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Certain class of p-valent Functions defined by Dziok-Srivastava Linear Operator [1]

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Abstract

In this paper, we introduce a new class of multivalent functions defined by Dziok-Srivastava operator to study some of the interesting properties like coefficient estimates, distortion bounds and to prove the class is closed under convolution product and integral representation.

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1 Introduction

Let $\mathcal{A}_p$ be the class of p-valent analytic functions with positive coefficients of the form

(1) $$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad z \in \Delta = \{z : |z| < 1\}.$$

For functions $f(z)$ given by (1) and

(2) $$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ denoted by $(f * g)(z) = (g * f)(z)$ is defined by

(3) $$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

For $\{\alpha_1, \alpha_2, \cdots, \alpha_m\} \subseteq \boldsymbol{C}$ and $\{\beta_1, \beta_2, \cdots, \beta_n\} \subseteq \boldsymbol{C} - \{0, -1, -2, \cdots\}$ the generalized hypergeometric function ${}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z)$ is defined by

(4) $${}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k z^k}{(\beta_1)_k \cdots (\beta_n)_k k!}$$

$$(m \le n+1, m, n, \in I\!N_0 = \{0, 1, 2, \cdots\})$$

where $(\lambda)_k$ is the pochhammer symbol defined by

(5) $$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(k)} = \begin{cases} 1 & k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & k \in I\!N \end{cases}$$

Using Dziok - Srivastava operator [2] , $f(z) \in \mathcal{A}_p$ we have

$$
\begin{aligned}
(6) \qquad \mathcal{DS}_p^{m,n} &= \mathcal{DS}_p^{(m,n)}(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) f(z) \\
&= h_p(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) * f(z) \\
&= z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_m)_{k-p} a_k z^k}{(\beta_1)_{k-p} \cdots (\beta_n)_{k-p} (k-p)!}
\end{aligned}
$$

where

$$h_p(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z) = z^p \; {}_mF_n(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n; z).$$

For $1 < \gamma < 1 + \frac{1}{2p}$, $z \in \Delta$ and let $g(z)$ given by (2) we define the class

$$\mathcal{A}_p(g(z), \alpha_1, \cdots, \alpha_m; \beta_1, \beta_2, \cdots, \beta_n, \gamma) = \mathcal{A}_p^{g(z)}(m, n, \gamma)$$

by

$$\mathcal{A}_p^{g(z)}(m, n, \gamma) = \left\{ f(z) \in \mathcal{A}_p : Re \left\{ 1 + \frac{z(\mathcal{DS}_p^{m,n}(f*g)(z))''}{(\mathcal{DS}_p^{m,n}(f*g)(z))'} \right\} < p\gamma, \right.$$

$$(7) \qquad \left. (1 < \gamma < 1 + \frac{1}{2p}, \quad z \in \Delta) \right\}$$

2 Main Results

In this section we obtain a necessary and sufficient condition for functions to be in the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 2.1. *$f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ if and only if*

$$\sum_{k=p+1}^{\infty} \frac{k(k-p\gamma)}{p^2(\gamma-1)} \ \theta(k,p) \ a_k b_k \leq 1. \tag{8}$$

where

$$\theta(k,p) = \frac{(\alpha_1)_{k-p}\cdots(\alpha_m)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_n)_{k-p}(k-p)!}.$$

Proof. If $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, then by using (6) and (7) we obtain

$$Re\left\{1+\frac{z(p(p-1)z^{p-2}+\sum\limits_{k=p+1}^{\infty}\theta(k,,p)k(k-1)a_kb_kz^{k-2}}{pz^{p-1}+\sum\limits_{k=p+1}^{\infty}\theta(k,p)ka_kb_kz^{k-1}}\right\} < p\gamma.$$

Choosing values of z on real axis and letting $z \to 1^-$ we have

$$\frac{p^2+\sum\limits_{k=p+1}^{\infty}\theta(k,p)k^2a_kb_k}{p+\sum\limits_{k=p+1}^{\infty}\theta(k,p)ka_kb_k} < p\gamma$$

or equivalently

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_kb_k \leq p^2(\gamma-1).$$

To prove the "if" part, let (8) holds true, so

$$\left|\frac{z(\mathcal{DS}_p^{m,n}(f*g)(z))''-(p-1)(\mathcal{DS}_p^{m,n}(f*g)(z))'}{z(\mathcal{DS}_p^{m,n}(f*g)(z)''-[2p(1-\gamma)-1+p](\mathcal{DS}_p^{m,n}(f*g)(z))'}\right|$$
$$\leq \frac{\sum\limits_{k=p+1}^{\infty}k(k-p)a_kb_k}{2p^2(\gamma-1)-\sum\limits_{k=p+1}^{\infty}[k(k-p)(1-2(1-\gamma)))]a_kb_k} \leq 1$$

or equivalently $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 2.2. *If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then*

$$a_k \leq \frac{p^2(\gamma - 1)}{k(k - p\gamma)b_k\theta(k, p)} \tag{9}$$

the result is sharp for functions of the form

$$f_k(z) = z^p + \frac{p^2(\gamma - 1)}{k(k - p\gamma)b_k\theta(k, p)}z^k \quad k = p + 1, p + 2, \cdots.$$

Proof. Since $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, by (8) we have

$$k(k - p\gamma)\theta(k, p)a_kb_k \leq \sum_{k=p+1}^{\infty} k(k - p\gamma)\theta(k, p)a_kb_k \leq p^2(\gamma - 1)$$

or

$$a_k \leq \frac{p^2(\gamma - 1)}{k(k - p\gamma)\theta(k, p)b_k}.$$

The sharpness is trivial and so omitted.

3 Distortion Bounds

In this section we obtain the distortion bounds for $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 3.1. *If $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then*

$$\begin{aligned} r^p - \frac{p^2(\gamma - 1)}{(p + 1)(p + 1 - p\gamma)\theta(p + 1, p)b_{p+1}}r^{p+1} &\leq |f(z)| \\ \leq r^p + \frac{p^2(\gamma - 1)}{(p + 1)(p + 1 - p\gamma)\theta(p + 1, p)b_{p+1}}r^{p+1} & \end{aligned} \tag{10}$$

where

$$\theta(p + 1, p) = \frac{\prod_{i=1}^{m} \alpha_i}{\prod_{j=1}^{n} \beta_j}, \quad |z| = r < 1.$$

The result is sharp for the function

$$f(z) = z^p + \frac{p^2(\gamma - 1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}} z^{p+1}. \tag{11}$$

Proof. By using (8), (9) we obtain

$$b_{p+1}\theta(p+1,p)(p+1)(p+1-p\gamma) \sum_{k=p+1}^{\infty} a_k \leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1)$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{p^2(\gamma - 1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}}. \tag{12}$$

For the function $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and using (12) and $|z| = r$ we have

$$\begin{aligned} |f(z)| &\leq r^p + \sum_{k=p+1}^{\infty} a_k r^k \\ &< r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k \\ &\leq r^p + \frac{p^2(\gamma - 1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}} r^{p+1}, \end{aligned}$$

also

$$\begin{aligned} |f(z)| &\geq r^p - \sum_{k=p+1}^{\infty} a_k r^k \\ &\geq r^p - \frac{p^2(\gamma - 1)}{(p+1)(p+1-p\gamma)\theta(p+1,p)b_{p+1}} r^{p+1}. \end{aligned}$$

Hence the proof is complete.

Corollary. *If* $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, *then*

$$pr^{p-1} - \frac{p^2(\gamma - 1)}{(p+1-p\gamma)\theta(p+1,p)b_{p+1}} r^p \leq |f'(z)|$$
$$\leq pr^{p-1} + \frac{p^2(\gamma - 1)}{(p+1-p\gamma)\theta(p+1,p)b_{p+1}} r^p.$$

The result is sharp for the function given by (11).

4 Integral Representation

In this section we obtain integral representation for $\mathcal{DS}_p^{m,n}(f * g)(z)$.

Theorem 4.1. *If* $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ *then*

$$\mathcal{DS}_p^{m,n}(f * g)(z) = (p\gamma - 1) \int_0^z e^{\int_0^z \frac{Q(t)}{t} dt} dt.$$

Proof. By letting $\mathcal{DS}_p^{m,n}(f * g)(z) = M(z)$ in (7) we have

$$Re \left\{ 1 + \frac{zM''(z)}{M'(z)} \right\} < p\gamma.$$

Thus

$$\frac{zM''(z)}{M'(z)} < p\gamma - 1$$

or

$$\frac{zM''(z)}{M'(z)} = Q(z)(p\gamma - 1)$$

where $|Q(z)| < 1, \quad z \in \Delta$.

So $\frac{M''(z)}{M'(z)} = \frac{Q(z)}{z}(p\gamma - 1)$, after integration we obtain

$$\log(M'(z)) = \int_0^z \frac{Q(t)}{t}(p\gamma - 1) dt$$

thus

$$M'(z) = \exp\left[\int_0^z \frac{Q(t)}{t}(p\gamma - 1)dt\right].$$

After integration we have

$$M(z) = \mathcal{DS}_p^{m,n}(f * g) = \int_0^z \exp\left[\int_0^z \frac{Q(t)}{t}(p\gamma - 1)dt\right] dt$$

and this gives the result.

5 Closure Theorems

In this section, we discuss certain inclusion properties of the class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Theorem 5.1. *Let* $F_j(z) = z^p + \sum\limits_{k=p+1}^{\infty} a_{k,j} z^k \quad (j = 1, 2, \cdots, q)$ *be in the class* $\mathcal{A}_p^{g(z)}(m, n, \gamma)$ *and* $\eta_j \geq 0$ *for* $j = 1, 2, \cdots, q$ *and* $\sum\limits_{j=1}^{q} \eta_j \leq 1$ *then the function*

$$f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\sum_{j=1}^{q} \eta_j a_{k,j} \right) z^k$$

belongs to $\mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Proof. Since $F_j(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then from Theorem 2.1 for every $j = 1, 2, \cdots, q$ we have

$$\sum_{k=p+1}^{\infty} k(k - p\gamma)\theta(k, p)b_k a_{k,j} \leq p^2(\gamma - 1).$$

Also

$$\sum_{k=p+1}^{\infty} k(k - p\gamma)\theta(k, p)b_k \left(\sum_{j=1}^{q} \eta_j a_{k,j} \right)$$
$$= \sum_{j=1}^{q} \eta_j (\sum_{k=p+1}^{\infty} k(k - p\gamma)\theta(k, p)b_k a_{k,j})$$

$$\leq \sum_{j=1}^{q} \eta_j p^2(\gamma - 1)$$
$$\leq p^2(\gamma - 1).$$

So by Theorem 2.1 $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$.

Corollary. *The class $\mathcal{A}_p^{g(z)}(m, n, \gamma)$ is closed under convex linear combination.*

Theorem 5.2. *Let $F_p(z) = z^p$ and*

$$F_k(z) = z^p + \frac{p^2(\gamma - 1)}{k(k - p\gamma)\theta(k, p)b_k} z^k, \quad (k = p + 1, \cdots).$$

Then $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$ if and only if

$$f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k F_k(z)$$

where $\sum\limits_{k=p}^{\infty} \eta_k = 1$ and $\eta_k \geq 0$.

Proof. Let $f(z) \in \mathcal{A}_p^{g(z)}(m, n, \gamma)$, then from Theorem 2.2, we have

$$a_k \leq \frac{p^2(\gamma - 1)}{k(k - p\gamma)\theta(k, p)b_k} \quad (k = p + 1, p + 2, \cdots)$$

therefore by letting

$$\eta_k = \frac{k(k - p\gamma)\theta(k, p)b_k a_k}{p^2(\gamma - 1)} \quad (k = p + 1, p + 2, \cdots)$$

and $\eta_p = 1 - \sum\limits_{k=p+1}^{\infty} \eta_k$.

We conclude the required result.

Conversely, let $f(z) = \eta_p z^p + \sum\limits_{k=p+1}^{\infty} \eta_k F_k(z)$, then

$$f(z) = \eta_p z^p + \sum_{k=p+1}^{\infty} \eta_k \left(z^p + \frac{p^2(\gamma - 1)}{k(k - p\gamma)\theta(k, p)b_k} z^k \right)$$

$$= \quad z^p + \sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} z^k.$$

Therefore

$$\sum_{k=p+1}^{\infty} \frac{\eta_k p^2(\gamma-1)}{k(k-p\gamma)\theta(k,p)b_k} \; \frac{k(k-p\gamma)}{p^2(\gamma-1)}\theta(k,p)b_k$$
$$= \sum_{k=p+1}^{\infty} \eta_k = 1 - \eta_p \leq 1.$$

Hence by Theorem 2.1, we have $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$.

6 Convolution Property and Integral Operator

In this section we show that the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$ is closed under convolution and integral operator.

Theorem 6.1. *Let* $h(z) = z^p + \sum\limits_{k=p+1}^{\infty} c_k z^k$ *be analytic in unit disk* Δ *and* $0 \leq c_k \leq 1$. *If* $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, *then* $(f*h)(z)$ *is also in the class* $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Proof. Since $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$ then by Theorem 2.1 we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1).$$

By using the last inequality and the fact that

$$(f*h)(z) = z^p + \sum_{k=p+1}^{\infty} a_k c_k z^k$$

we have

$$\sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k c_k b_k$$
$$\leq \sum_{k=p+1}^{\infty} k(k-p\gamma)\theta(k,p)a_k b_k \leq p^2(\gamma-1)$$

and hence by Theorem 2.1 result follows.

Theorem 6.2. *If* $f(z) \in \mathcal{A}_p^{g(z)}(m,n,\gamma)$, *then*

$$F(z) = \frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt \quad (\lambda > -1; \quad z \in \Delta)$$

is also in the class $\mathcal{A}_p^{g(z)}(m,n,\gamma)$.

Proof. Since $F(z) = f(z) * \left(z^p + \sum\limits_{k=p+1}^{\infty} \frac{\lambda+p}{\lambda+k} z^k\right)$ and $\frac{\lambda+p}{\lambda+k} \leq 1$, by Theorem 6.1, the proof is trivial.

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