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Error Inequalities for a Generalized Quadrature Rule

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Dedicated to Professor Dumitru Acu on his 60th anniversary

Abstract

Error inequalities for a generalized quadrature rule are derived. A summation formula for the special function Si(x) is given.

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1 Introduction

In recent years a number of authors have considered generalizations of some known and some new quadrature rules. For example, generalizations of the trapezoid, mid-point and Simpson's quadrature rules are considered in [1],

[2], [3], [5] and [9]. As an illustration we give a generalization of the mid-point quadrature rule (see [3]),

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} \left[1 + (-1)^k\right] \frac{(b-a)^{k+1}}{2^{k+1}(k+1)!} f^{(k)}(\frac{a+b}{2}) + (-1)^n \int_a^b K_n(t) f^{(n)}(t)dt,$$

where

$$K_n(t) = \begin{cases} \frac{(t-a)^n}{n!}, & t \in \left[a, \frac{a+b}{2}\right] \\ \frac{(t-b)^n}{n!}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} .$$

For $n = 1$ we get the mid-point rule

$$\int_a^b f(t)dt = (b-a) f(\frac{a+b}{2}) - \int_a^b K_1(t) f'(t)dt.$$

In this paper we consider a generalization of a simple quadrature rule of open type which has the form

$$\int_a^b f(t)dt = \frac{b-a}{2} \left[f(\frac{3a+b}{4}) + f(\frac{a+3b}{4}) \right] + R(f). \tag{1}$$

In [14] it is shown that the above 2-point quadrature rule of open type is optimal with respect to a given way of estimation of the remainder term (error) $R(f)$. We have (see [14])

$$|R(f)| \leq \frac{\Gamma - \gamma}{16} (b-a)^2, \tag{2}$$

where $\gamma \leq f'(t) \leq \Gamma$, $t \in [a, b]$.

On the other hand, the well-known 2-point Gauss quadrature rule

$$\int_a^b f(t)dt = \frac{b-a}{2} \left[f(\frac{a+b}{2} - \frac{\sqrt{3}}{6}(b-a)) + f(\frac{a+b}{2} + \frac{\sqrt{3}}{6}(b-a)) \right] + R_1(f) \tag{3}$$

has the estimation of the error (see [10])

$$|R_1(f)| \leq \frac{\Gamma - \gamma}{24}(5 - 2\sqrt{3})(b-a)^2. \tag{4}$$

Since $\frac{1}{16} = 0.0625 < \frac{1}{24}(5 - 2\sqrt{3}) = 0.06399$ we conclude that (2) is better than (4).

In [14] we can also find various error inequalities for this rule. Here we also give various error bounds for the generalization of this rule. These error bounds are generalizations of the error bounds obtained in [14] and they are similar to error bounds obtained in [15].

Finally, we give a numerical example. In fact, we derive a summation formula for the special function $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$.

2 Main results

Lemma 1. *Let $f : [a,b] \to R$ be a function such that $f^{(n-1)}$ is absolutely continuous. Then*

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})\right] - \tag{5}$$

$$-2\sum_{i=1}^{m} \frac{(b-a)^{2i+1}}{4^{2i+1}(2i+1)!}\left[f^{(2i)}(\frac{3a+b}{4}) + f^{(2i)}(\frac{a+3b}{4})\right] + R(f),$$

where $m = \left[\frac{n-1}{2}\right]$, the integer part of $(n-1)/2$,

$$R(f) = (-1)^n \int_a^b S_n(t) f^{(n)}(t)dt \tag{6}$$

and

$$S_n(t) = \begin{cases} \frac{1}{n!}(t-a)^n, & t \in \left[a, \frac{3a+b}{4}\right] \\ \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n, & t \in \left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \\ \frac{1}{n!}(t-b)^n, & t \in \left[\frac{a+3b}{4}, b\right] \end{cases}. \tag{7}$$

Proof. We prove (5) by induction. First we note that

$$S_1(t) = \begin{cases} t-a, & t \in \left[a, \frac{3a+b}{4}\right] \\ t-\frac{a+b}{2}, & t \in \left(\frac{3a+b}{4}, \frac{a+3b}{4}\right) \\ t-b, & t \in \left[\frac{a+3b}{4}, b\right] \end{cases}$$

is a Peano kernel for the quadrature rule of open type, that is, we have

$$-\int_a^b S_1(t)f'(t)dt = -\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2}(b-a) + \int_a^b f(t)dt.$$

We easily show that (5) holds for $n = 2$. Now suppose that (5) holds for an arbitrary n. We have to prove that (5) holds for $n \to n+1$. To simplify the proof we introduce the notations

$$P_n(t) = \frac{(t-a)^n}{n!}, \tag{8}$$

$$Q_n(t) = \frac{1}{n!}\left(t - \frac{a+b}{2}\right)^n, \tag{9}$$

$$R_n(t) = \frac{(t-b)^n}{n!}. \tag{10}$$

We see that P_n, Q_n and R_n form Appell sequences of polynomials, that is

$$\begin{aligned} P_n'(t) &= P_{n-1}(t),\ Q_n'(t) = Q_{n-1}(t),\ R_n'(t) = R_{n-1}(t), \\ P_0(t) &= Q_0(t) = R_0(t) = 1. \end{aligned}$$

We have

$$(-1)^{n+1}\int_a^b S_{n+1}(t)f^{(n+1)}(t)dt$$

$$
\begin{aligned}
= \quad & (-1)^{n+1}\int_a^{\frac{3a+b}{4}} P_{n+1}(t)f^{(n+1)}(t)dt + (-1)^{n+1}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} Q_{n+1}(t)f^{(n+1)}(t)dt \\
& (-1)^{n+1}\int_{\frac{a+3b}{4}}^{b} R_{n+1}(t)f^{(n+1)}(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{n+1}\left[P_{n+1}(\frac{3a+b}{4})f^{(n)}(\frac{3a+b}{4}) - P_{n+1}(a)f^{(n)}(a)\right] \\
&+(-1)^{n+1}\left[Q_{n+1}(\frac{a+3b}{4})f^{(n)}(\frac{a+3b}{4}) - Q_{n+1}(\frac{3a+b}{4})f^{(n)}(\frac{3a+b}{4})\right] \\
&+(-1)^{n+1}\left[R_{n+1}(b)f^{(n)}(b) - R_{n+1}(\frac{a+3b}{4})f^{(n)}(\frac{a+3b}{4})\right]
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{n}\int_a^{\frac{3a+b}{4}} P_{n}(t)f^{(n)}(t)dt + (-1)^{n}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} Q_{n}(t)f^{(n)}(t)dt \\
&+(-1)^{n}\int_{\frac{a+3b}{4}}^{b} R_{n}(t)f^{(n)}(t)dt
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-1)^{n}\int_a^b S_n(t)f^{(n)}(t)dt \\
& +(-1)^{n+1}\left[P_{n+1}(\frac{3a+b}{4}) - Q_{n+1}(\frac{3a+b}{4})\right] f^{(n)}(\frac{3a+b}{4}) \\
& +(-1)^{n+1}\left[Q_{n+1}(\frac{a+3b}{4}) - R_{n+1}(\frac{a+3b}{4})\right] f^{(n)}(\frac{a+3b}{4})
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2}(b-a) + \int_a^b f(t)dt \\
& + \sum_{i=1}^{m} \frac{2(b-a)^{2i+1}}{4^{2i+1}(2i+1)!}\left[f^{(2i)}(\frac{3a+b}{4}) + f^{(2i)}(\frac{a+3b}{4})\right] \\
& +(-1)^{n+1}\left[P_{n+1}(\frac{3a+b}{4}) - Q_{n+1}(\frac{3a+b}{4})\right] f^{(n)}(\frac{3a+b}{4})
\end{aligned}
$$

$$+(-1)^{n+1}\left[Q_{n+1}(\frac{a+3b}{4})-R_{n+1}(\frac{a+3b}{4})\right]f^{(n)}(\frac{a+3b}{4})$$

$$\begin{aligned} &= -\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}(b-a)+\int_a^b f(t)dt \\ &\quad +\sum_{i=1}^{m_1}\frac{2(b-a)^{2i+1}}{4^{2i+1}(2i+1)!}\left[f^{(2i)}(\frac{3a+b}{4})+f^{(2i)}(\frac{a+3b}{4})\right] \end{aligned}$$

where $m_1=\left[\frac{n}{2}\right]$, since

$$+(-1)^{n+1}\left[P_{n+1}(\frac{3a+b}{4})-Q_{n+1}(\frac{3a+b}{4})\right]f^{(n)}(\frac{3a+b}{4})$$

$$+(-1)^{n+1}\left[Q_{n+1}(\frac{a+3b}{4})-R_{n+1}(\frac{a+3b}{4})\right]f^{(n)}(\frac{a+3b}{4})$$

$$=\frac{(b-a)^{n+1}}{4^{n+1}(n+1)!}\left[1-(-1)^{n+1}\right]\left[f^{(n)}(\frac{3a+b}{4})+f^{(n)}(\frac{a+3b}{4})\right].$$

This completes the proof.

Lemma 2. *The Peano kernels $S_n(t)$, $n>1$, satisfy:*

(11) $$\int_a^b S_n(t)dt=0, \quad \text{if } n \text{ is odd},$$

(12) $$\int_a^b |S_n(t)|\,dt=\frac{(b-a)^{n+1}}{4^n(n+1)!},$$

(13) $$\max_{t\in[a,b]}|S_n(t)|=\frac{(b-a)^n}{4^n n!}.$$

Proof. A simple calculation gives

$$\int_a^b S_n(t)dt=\frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!}\left[1-(-1)^{n+1}\right].$$

From the above relation we see that (11) holds, since $1-(-1)^{n+1}=0$ if n is odd.

We now consider some properties of the Appell sequences of polynomials $P_n(t)$, $Q_n(t)$ and $R_n(t)$, given by (8), (9) and (10), respectively. We have that $(t-a)^n \geq 0$, for each n, such that $P_n(t) \geq 0$, $\forall n$ and $t \in \left[a, \frac{3a+b}{2}\right]$. Since $P_n'(t) = P_{n-1}(t)$ we conclude that $P_n(t)$ are increasing functions. If n is even then $\left(t - \frac{a+b}{2}\right)^n \geq 0$. If n is odd then $\left(t - \frac{a+b}{2}\right)^n \geq 0$, for $t \in \left[\frac{a+b}{2} \frac{a+3b}{4}\right]$ and $\left(t - \frac{a+b}{2}\right)^n \leq 0$, for $t \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$.

Since $Q_n'(t) = Q_{n-1}(t)$ we conclude that $Q_n(t)$ is increasing function if n is even and $Q_n(t)$ is decreasing function for $t \in \left[\frac{3a+b}{4}, \frac{a+b}{2}\right]$, while $Q_n(t)$ is increasing function for $t \in \left[\frac{a+b}{2} \frac{a+3b}{4}\right]$, if n is odd.

We have that $(t-b)^n \leq 0$ if $n \geq 1$, n is odd and $(t-b)^n \geq 0$ if $n \geq 0$, n is even. Thus, we have that $R_n(t) \leq 0$ if n is odd and $R_n(t) \geq 0$ if n is even. As we know $R_n'(t) = R_{n-1}(t)$ such that $R_n(t)$ are decreasing functions if n is even and $R_n(t)$ are increasing functions if n is odd. We use these properties to prove (12) and (13).

We have

$$\begin{aligned}
\int_a^b |S_n(t)|\,dt &= \int_a^{\frac{3a+b}{4}} |P_n(t)|\,dt + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} |Q_n(t)|\,dt + \int_{\frac{a+3b}{4}}^{b} |R_n(t)|\,dt \\
&= \frac{(b-a)^{n+1}}{4^n(n+1)!}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\max_{t\in[a,b]}|S_n(t)| &= \max\left\{\max_{t\in\left[a,\frac{3a+b}{4}\right]}|P_n(t)|, \max_{t\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}|Q_n(t)|, \max_{t\in\left[\frac{a+3b}{4},b\right]}|R_n(t)|\right\}\\
&= \max\left\{\left|P_n(\frac{3a+b}{4})\right|, \left|Q_n(\frac{3a+b}{4})\right|, \left|Q_n(\frac{a+3b}{4})\right|, \left|R_n(\frac{a+3b}{4})\right|\right\}\\
&= \frac{(b-a)^n}{4^n n!}.
\end{aligned}$$

We introduce the notations

$$I=\int_a^b f(t)dt,$$

$$\begin{aligned}
F &= -\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}(b-a)\\
&\quad+\sum_{i=1}^{m}\frac{2(b-a)^{2i+1}}{4^{2i+1}(2i+1)!}\left[f^{(2i)}(\frac{3a+b}{4})+f^{(2i)}(\frac{a+3b}{4})\right].
\end{aligned}$$

Theorem 3. *Let $f:[a,b]\to R$ be a function such that $f^{(n-1)}$, $n>1$, is absolutely continuous and there exist real numbers γ_n,Γ_n such that $\gamma_n\le f^{(n)}(t)\le\Gamma_n$, $t\in[a,b]$. Then*

$$|I-F|\le\frac{1}{2}\frac{\Gamma_n-\gamma_n}{(n+1)!}\frac{1}{4^n}(b-a)^{n+1}\quad \textit{if } n \textit{ is odd} \tag{14}$$

and

$$|I-F|\le\frac{(b-a)^{n+1}n}{4^n(n+1)!}\left\|f^{(n)}\right\|_\infty \textit{ if } n \textit{ is even.} \tag{15}$$

Proof. Let n be odd. From (6) and (11) we get

$$R(f)=(-1)^n\int_a^b S_n(t)f^{(n)}(t)dt=(-1)^n\int_a^b S_n(t)\left[f^{(n)}(t)-\frac{\gamma_n+\Gamma_n}{2}\right]dt$$

such that we have

$$|R(f)| = |I - F| \leq \max_{t\in[a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^b |S_n(t)|\, dt. \tag{16}$$

We also have

$$\max_{t\in[a,b]} \left| f^{(n)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \leq \frac{\Gamma_n - \gamma_n}{2}. \tag{17}$$

From (16), (17) and (12) we get

$$|I - F| \leq \frac{1}{2}\frac{\Gamma_n - \gamma_n}{(n+1)!}\frac{1}{4^n}(b-a)^{n+1}.$$

Let n be even. Then we have

$$|R(f)| = |I - F| \leq \int_a^b |S_n(t)|\, dt \left\| f^{(n)} \right\|_\infty = \frac{(b-a)^{n+1} n}{4^n (n+1)!} \left\| f^{(n)} \right\|_\infty .$$

Theorem 4. *Let $f : [a,b] \to R$ be a function such that $f^{(n-1)}$, $n > 1$, is absolutely continuous and let n be odd. If there exists a real number γ_n such that $\gamma_n \leq f^{(n)}(t)$, $t \in [a,b]$ then*

$$|I - F| \leq (T_n - \gamma_n) \frac{(b-a)^{n+1}}{4^n n!}, \tag{18}$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

If there exists a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n$, $t \in [a,b]$ then

$$|I - F| \leq (\Gamma_n - T_n) \frac{(b-a)^{n+1}}{4^n n!}. \tag{19}$$

Proof. We have

$$|R(f)| = |I - F| = \left| \int_a^b (f^{(n)}(t) - \gamma_n) S_n(t) dt \right|,$$

since (11) holds. Then we have

$$\begin{aligned}\left|\int_a^b (f^{(n)}(t)-\gamma_n)S_n(t)dt\right| &\leq \max_{t\in[a,b]}|S_n(t)|\int_a^b (f^{(n)}(t)-\gamma_n)dt \\ &= \frac{(b-a)^n}{4^n n!}\left[f^{(n-1)}(b)-f^{(n-1)}(a)-\gamma_n(b-a)\right] \\ &= \frac{(b-a)^{n+1}}{4^n n!}\left(T_n-\gamma_n\right).\end{aligned}$$

In a similar way we can prove that (19) holds.

Remark 5. *Note that we can apply the estimations (14) and (15) only if $f^{(n)}$ is bounded. On the other hand, we can apply the estimation (18) if $f^{(n)}$ is unbounded above and we can apply the estimation (19) if $f^{(n)}$ is unbounded below.*

3 A numerical example

Here we consider the integral (special function) $\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}dt$ and apply the summation formula (5) to this integral. We get the summation formula $\mathrm{Si}(x)=F(x)+R(x)$, where

$$F(x)=2\sin\frac{x}{4}+\frac{2}{3}\sin\frac{3x}{4}+\sum_{i=1}^{m}\frac{2x^{2i+1}}{4^{2i+1}(2i+1)!}\left[f^{(2i)}(\frac{x}{4})+f^{(2i)}(\frac{3x}{4})\right] \tag{20}$$

and $f(t)=(\sin t)/t$. We calculate the derivatives $f^{(j)}(t)$ as follows. We have

$$(g(t)h(t))^{(j)}=\sum_{k=0}^{j}\binom{j}{k}g^{(k)}(t)h^{(j-k)}(t).$$

If we choose $g(t) = \sin t$ and $h(t) = 1/t$ then we get

$$\begin{aligned} f^{(j)}(\frac{x}{4}) &= \sum_{i=0}^{[\frac{j-1}{2}]} \binom{j}{2i+1} (-1)^{j-i+1} \frac{(j-2i-1)!4^{j-2i}}{x^{j-2i}} \cos\frac{x}{4} \\ &+ \sum_{i=0}^{[\frac{j}{2}]} \binom{j}{2i} (-1)^{j-i} \frac{(j-2i)!4^{j-2i+1}}{x^{j-2i+1}} \sin\frac{x}{4}, \end{aligned}$$

$$\begin{aligned} f^{(j)}(\frac{3x}{4}) &= \sum_{i=0}^{[\frac{j-1}{2}]} \binom{j}{2i+1} (-1)^{j-i+1} \frac{(j-2i-1)!4^{j-2i}}{3^{j-2i}x^{j-2i}} \cos\frac{3x}{4} \\ &+ \sum_{i=0}^{[\frac{j}{2}]} \binom{j}{2i} (-1)^{j-i} \frac{(j-2i)!4^{j-2i+1}}{3^{j-2i+1}x^{j-2i+1}} \sin\frac{3x}{4}. \end{aligned}$$

We now compare the summation formula (20) with the known compound formula (for the given quadrature rule of open type),

$$\int_0^x f(t)dt = \frac{h}{2} \sum_{i=0}^{n-1} \left[f(\frac{3x_i + x_{i+1}}{4}) + f(\frac{x_i + 3x_{i+1}}{4}) \right] + R(x), \tag{21}$$

where $x_i = ih$, $h = x/n$, $f(t) = (\sin t)/t$.

Let us choose $x = 1$. The "exact" value is $\mathrm{Si}(1) = 0.946083070367$. If we use (20) with $m = 5$ then we get $\mathrm{Si}(1) \approx 0.946083070363$. If we use (21) with $n = 40000$ then we get $\mathrm{Si}(1) \approx 0.946083070369$. All calculations are done in double precision arithmetic. The first approximate result is obtained much faster than the second approximate result. The same is valid if we use some quadrature rule of higher order, for example Simpson's rule. This is a consequence of the fact that we have to calculate the function $\sin t$ many times when we apply the compound formula and we have only to calculate $\sin(x/4)$, $\cos(x/4)$, $\sin(3x/4)$ and $\cos(3x/4)$ when we apply the summation formula.

Similar summation formulas can be obtained for the integrals (special functions): $\int_0^x [(e^t - 1)/t]\, dt$, $\int_0^x [(\cos t - 1)/t]\, dt$, $\int_0^x \exp(-t^2) dt$, etc.

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