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An intermediate point property in the quadrature formulas of type Gauss

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Abstract

In this paper we study a property of the intermediate point (see [1], [2], [3], [6], [7]) from the quadrature formulas of Gauss type.

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1. In [6] B. Jacobson studied a property of the intermediate point which appear in the mean-value theorem for integrals. This property has been studied for others mean-value formulas in the articles [1], [2], [3], and [7]. In this paper we will study this property for two particular cases of the quadrature formulas of Gauss type.

The quadrature formulas of Gauss type have the form (see [5]).

$$(1)\qquad \int_a^b f(x)dx = (b-a)[c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n)] + R_n(f)$$

where $f : [a, b] \to \mathbb{R},\ f \in C^{2n}[a, b]$.

$$R_n(f) = \frac{(n!)^4 \cdot 2^{2n+1}}{[(2n)!]^3(2n+1)} \cdot \left(\frac{b-1}{2}\right)^{2n+1} \cdot f^{(2n)}(\xi_b),\ \xi_b \in (a, b), \tag{2}$$

the nodes $x_i,\ i = \overline{1, n}$ which appears in (1) are given by

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} y_i,\ i = \overline{1, n} \tag{3}$$

where $y_i, i = \overline{1, n}$ are the zeros of the Legendre polynomial

$$L_n(y) = \frac{1}{2^n \cdot n!}[(y^2 - 1)^n]^{(n)},\ n \geq 1 \tag{4}$$

and the coefficients $c_1, c_2, ..., c_n$ from (1) are the solution of the system

$$\begin{cases} c_1 + c_2 + ... + c_n = 1 \\ c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0 \\ c_1 y_1^2 + c_2 y_2^2 + ... + c_n y_n^2 = \frac{1}{3} \\ c_1 y_1^3 + c_2 y_2^3 + ... + c_n y_n^3 = 0 \\ c_1 y_1^4 + c_2 y_2^4 + ... + c_n y_n^4 = \frac{1}{9} \\ .... \end{cases} \tag{5}$$

Remark 1. *The quadrature formula (1) has the algebraic degree of exactness* $2n - 1$.

2. Now, let us consider the quadrature formula (1) for the particular case $n = 2$. Taking into account of the formulas (2), (3), (4) and (5) we obtain:

$$c_1 = c_2 = \frac{1}{2},\ x_1 = \frac{a+b}{2} + \frac{b-a}{2} y_1,\ x_2 = \frac{a+b}{2} + \frac{b-a}{2} y_2,$$

where y_1, y_2 are the zeros of the polynom $L_2(y) = \frac{1}{2}(3y^2 - 1)$, and $R_2(f) = \frac{1}{135}\left(\frac{b-a}{2}\right)^5 \cdot f^{(IV)}(\xi)$, $\xi \in (a, b)$.

In this case we have that if $f : [a,b] \to \mathbb{R}$, $f \in C^4[a,b]$ then for any $x \in (a,b]$ there is $c_x \in (a,x)$ such that

$$\int_a^x f(t)dt = \tag{6}$$

$$= (x-a)\left[\frac{1}{2}f\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right) + \frac{1}{2}f\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\right] +$$

$$+\frac{1}{135}\left(\frac{x-a}{2}\right)^5 f^{(IV)}(c_x)$$

with $y_1^2 = \frac{1}{3}$.

We now prove the following theorem

Theorem 1. *If $f \in C^6[a,b]$ and $f^{(5)}(a) \neq 0$ then for the intermediate point c_x which appears in formula (6) we have*

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}$$

Proof. Let us consider $F, G : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt - (x-a)\left[\frac{1}{2}f\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right) + \frac{1}{2}f\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\right] -$$

$$-\frac{1}{135}\left(\frac{x-a}{2}\right)^5 f^{(IV)}(a),$$

$$G(x) = (x-a)^6.$$

We have that F and G are six times derivable on $[a,b]$, $G^{(i)}(x) \neq 0$, $i = \overline{1,5}$ for any $x \in (a,b]$ and $F^{(k)}(a) = 0$, $G^{(k)}(a) = 0$, $k = \overline{1,5}$.

By using successive l'Hospital rule, we obtain

$$\lim_{x \to a} \frac{F(x)}{G(x)} = \lim_{x \to a} \frac{F^{(VI)}(x)}{G^{(VI)}(x)}.$$

Now, from

$$F^{(VI)}(x) = f^{(V)}(x) - 3f^{(V)}\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right)\left(\frac{1}{2} - \frac{y_1}{2}\right)^5 -$$

$$-3f^{(V)}\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\left(\frac{1}{2} + \frac{y_1}{2}\right)^5 -$$

$$-\frac{(x-a)}{2}\left[f^{(VI)}\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right)\left(\frac{1}{2} - \frac{y_1}{2}\right)^6\right] -$$

$$-\frac{(x-a)}{2}\left[f^{(VI)}\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\left(\frac{1}{2} + \frac{y_1}{2}\right)^6\right],$$

$$G^{(VI)}(x) = 6!$$

we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{1}{6!}F^{(VI)}(a) \tag{7}$$

It is easily to see that

$$F^{(VI)}(a) = f^{(V)}(a)\left[1 - 3\left(\frac{1}{2} - \frac{y_1}{2}\right)^5 - 3\left(\frac{1}{2} + \frac{y_1}{2}\right)^5\right] =$$

$$= f^{(V)}(a)\left[1 - \frac{3}{16}\left(1 + 10y_1^2 + 5y_1^4\right)\right] = \frac{1}{12}f^{(V)}(a).$$

Therefore

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{1}{6!12}\cdot f^{(V)}(a). \tag{8}$$

On the other hand, we have

$$\frac{F(x)}{G(x)} = \frac{\frac{1}{135}\left(\frac{x-a}{2}\right)^5\left[f^{(IV)}(c_x) - f^{(IV)}(a)\right]}{(x-a)^6} =$$

$$= \frac{1}{135\cdot 2^5}\cdot\frac{f^{(IV)}(c_x) - f^{(IV)}(a)}{c_2 - a}\cdot\frac{c_x - a}{x-a}$$

whence

$$\lim_{x\to a}\frac{F(x)}{G(x)}=\frac{1}{135\cdot 2^5}\cdot f^{(V)}(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{9}$$

From the relation (8) and (9) we obtain

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}$$

which is exactly the assertion Theorem 1.

3. The second particular case of the quadrature formula (1) is that one in which $n=3$.

The nodes and coefficients of the corresponding quadrature formula can be obtained, from (3), (4) and (5). We find $L_3(y)=\frac{1}{2}(5y^3-3y)$

$$c_1=\frac{5}{18},\ c_2=\frac{8}{18}, c_3=\frac{5}{18}$$

and

$$x_1=\frac{a+b}{2}-\frac{b-a}{2}y_1,\ x_2=\frac{a+b}{2},\ x_3=\frac{a+b}{2}+\frac{b-a}{2}y_1$$

with $y_1^2=\frac{3}{5}$.

From relation (2) we obtain

$$R_3(f)=\frac{1}{15750}\left(\frac{b-a}{2}\right)^7 f^{(6)}(\xi).$$

In this case we have that if $f:[a,b]\to\mathbb{R}$, $f\in C^6[a,b]$ then for any $x\in(a,b]$ there is $c_x\in(a,x)$ such that

$$\int_a^x f(t)dt= \tag{10}$$

$$=\frac{(x-a)}{18}\left[5f\left(\frac{a+x}{2}-\frac{x-a}{2}y_1\right)+8f\left(\frac{a+x}{2}\right)+5f\left(\frac{a+x}{2}+\frac{x-a}{2}y_1\right)\right]+$$
$$+\frac{1}{15750}\left(\frac{x-a}{2}\right)^7 f^{(6)}(c_x).$$

Our main result is contained in the following theorem.

Theorem 2. *If $f \in C^8[a,b]$ and $f^{(7)}(a) \neq 0$ then for the intermediate point c_x which appears in formula (10) we have:*

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

Proof. Let us consider $F, G : [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt -$$

$$-\frac{(x-a)}{18}\left[5f\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right) + 8f\left(\frac{a+x}{2}\right) + 5f\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\right] -$$

$$-\frac{1}{15750}\left(\frac{x-a}{2}\right)^7 f^{(6)}(a),$$

$$G(x) = (x-a)^8.$$

Since F and G are eight times derivable on $[a,b]$, $G^{(i)} \neq 0$, $i = \overline{1,7}$ for any $x \in (a,b]$ and $F^{(k)}(a) = 0$, $G^{(k)}(a) = 0$, $k = \overline{1,7}$.

By using successive l'Hospital rule, we obtain

$$\lim_{x \to a} \frac{F(x)}{G(x)} = \lim_{x \to a} \frac{F^{(8)}(x)}{G^{(8)}(x)}$$

Now, from

$$F^{(8)}(x) = f^{(7)}(x) - \frac{4}{9}\left[5f^{(7)}\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right)\left(\frac{1}{2} - \frac{y_1}{2}\right)^7 + \right.$$

$$\left. + \frac{1}{16}f^{(7)}\left(\frac{a+x}{2}\right) + 5f^{(7)}\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\left(\frac{1}{2} + \frac{y_1}{2}\right)^7\right] -$$

$$-\frac{(x-a)}{18}\left[5f^{(8)}\left(\frac{a+x}{2} - \frac{x-a}{2}y_1\right)\left(\frac{1}{2} - \frac{y_1}{2}\right)^8 + \frac{1}{32}f^{(8)}\left(\frac{a+x}{2}\right) + \right.$$

$$\left. + +5f^{(8)}\left(\frac{a+x}{2} + \frac{x-a}{2}y_1\right)\left(\frac{1}{2} + \frac{y_1}{2}\right)^8\right],$$

$$G^{(8)}(x) = 8!$$

we obtain

(11) $$\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{1}{8!} F^{(8)}(a),$$

where

$$F^{(8)}(a) = f^{(7)}(a)\left\{1 - \frac{4}{9}\left[5\left(\frac{1}{2} - \frac{y_1}{2}\right)^7 + 5\left(\frac{1}{2} + \frac{y_1}{2}\right)^7 + \frac{1}{16}\right]\right\} =$$

$$= f^{(7)}(a)\left\{1 - \frac{4}{9}\left[\frac{5}{2^7}(1-y_1)^7 + \frac{5}{2^7}(1+y_1)^7 + \frac{1}{16}\right]\right\} =$$

$$= f^{(7)}(a)\left\{1 - \frac{4}{9}\left[\frac{5}{2^6}(1 + 21y_1^2 + 35y_1^4 + 7y_1^6) + \frac{1}{16}\right]\right\} =$$

$$= f^{(7)}(a)\left(1 - \frac{4}{9}\cdot\frac{891}{400}\right) = \frac{1}{100} f^{(7)}(a).$$

Hence

(12) $$\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{1}{100\cdot 8!}\cdot f^{(7)}(a).$$

On the other hand

$$\frac{F(x)}{G(x)} = \frac{\frac{1}{15750}\left(\frac{x-a}{2}\right)^7\left[f^{(6)}(c_x) - f^{(6)}(a)\right]}{(x-a)^8} =$$

$$= \frac{1}{15750\cdot 2^7}\cdot\frac{f^{(6)}(c_x) - f^{(6)}(a)}{c_x - a}\cdot\frac{c_x - a}{x-a},$$

whence

(13) $$\lim_{x\to a} \frac{F(x)}{G(x)} = \frac{1}{15750\cdot 2^7}\cdot f^{(7)}(a)\cdot \lim_{x\to a}\frac{c_x - a}{x - a}$$

From the relation (12) and (13) we obtain

$$\lim_{x\to a}\frac{c_x - a}{x-a} = \frac{1}{2}.$$

4. Open problem. For intermediate point which appear in quadrature formulas of Gauss type (1) - (2) we have

$$\lim_{b\to a}\frac{\xi_b - a}{b-a} = \frac{1}{2},$$

any natural number n.

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