2}, \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0. \end{aligned} \tag{6}$$

From the second equation of (6), we have $r^2\dot{\theta} = H$, where H is the constant of the area's first integral.

Another form of (6) can be obtained using the following transformation (Valeev et al. (1965); Barbosu (2000), Mioc et al. (2001)):

$$x = \frac{k}{r^3\dot{\theta}^2}, \quad y = \frac{\dot{r}}{r\dot{\theta}} \tag{7}$$

Using the area's first integral and (7), equations (6) become:

$$\begin{aligned} \frac{dx}{d\theta} &= P(x, y) = xy, \\ \frac{dy}{d\theta} &= Q(x, y) = 1 - x - \varepsilon y + y^2, \end{aligned} \tag{8}$$

where $\epsilon = l/H$. Note that the above system of equations is autonomous, i.e. the independent variable θ does not appear explicitly in the right hand side of (8).

3 Equilibrium points and stability of motion

The equilibrium points of (8) are given by:

$$\begin{aligned} xy &= 0, \\ 1 - x - \varepsilon y + y^2 &= 0. \end{aligned} \tag{9}$$

whose solutions are:

$$\{x = 1,\, y = 0\}, \quad \{x = 0,\, y = \frac{\varepsilon - \sqrt{\varepsilon^2 - 4}}{2}\}, \tag{10}$$

$$\{x = 0,\ y = \frac{\varepsilon + \sqrt{\varepsilon^2 - 4}}{2}\}$$

According to the possible values of ε, we have the following cases:

a) For $\varepsilon > 2$ there are three equilibrium points:

$$\{0, 1\}, \quad \{0, \frac{\varepsilon - \sqrt{\varepsilon^2 - 4}}{2}\}, \quad \{0, \frac{\varepsilon + \sqrt{\varepsilon^2 - 4}}{2}\}$$

b) For $\varepsilon = 2$ there are two equilibrium points:

$$\{1, 0\}, \quad \{0, 1\}$$

c) For $\varepsilon < 2$ there is one equilibrium point:

$$\{1, 0\}$$

In what follows we will investigate the stability around the equilibrium points. Let us consider small variations around the equilibrium point $\{x_0, y_0\}$. We have (Jordan et al. (1986), Zeldovich et al. (1985)):

$$x = x_0 + \xi, \quad y = y_0 + \eta, \tag{11}$$

Thus,

$$\begin{aligned} \frac{d\xi}{d\theta} &= a\xi + b\eta, \\ \frac{d\eta}{d\theta} &= c\eta + d\eta, \end{aligned} \tag{12}$$

where $a = y_0, \quad b = x_0, \quad c = -1, \quad d = 2y_0 - \varepsilon$.

Let:

$$\xi = \alpha e^{\lambda\theta}, \quad \eta = \beta e^{\lambda\theta}, \tag{13}$$

be the solutions of (12), where α and β are constants. The characteristic equation for λ is:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0, \tag{14}$$

or

$$\lambda^2 - (3y_0 - \varepsilon)\lambda + ({2y_0}^2 - \varepsilon y_0 + x_0) = 0, \tag{15}$$

with:

$$\lambda_{1,2} = \frac{(3y_0 - \varepsilon) \pm \sqrt{(y_0 - \varepsilon)^2 - 4x_0}}{2} \tag{16}$$

For the equilibrium points situated on the Oy-axis ($x_0 = 0$, $y_0 \neq 0$),

$$\lambda_1 = 2y_0 - \varepsilon, and \quad \lambda_2 = y_0, \tag{17}$$

For the equilibrium points situated on the Ox-axis ($x_0 \neq 0$, $y_0 = 0$),

$$\lambda_{1,2} = \frac{-\varepsilon \pm \sqrt{\varepsilon - 4x_0}}{2} \tag{18}$$

According to (17) and (18) we can draw the following conclusions:

For $\varepsilon < 2$, the equilibrium point $(1,0)$ is asymptotically stable.

For $\varepsilon = 2$, the equilibrium point $(1,0)$ is asymptotically stable and the point $(0,1)$ is unstable.

For $\varepsilon > 2$, the point $(1,0)$ is asymptotically stable and the other two points $(0, \frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2})$ and $(0, \frac{\varepsilon-\sqrt{\varepsilon^2-4}}{2})$ are unstable.

4 Phase portrait

In what follows we'll present the phase portraits for the three cases described in the previous section.

For $\varepsilon < 2$, we have:

One can see here the asymptotically stable equilibrium point: $(1,0)$.

For $\varepsilon = 2$, the phase portrait is:

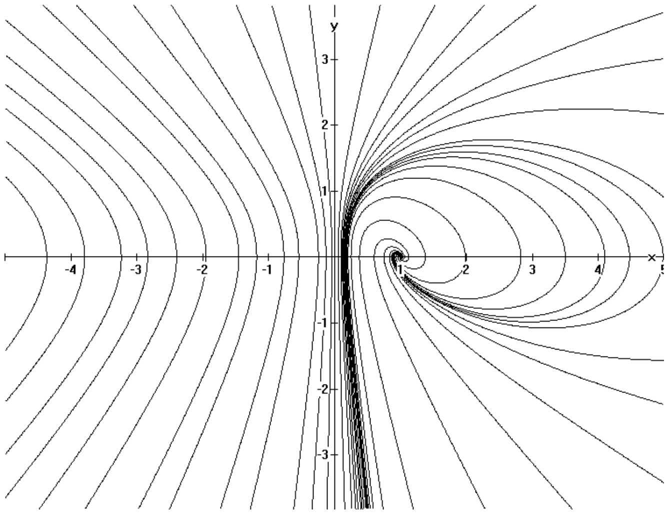

Figure 1: $\varepsilon < 2$

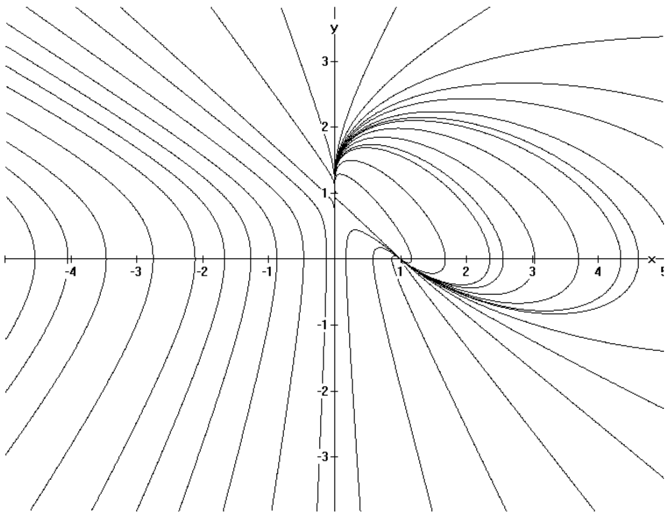

Figure 2: $\varepsilon = 2$

Here, the two equilibrium points are $(1, 0)$ and $(0, 1)$.

For $\varepsilon > 2$, we can see that we have three equilibrium points: the first $(1, 0)$, which is asymptotically stable and the other two $(0, \frac{\varepsilon+\sqrt{\varepsilon^2-4}}{2})$ and $(0, \frac{\varepsilon-\sqrt{\varepsilon^2-4}}{2})$ that are unstable:

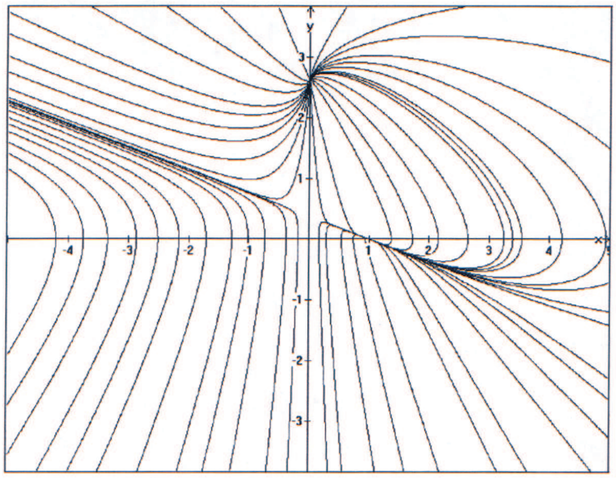

Figure 3: $\varepsilon > 2$

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