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Some Cancellation Ideal Rings

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

An ideal I of a commutative ring R is called a *cancellation ideal* of R if for any ideals A, B of $R, AI = BI$ implies $A = B$, and we call R a *cancellation ideal ring* if every nonzero ideal of R is a cancellation ideal. Our purpose is to show that the ring $(m\mathbb{Z}, +, \cdot)$ is always a cancellation ideal ring and the nontrivial ring $(m\mathbb{Z}_n, +, \cdot)$ is a cancellation ideal ring if and only if $\dfrac{n}{(m,n)}$ is a prime and $n \nmid (m,n)^2$ where (m,n) denotes the g.c.d. of m and n.

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1 Introduction

Let $\mathbb{Z}$ denote the ring of integers and $\mathbb{Z}_n$ the ring of integers modulo n. The set of all positive integers will be denoted by $\mathbb{N}$. Recall that $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}$. The cardinality of a set X is denoted by $|X|$.

For nonempty subsets A and B of a ring R, let AB be the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$. Note that if A and B are ideals of R, then so is AB.

An introduction to cancellation ideals may be found in [2]. The following definition of cancellation ideals was provided in [1] by D. D. Anderson and M. Roitman. An ideal I of a commutation ring R with identity is called a *cancellation ideal* of R if for any ideals A and B of R, $AI = BI$ implies $A = B$. D. D. Anderson and M. Roitman [1] mentioned the following fact which is easily seen.

Proposition 1.1. ([1]) *Let R be a commutative ring with identity. Then for $a \in R \smallsetminus \{0\}$, the principal ideal aR of R is a cancellation ideal if and only if a is not a zero divisor of R.*

In fact, D. D. Anderson and M. Roitman [1] gave a necessary and sufficient condition for an ideal of a commutative ring with identity to be a cancellation ideal. However, it is not easily seen from this characterization whether a given ideal of R is a cancellation ideal. We can see from the definition of cancellation ideals of a commutative ring with identity that R itself is a cancellation ideal of R since $AR = A$ for every ideal A of R. Some results of cancellation ideals of certain commutative rings with identity can be seen in [3].

In this paper, the definition of cancellation ideals is given analogously for any commutative ring with or without identity. Then a commutative ring R need not be a cancellation ideal of itself. An obvious example is a nontrivial zero ring. If R is a Boolean ring, then $AR = A$ for every ideal A of R, so R is a cancellation ideal of itself. A Boolean ring is known to be a commutative ring. However, it need not have an identity. If R is the subring of the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ consisting of all finite sequences $(x_1, x_2, x_3, \ldots) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$, that is, $x_i = 0$ for all but a finite number of $i \in \mathbb{N}$, then R is a Boolean ring without identity.

Observe that the ideal $\{0\}$ of a nontrivial commutative ring R is not a cancellation ideal of R. By a *cancellation ideal ring* we mean a commutative ring of which all nonzero ideals are cancellation ideals. It then follows from Proposition 1.1 that every PID is a cancellation ideal ring. In particular, the ring $\mathbb{Z}$ and the polynomial ring $F[x]$ over a field F are cancellation ideal rings with identity.

It is well-known that the set of subrings and the set of ideals of the ring $\mathbb{Z}$ coincide, and it is precisely $\{m\mathbb{Z} \mid m \in \mathbb{N} \cup \{0\}\}$. Let $\varphi : \mathbb{Z} \to \mathbb{Z}_n$ be defined by $\varphi(x) = \overline{x}$ for all $x \in \mathbb{Z}$. Then φ is an epimorphism of the ring $\mathbb{Z}$ onto the ring $\mathbb{Z}_n$. Thus $\varphi(m\mathbb{Z}) = m\mathbb{Z}_n$ for all $m \in \mathbb{N} \cup \{0\}$. If I is an ideal of $\mathbb{Z}_n$, then $\varphi^{-1}(I) = m\mathbb{Z}$ for some $m \in \mathbb{N} \cup \{0\}$, so $I = \varphi(\varphi^{-1}(I)) = \varphi(m\mathbb{Z}) = m\mathbb{Z}_n$. Hence the set of subrings and the set of ideals of the ring $\mathbb{Z}_n$ are identical and it is $\{m\mathbb{Z}_n \mid m \in \mathbb{N} \cup \{0\}\}$. Observe that for every $m \in \mathbb{N} \cup \{0\}$, $(-m)\mathbb{Z} = m(-\mathbb{Z}) = m\mathbb{Z}$ and $(-m)\mathbb{Z}_n = m(-\mathbb{Z}_n) = m\mathbb{Z}_n$. It is easily seen that $m\mathbb{Z}_n = \{\overline{0}\}$ if and only if $n \mid m$.

Our purpose is to show that the ring $m\mathbb{Z}$ is always a cancellation ideal ring and a nontrivial ring $m\mathbb{Z}_n$, that is, $n \nmid m$, is a cancellation ideal ring if and only if $\dfrac{n}{(m,n)}$ is a prime and $n \nmid (m,n)^2$ where (m,n) denotes the g.c.d. of m and n.

We give here a basic property of $m\mathbb{Z}_n$ which will be used.

Proposition 1.2. *For any $n, m \in \mathbb{N}$,*

$$m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \left\{\overline{0}, \overline{(m,n)}, 2\overline{(m,n)}, \ldots, \left(\frac{n}{(m,n)} - 1\right)\overline{(m,n)}\right\},$$
$$|m\mathbb{Z}_n| = \frac{n}{(m,n)}.$$

Proof. We know that $(m,n) = mx + ny$ for some $x, y \in \mathbb{Z}$. Then

$$\begin{aligned} m\mathbb{Z}_n &= (m,n)\big(\frac{m}{(m,n)}\big)\mathbb{Z}_n \\ &\subseteq (m,n)\mathbb{Z}_n \\ &= (mx+ny)\mathbb{Z}_n \\ &= m(x\mathbb{Z}_n) \\ &\subseteq m\mathbb{Z}_n. \end{aligned}$$

Hence $m\mathbb{Z}_n = (m,n)\mathbb{Z}_n$. We can see that $(m,n)\mathbb{Z}_n = \{x\overline{(m,n)} \mid x \in \mathbb{Z}\}$. If $x \in \mathbb{Z}$, then $x = \frac{n}{(m,n)}q + r$ for some $q, r \in \mathbb{Z}$ and $0 \le r < \frac{n}{(m,n)}$, so $x\overline{(m,n)} = \big(\frac{n}{(m,n)}q + r\big)\overline{(m,n)} = r\overline{(m,n)}$. Thus the second equality holds. If $i, j \in \big\{0, 1, 2, \ldots, \frac{n}{(m,n)} - 1\big\}$ are such that $i \ge j$ and $i\overline{(m,n)} = j\overline{(m,n)}$, then $0 \le i - j < \frac{n}{(m,n)}$ and $n \mid (i-j)(m,n)$ and hence $\frac{n}{(m,n)} \mid i - j$. This implies that $i - j = 0$, so $i = j$. Therefore we deduce that $|m\mathbb{Z}_n| = \frac{n}{(m,n)}$.

2 The Rings $m\mathbb{Z}$ and $m\mathbb{Z}_n$

To prove that the ring $m\mathbb{Z}$ is a cancellation ideal ring, the following lemma is needed.

Lemma 2.1. *Let $m \in \mathbb{N}$ and $I \subseteq m\mathbb{Z}$. Then I is an ideal of the ring $m\mathbb{Z}$ if and only if $I = mk\mathbb{Z}$ for some $k \in \mathbb{N} \cup \{0\}$.*

Proof. Assume that I is an ideal of the ring $m\mathbb{Z}$. Then I is a subring of the ring $m\mathbb{Z}$. But $m\mathbb{Z}$ is a subring of the ring $\mathbb{Z}$, thus I is a subring of the ring $\mathbb{Z}$. This implies that I is an ideal of the ring $\mathbb{Z}$. Hence $I = x\mathbb{Z}$ for some $x \in \mathbb{N} \cup \{0\}$. Since $I = x\mathbb{Z} \subseteq m\mathbb{Z}$, it follows that $x = mk$ for some $k \in \mathbb{N} \cup \{0\}$. Consequently, $I = mk\mathbb{Z}$.

Since $mk\mathbb{Z}$ is an ideal of the ring $\mathbb{Z}$ and $mk\mathbb{Z} \subseteq m\mathbb{Z}$, the converse holds.

Theorem 2.2. *For every $m \in \mathbb{N}$, the ring $m\mathbb{Z}$ is a cancellation ideal ring.*

Proof. Let I be a nonzero ideal of the ring $m\mathbb{Z}$ and let A and B be ideals of $m\mathbb{Z}$ such that $AI = BI$. By Lemma 2.1, $I = mx\mathbb{Z}, A = my\mathbb{Z}$ and $B = mz\mathbb{Z}$ for some $x, y, z \in \mathbb{Z}$ and $x \neq 0$. Thus $mymx\mathbb{Z} = AI = BI = mzmx\mathbb{Z}$. But $mx \neq 0$, so $my\mathbb{Z} = mz\mathbb{Z}$, that is, $A = B$.

Remark 2.3. Since the subrings and the ideals of $\mathbb{Z}_n$ coincide and $m\mathbb{Z}_n = \overline{m}\mathbb{Z}_n$, it is easy to see from the proof of Lemma 2.1 that Lemma 2.1 still holds if we replace $\mathbb{Z}$ by $\mathbb{Z}_n$. However, $m\mathbb{Z}_n$ need not be a cancellation ideal ring. This can be seen later.

Next, we shall characterize when the ring $m\mathbb{Z}_n$ is a cancellation ideal ring. First, we determine when it has an identity.

Lemma 2.4. *For every $m, n \in \mathbb{N}$, the ring $m\mathbb{Z}_n$ has an identity if and only if $\dfrac{n}{(m,n)}$ and m are relatively prime.*

Proof. Note that $m\mathbb{Z}_n = \{\overline{mx} \mid x \in \mathbb{Z}\}$. First assume that the ring $m\mathbb{Z}_n$ has an identity, say $\overline{ma}$. Then $\overline{ma}\,\overline{m} = \overline{m}$ which implies that $n \mid m^2a - m$. Hence $\dfrac{n}{(m,n)} \mid \dfrac{m}{(m,n)}(ma-1)$. But $\dfrac{n}{(m,n)}$ and $\dfrac{m}{(m,n)}$ are relatively prime, so $\dfrac{n}{(m,n)} \mid ma - 1$. Consequently, $\dfrac{n}{(m,n)}x + ma = 1$ for some $x \in \mathbb{Z}$. Thus $\dfrac{n}{(m,n)}$ and m are relatively prime.

Conversely, assume that $\dfrac{n}{(m,n)}$ and m are relatively prime. Then there are $s, t \in \mathbb{Z}$ such that $\frac{n}{(m,n)}s + mt = 1$. Hence for every $x \in \mathbb{Z}$,

$$\begin{aligned}\overline{mx}\ \overline{mt} &= \overline{mx}\ \overline{\left(1 - \frac{n}{(m,n)}s\right)} \\ &= \overline{mx} - \overline{n\frac{m}{(m,n)}xs} \\ &= \overline{mx}\end{aligned}$$

which implies that $\overline{mt}$ is the identity of the ring $m\mathbb{Z}_n$.

Theorem 2.5. *Let $m, n \in \mathbb{N}$ be such that $n \nmid m$. Then the ring $m\mathbb{Z}_n$ is a cancellation ideal ring if and only if $\frac{n}{(m,n)}$ is a prime and $n \nmid (m,n)^2$. If this is the case, $m\mathbb{Z}_n$ is the ring with identity and has exactly two ideals.*

Proof. Since $n \nmid m$, $\frac{n}{(m,n)} > 1$. Let $d = \frac{n}{(m,n)}$. By Proposition 1.2, $m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \{\overline{0}, \overline{(m,n)}, 2\overline{(m,n)}, \ldots, (d-1)\overline{(m,n)}\}$ and $|m\mathbb{Z}_n| = d$.

First, assume that the ring $m\mathbb{Z}_n$ is a cancellation ideal ring. Since $(m,n)\mathbb{Z}_n = m\mathbb{Z}_n \neq \{\overline{0}\}$, it follows that $((m,n)\mathbb{Z}_n)^2 \neq \{\overline{0}\}$. But $((m,n)\mathbb{Z}_n)^2 = (m,n)^2\mathbb{Z}_n$, thus $n \nmid (m,n)^2$. To show that d is a prime, suppose not. Then $d = lk$ for some $l, k \in \mathbb{N}$ with $1 < l, k < d$. Let $I = l(m,n)\mathbb{Z}_n$ and $J = k(m,n)\mathbb{Z}_n$. Then by Lemma 2.1, I and J are ideals of the ring $(m,n)\mathbb{Z}_n$. Since $n = d(m,n) > l(m,n) > 0$ and $n > k(m,n) > 0$, it follows that $I = l(m,n)\mathbb{Z}_n \neq \{0\}$ and $J = k(m,n)\mathbb{Z}_n \neq \{0\}$. But $IJ = l(m,n)k(m,n)\mathbb{Z}_n = d(m,n)(m,n)\mathbb{Z}_n = n(m,n)\mathbb{Z}_n = \{\overline{0}\}$, so I and J are not cancellation ideals of $(m,n)\mathbb{Z}_n$. This shows that d must be a prime.

Conversely, assume that d is a prime and $n \nmid (m,n)^2$. Since $|m\mathbb{Z}_n| = d$, it follows that $\{\overline{0}\}$ and $m\mathbb{Z}_n$ are the only ideals of the ring $m\mathbb{Z}_n$. We shall show that $m\mathbb{Z}_n$ has an identity. This result implies that $m\mathbb{Z}_n$ is a cancellation ideal of itself. Since $n \nmid (m,n)^2$, we have that $\frac{n}{(m,n)} \nmid (m,n)$. But $\frac{n}{(m,n)}$ and $\frac{m}{(m,n)}$ are relatively prime, so $\frac{n}{(m,n)} \nmid \frac{m}{(m,n)}(m,n)$. Hence

$\dfrac{n}{(m,n)} \nmid m$. Since $\dfrac{n}{(m,n)}$ is a prime, it follows that $\dfrac{n}{(m,n)}$ and m are relatively prime. Hence by Lemma 2.4, the ring $m\mathbb{Z}_n$ has an identity.

Therefore the proof is complete.

A direct consequence of Theorem 2.5 is as follows :

Corollary 2.6. *The ring $\mathbb{Z}_n$ is a cancellation ideal ring if and only if either $n = 1$ or n is a prime.*

Example 2.7. Since $2\mathbb{Z}_4 = \{\overline{0}, \overline{2}\}$ and $3\mathbb{Z}_6 = \{\overline{0}, \overline{3}\}$, we can check directly that $2\mathbb{Z}_4$ is a zero ring and $3\mathbb{Z}_6$ has an identity. These imply that $2\mathbb{Z}_4$ is not a cancellation ideal ring but $3\mathbb{Z}_6$ is a cancellation ideal ring.

Next, consider the ring $4\mathbb{Z}_{30}$. Then $4\mathbb{Z}_{30} = 2\mathbb{Z}_{30}$ by Proposition 1.2. Since $\dfrac{30}{(4,30)} = 15$ and 4 are relatively prime, by Lemma 2.4, the ring $4\mathbb{Z}_{30}$ has an identity. From that $-15 + (4 \times 4) = 1$ and the proof of Lemma 2.4, we have that $\overline{4 \times 4} = \overline{16}$ is the identity of $4\mathbb{Z}_{30}$. But $\dfrac{30}{(4,30)} = 15$ is not a prime, so by Theorem 2.5, $4\mathbb{Z}_{30}$ is not a cancellation ideal ring. Since $15 = 3 \times 5$, we can see from the proof of Theorem 2.5 that the nonzero ideals $6\mathbb{Z}_{30}$ and $10\mathbb{Z}_{30}$ of $4\mathbb{Z}_{30}$ are not cancellation ideals.

Remark 2.8. As was mentioned in Section 1, the subring R of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ consisting of all finite sequences of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ is a Boolean ring without identity. Let $\overline{R}$ be the ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$. Then $\overline{R}$ is a Boolean ring with identity. Hence R and $\overline{R}$ are cancellation ideals of R and $\overline{R}$, respectively (see page 2). The ring R is clearly a proper ideal of $\overline{R}$ and $RR = \overline{R}R$. Therefore R is not a cancellation ideal of $\overline{R}$. More generally, every Boolean ring S has no proper cancellation ideal since $II = I = SI$ for every ideal I of S. Since R is a nonzero proper ideal of $\overline{R}$, we have that $\overline{R}$ is not a cancellation ideal ring. In fact, the direct product $\prod_{i \in I} R_i$ of nonzero rings R_i with $|I| > 1$ is never a cancellation ideal ring. That is because for every $k \in I$,

$$\Big(R_k \times \prod_{i\in I\setminus\{k\}}\{0\}\Big)\Big(\{0\}\times \prod_{i\in I\setminus\{k\}} R_i\Big) = \prod_{i\in I}\{0\} = \Big(\prod_{i\in I}\{0\}\Big)\Big(\{0\}\times \prod_{i\in I\setminus\{k\}} R_i\Big)$$

and $R_k \times \prod_{i\in I\setminus\{k\}}\{0\}$ and $\{0\} \times \prod_{i\in I\setminus\{k\}} R_i$ are both nonzero ideals of the ring $\prod_{i\in I} R_i$.

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