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Fekete-Szegö Inequality for Certain Subclass of Analytic Functions [1]

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Abstract

In this present investigation, the authors obtain Fekete-Szegö inequality for certain normalized analytic function $f(z)$ defined on the open unit disc for which

$$\frac{(1-\alpha)z(D^n f(z))' + \alpha z(D^{n+1} f(z))'}{(1-\alpha)D^n f(z) + \alpha D^{n+1} f(z)}$$

$(0 \leq \alpha)$ lines in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szegö inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Srivastava and Mishra by making use of Salagean differential operator.

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1 Introduction

Let A be class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open disc $E = \{z : z \in C \text{ and } |\text{z}| < 1\}$. Further, let S denote the class of functions which are univalent in E. For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z), D^1 f(z) = Df(z) = zf'(z),$$

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in N = \{1, 2, 3, ...\}).$$

Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ (n \in N_0 = N \cup \{0\}). \tag{1.2}$$

The differential operator D^n was introduced by Sălăgean [4].

Let $\phi(z)$ be an analytic function with positive real part on E with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk E onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f(z) \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z \in E)$$

and $C(\phi)$ be the class of functions in $f(z) \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), \qquad (z \in E),$$

where $\prec$ denotes the subordination between analytic functions. These classes were investigated and studied by Ma and Minda [3]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for class of starlike, convex, and close-to convex functions, see the recent paper by Srivastava *et al.* [2].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $M_{\alpha,n}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M^{\lambda}_{\alpha,n}(\phi)$ of functions defined by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [1].

Definition 1.1. *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disc E onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is in the class $M_{\alpha,n}(\phi)$ if*

$$\frac{(1-\alpha)z(D^n f(z))' + \alpha z(D^{n+1} f(z))'}{(1-\alpha)D^n f(z) + \alpha D^{n+1} f(z)} \prec \phi(z) \qquad (\alpha \geq 0).$$

For fixed $g \in A$, we define the class $M^g_{\alpha,n}(\phi)$ to be the class of functions $f \in A$ for which $(f * g) \in M_{\alpha,n}(\phi)$.

In order to derive our main results, we have to recall here the following Lemma [3].

Lemma 1.2. If $p_1 = 1 + c_1 z + c_2 z^2 + \ldots$ *is an analytic function with positive real part in E, then*

$$\left|c_2 - vc_1^2\right| \leq \begin{cases} -4v + 2 & \text{if} \quad v \leq 0; \\ 2 & \text{if} \quad 0 \leq v \leq 1; \\ 4v - 2 & \text{if} \quad v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1 + z)/(1 - z)$ or one of its rotations. If $0 < v < 1$, then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = (\frac{1}{2} + \frac{1}{2}\lambda)\frac{1+z}{1-z} + (\frac{1}{2} - \frac{1}{2}\lambda)\frac{1-z}{1+z} \qquad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$.

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \leq 2 \qquad (0 < v \leq \frac{1}{2})$$

and

$$\left|c_2 - vc_1^2\right| + (1 - v)\left|c_1\right|^2 \leq 2 \qquad (\frac{1}{2} < v \leq 1).$$

2 Fekete-Szegö Problem

Our main result is the following:

Theorem 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. *If*

$$f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \; n \in N_0 = \{0\} \cup N$$

belongs to $M_{\alpha,n}(\phi)$, *then*

(2.1)
$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_2}{3^n(2+4\alpha)} - \frac{\mu}{2^{2n}(1+\alpha)^2}B_1^2 + \frac{1}{3^n(2+4\alpha)}B_1^2 & \text{if } \mu \leq \sigma_1; \\ \frac{B_1}{3^n(2+4\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ -\frac{B_2}{3^n(2+4\alpha)} + \frac{\mu}{2^{2n}(1+\alpha)^2}B_1^2 - \frac{1}{3^n(2+4\alpha)}B_1^2 & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{2^{2n}(1+\alpha)^2\{(B_2 - B_1) + B_1^2\}}{3^n(2+4\alpha)B_1^2},$$
$$\sigma_2 := \frac{2^{2n}(1+\alpha)^2\{(B_2 + B_1) + B_1^2\}}{3^n(2+4\alpha)B_1^2}.$$

The result is sharp.

Proof. For $f(z) \in M_{\alpha,n}(\phi)$, let

$$p(z) = \frac{(1-\alpha)z(D^n f(z))' + \alpha z(D^{n+1} f(z))'}{(1-\alpha)D^n f(z) + \alpha D^{n+1} f(z)} = 1 + b_1 z + b_2 z^2 + ... \tag{2.2}$$

From (2.2), we obtain

$$2^n(1+\alpha)a_2 = b_1 \text{ and } 3^n(2+4\alpha)a_3 = b_2 + 2^{2n}(1+\alpha)^2 a_2^2. \tag{2.3}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + ...$$

is analytic and has a positive real part in E. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{2.4}$$

and from this equation (2.2),

$$1+b_1z+b_2z^2+...=\phi(\frac{c_1z+c_2z^2+...}{2+c_1z+c_2z^2+...})=$$
$$=\phi[\frac{1}{2}c_1z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+...]=$$
$$=1+B_1\frac{1}{2}c_1z+B_1\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+...+B_2\frac{1}{4}c_1^2z^2+...$$

we obtain

$$b_1=\frac{1}{2}B_1c_1 \text{ and } b_2=\frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2.$$

Therefore we have

$$a_3-\mu a_2^2=\frac{B_1}{2.3^n(2+4\alpha)}\{c_2-c_1^2[\frac{1}{2}(1-\frac{B_2}{B_1}+\frac{3^n(2+4\alpha)\mu-2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2}B_1)]\},$$

$$a_3-\mu a_2^2=\frac{B_1}{2.3^n(2+4\alpha)}\{c_2-vc_1^2\}, \tag{2.5}$$

where

$$v:=\frac{1}{2}[1-\frac{B_2}{B_1}+\frac{3^n(2+4\alpha)\mu-2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2}B_1].$$

If $\mu\leq\sigma_1$,then by applying Lemma 1.2 , we get

$$\left|a_3-\mu a_2^2\right|=$$
$$=\frac{B_1}{2.3^n(2+4\alpha)}\left|c_2-c_1^2\{\frac{1}{2}(1-\frac{B_2}{B_1}+\frac{3^n(2+4\alpha)\mu-2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2}B_1)\}\right|$$
$$\left|a_3-\mu a_2^2\right|\leq\frac{B_2}{3^n(2+4\alpha)}-\frac{\mu}{2^{2n}(1+\alpha)^2}B_1^2+\frac{1}{3^n(2+4\alpha)}B_1^2,$$

which is the first part of assertion (2.1).

Next, if $\mu \geq \sigma_2$,by applying Lemma 1.2 , we write

$$\left|a_3 - \mu a_2^2\right| =$$

$$= \frac{B_1}{2.3^n(2+4\alpha)} \left| c_2 - c_1^2\{\frac{1}{2}(1 - \frac{B_2}{B_1} + \frac{3^n(2+4\alpha)\mu - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2} B_1)\} \right|$$

$$\left|a_3 - \mu a_2^2\right| \leq -\frac{B_2}{3^n(2+4\alpha)} + \frac{\mu}{2^{2n}(1+\alpha)^2} B_1^2 - \frac{1}{3^n(2+4\alpha)} B_1^2.$$

If $\mu = \sigma_1$,then equality holds if and only if

$$p_1(z) = (\frac{1+\lambda}{2})\frac{1+z}{1-z} + (\frac{1-\lambda}{2})\frac{1-z}{1+z} \qquad (0 \leq \lambda \leq 1;\ z \in E)$$

or one of its rotations.

If $\mu = \sigma_2$,then

$$\frac{1}{2}[1 - \frac{B_2}{B_1} + \frac{3^n(2+4\alpha)\mu - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2} B_1] = 0.$$

Therefore,

$$\frac{1}{p_1(z)} = (\frac{1+\lambda}{2})\frac{1+z}{1-z} + (\frac{1-\lambda}{2})\frac{1-z}{1+z} \qquad (0 < \lambda < 1; z \in E).$$

Finally, we see that

$$\left|a_3 - \mu a_2^2\right| =$$

$$= \frac{B_1}{2.3^n(2+4\alpha)} \left| c_2 - c_1^2\{\frac{1}{2}(1 - \frac{B_2}{B_1} + \frac{3^n(2+4\alpha)\mu - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2} B_1)\} \right|$$

and

$$\max \left| \frac{1}{2}[1 - \frac{B_2}{B_1} + \frac{3^n(2+4\alpha)\mu - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2} B_1] \right| \leq 1, \qquad (\sigma_1 \leq \mu \leq \sigma_2).$$

Therefore using Lemma 1.2. , we get

$$\left|a_3 - \mu a_2^2\right| = \frac{B_1 \left|c_1\right|}{2.3^n(2+4\alpha)} \le \frac{B_1}{3^n(2+4\alpha)}, \qquad (\sigma_1 \le \mu \le \sigma_2).$$

If $\sigma_1 < \mu < \sigma_2$,then we have

$$p_1(z) = \frac{1+\lambda z^2}{1-\lambda z^2}, \qquad (0 \le \lambda \le 1).$$

Our result now follows by an application of Lemma 1.2. To show that the bounds are sharp, we define the functions $K_\alpha^{\phi_\delta}(\delta = 2, 3, ...)$ by

$$\frac{(1-\alpha)z[D^n K_\alpha^{\phi_\delta}]'(z) + \alpha z[D^{n+1}K_\alpha^{\phi_\delta}]'(z)}{(1-\alpha)[D^n K_\alpha^{\phi_\delta}](z) + \alpha[D^{n+1}K_\alpha^{\phi_\delta}](z)} = \phi(z^{\delta-1}),$$
$$K_\alpha^{\phi_\delta}(0) = 0 = [K_\alpha^{\phi_\delta}]'(0) - 1$$

and function F_α^λ and $G_\alpha^\lambda(0 \le \lambda \le 1)$ by

$$\frac{(1-\alpha)z[D^n F_\alpha^\lambda]'(z) + \alpha z[D^{n+1}F_\alpha^\lambda]'(z)}{(1-\alpha)[D^n F_\alpha^\lambda](z) + \alpha[D^{n+1}F_\alpha^\lambda](z)} = \phi[\frac{z(z+\lambda)}{1+\lambda z}],$$
$$F^\lambda(0) = 0 = (F^\lambda)'(0) - 1$$

and

$$\frac{(1-\alpha)z[D^n G_\alpha^\lambda]'(z) + \alpha z[D^{n+1}G_\alpha^\lambda]'(z)}{(1-\alpha)[D^n G_\alpha^\lambda](z) + \alpha[D^{n+1}G_\alpha^\lambda](z)} = \phi[-\frac{z(z+\lambda)}{1+\lambda z}],$$
$$G^\lambda(0) = 0 = (G^\lambda)'(0) - 1.$$

Clearly the functions $K_\alpha^{\phi_n}, F_\alpha^\lambda, G_\alpha^\lambda \in M_{\alpha,n}(\phi)$. Also we write $K_\alpha^\phi := K_\alpha^{\phi_2}$. If $\mu < \sigma_1$or $\mu > \sigma_2$,then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$,the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_α^λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations.

Remark 2.2. *If* $\sigma_1 \leq \mu \leq \sigma_2$*, then, in view of Lemma 1.2, Theorem 2.1 can be improved. Let* σ_3 *be given by*

$$\sigma_3 := \frac{2^{2n}(1+\alpha)^2\{B_1^2 + B_2\}}{3^n(2+4\alpha)B_1^2}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$*,then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n}(1+\alpha)^2}{3^n(2+4\alpha)B_1^2}[B_1 - B_2 + \frac{3^n\mu(2+4\alpha) - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2}B_1^2]\left|a_2\right|^2 \leq$$

$$\leq \frac{B_1}{3^n(2+4\alpha)}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$*,then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2^{2n}(1+\alpha)^2}{3^n(2+4\alpha)B_1^2}[B_1 + B_2 - \frac{3^n\mu(2+4\alpha) - 2^{2n}(1+\alpha)^2}{2^{2n}(1+\alpha)^2}B_1^2]\left|a_2\right|^2 \leq$$

$$\leq \frac{B_1}{3^n(2+4\alpha)}.$$

Proof. For the values of $\sigma_1 \leq \mu \leq \sigma_3$,we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)\left|a_2\right|^2 =$$

$$= \frac{B_1}{3^n 4(1+2\alpha)}\left|c_2 - vc_1^2\right| + (\mu - \sigma_1)\frac{B_1^2}{4.2^{2n}(1+\alpha)^2}\left|c_1\right|^2 =$$

$$= \frac{B_1}{3^n 4(1+2\alpha)}\left|c_2 - vc_1^2\right| +$$

$$+(\mu - \frac{2^{2n}(1+\alpha)^2\{(B_2 - B_1) + B_1^2\}}{3^n(2+4\alpha)B_1^2})\frac{B_1^2}{4.2^{2n}(1+\alpha)^2}\left|c_1\right|^2 =$$

$$= \frac{B_1}{3^n(2+4\alpha)}\{\frac{1}{2}[\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2]\} \leq \frac{B_1}{3^n(2+4\alpha)}.$$

Similarly, for the values of $\sigma_3 \leq \mu \leq \sigma_2$,we write

$$\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)\left|a_2\right|^2=\frac{B_1}{3^n4(1+2\alpha)}\left|c_2-vc_1^2\right|+(\sigma_2-\mu)\frac{B_1^2}{4.2^{2n}(1+\alpha)^2}\left|c_1\right|^2$$

$$=\frac{B_1}{3^n4(1+2\alpha)}\left|c_2-vc_1^2\right|+(\frac{2^{2n}(1+\alpha)^2\{(B_2+B_1)+B_1^2\}}{3^n(2+4\alpha)B_1^2}-\mu)\frac{B_1^2}{4.2^{2n}(1+\alpha)^2}\left|c_1\right|^2$$

$$=\frac{B_1}{3^n(2+4\alpha)}\{\frac{1}{2}[\left|c_2-vc_1^2\right|+(1-v)\left|c_1\right|^2]\}$$

$$\leq\frac{B_1}{3^n(2+4\alpha)}.$$

Thus, the proof of Remark 2.2 is evidently completed.

3 Applications to Functions Defined by Fractional Derivatives

In order to introduce the class $M_{\alpha,n}^{\lambda}(\phi)$, we need the following:

Definition 3.1. *Let $f(z)$ be analytic in a simply connected region of the $z-$plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^{\lambda}f(z):=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \qquad (0\leq\lambda<1),$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta>0$. Using the above Definition 3.1. and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^{\lambda}:A\to A$ defined by

$$(\Omega^{\lambda}f)(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z), \qquad (\lambda\neq 2,3,4,...).$$

The class $M^{\lambda}_{\alpha,n}(\phi)$ consists of functions $f \in A$ for which $\Omega^{\lambda} f \in M_{\alpha,n}(\phi)$. Note that $M^0_0(\phi) \equiv S*(\phi)$ and $M^{\lambda}_{\alpha,n}(\phi)$ is the special case of the class $M^g_{\alpha,n}(\phi)$ when

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} z^k. \tag{3.1}$$

Let

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k \qquad (g_k > 0).$$

Since

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \in M^g_{\alpha,n}(\phi)$$

If and only if

$$(f * g) = z + \sum_{k=2}^{\infty} k^n g_k a_k z^k \in M_{\alpha,n}(\phi),$$

we obtain the coefficient estimate for functions in the class $M^g_{\alpha,n}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,n}(\phi)$. Applying Theorem 2.1 for the function $(f * g) = z + 2^n g_2 a_2 z^2 + 3^n g_3 a_3 z^3 + ...$, we get the following Theorem 3.2 after an obvious change of the parameter μ :

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ If D^n $f(z)$ given by (1.2) belongs to $M^g_{\alpha,n}(\phi)$, then*

$$\left| a_3 - \mu a_2^2 \right| \leq$$

$$\leq \begin{cases} \frac{1}{g_3}[\frac{B_2}{3^n(2+4\alpha)} - \frac{\mu g_3}{2^{2n}(1+\alpha)^2 g_2^2}B_1^2 + \frac{1}{3^n(2+4\alpha)}B_1^2] & \text{if } \mu \leq \sigma_1; \\ \frac{1}{g_3}[\frac{B_1}{3^n(2+4\alpha)}] \quad \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{g_3}[-\frac{B_2}{3^n(2+4\alpha)} + \frac{\mu g_3}{2^{2n}(1+\alpha)^2 g_2^2}B_1^2 - \frac{1}{3^n(2+4\alpha)}B_1^2] & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2(1+\alpha)^2 2^{2n}}{g_3}[\frac{(B_2 - B_1) + B_1^2}{3^n(2+4\alpha)B_1^2}],$$

$$\sigma_2 := \frac{g_2^2(1+\alpha)^2 2^{2n}}{g_3}[\frac{(B_2 + B_1) + B_1^2}{3^n(2+4\alpha)B_1^2}].$$

The result is sharp.

Since

$$(\Omega^\lambda D^n f)(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} k^n a_k z^k,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2 and g_3 given by (3.2) and (3.3), Theorem 3.2 reduces to the following:

Theorem 3.3. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $D^n f(z)$ given by (1.2) belongs to $M^g_{\alpha,n}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq$$

$$\leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{6}[\frac{B_2}{3^n(2+4\alpha)} - \frac{3(2-\lambda)}{2(3-\lambda)}\frac{\mu}{2^{2n}(1+\alpha)^2}B_1^2 + \frac{1}{3^n(2+4\alpha)}B_1^2], \\ \quad \textit{if} \;\; \mu \leq \sigma_1; \\ \frac{(2-\lambda)(3-\lambda)}{6}[\frac{B_1}{3^n(2+4\alpha)}] \;\; \textit{if}\, \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{(2-\lambda)(3-\lambda)}{6}[-\frac{B_2}{3^n(2+4\alpha)} + \frac{3(2-\lambda)}{2(3-\lambda)}\frac{\mu}{2^{2n}(1+\alpha)^2}B_1^2 - \frac{1}{3^n(2+4\alpha)}B_1^2], \\ \quad \textit{if} \;\; \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{2(3-\lambda)(1+\alpha)^2 2^{2n}}{3(2-\lambda)}[\frac{(B_2 - B_1) + B_1^2}{3^n(2+4\alpha)B_1^2}],$$

$$\sigma_2 := \frac{2(3-\lambda)(1+\alpha)^2 2^{2n}}{3(2-\lambda)}[\frac{(B_2 + B_1) + B_1^2}{3^n(2+4\alpha)B_1^2}].$$

The result is sharp.

Remark 3.3. *When $\alpha = 0$, $n = 0$, $B_1 = 8/\pi^2$ and $B_2 = 16/3\pi^2$ the above Theorem 3.1 reduces to a recent result of Srivastava and Mishra [1, Theorem 8, p. 64] for a class of functions for which $\Omega^\lambda f(z)$ is a parabolic starlike functions* [6, 7].

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