General Mathematics Vol. 16, No. 3 (2008), 19-27

On the Fekete-Szegö inequality for a class of analytic functions defined by using the generalized Sălăgean operator[1]

Dorina Răducanu

Abstract

In this paper we obtain the Fekete-Szegö inequality for a class of analytic functions $f(z)$ defined in the open unit disk for which $\left(\frac{D_\lambda^{n+1}f}{D_\lambda^n f}\right)^\alpha \left(\frac{D_\lambda^{n+2}f}{D_\lambda^{n+1}f}\right)^\beta$ $(\alpha\ ,\beta\ ,\lambda \geq 0)$ lies in a region starlike with respect to 1 and which is symmetric with respect to the real axis.

2000 Mathematics Subject Classification: 30C45

Key words: Analytic functions,generalized Sălăgean operator, Fekete-Szegö inequality

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let S be the subclass of $\mathcal{A}$ consisting of univalent functions.

[1] *Received 28 December, 2007*

Accepted for publication (in revised form) 3 January, 2008

The generalized Sălăgean differential operator is defined in [2] by

$$D_\lambda^0 f(z) = f(z) \ , \ \ D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z)$$

$$D_\lambda^n f(z) = D_\lambda^1 (D_\lambda^{n-1} f(z)) \ , \lambda \geq 0.$$

If f is given by (1) we see that

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k. \tag{1.2}$$

When $\lambda = 1$ we get the classic Sălăgean differential operator [6].

Let $\Phi(z)$ be an analytic function with positive real part on U with $\Phi(0) = 1$, $\Phi'(0) > 0$ which maps the unit disk U onto a region starlike with respect to 1 which is symmetric with respect to the real axis.

Denote by $S^*(\Phi)$ the class of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \Phi(z) \ , \ z \in U$$

and denote by $C(\Phi)$ the class of functions $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z) \ , \ z \in U$$

where " $\prec$ " stands for the usual subordination.The classes $S^*(\Phi)$ and $C(\Phi)$ where defined and studied by Ma and Minda [1]. They obtained the Fekete-Szegö inequality for functions in the class $S^*(\Phi)$ and also for functions in the class $C(\Phi)$.

By using the generalized Sălăgean differential operator we define the following class of functions:

Definition 1.1. *Let $\Phi(z)$ be a univalent stralike function with respect to 1 which maps the unit disk onto a region in the right halfplane symmetric with respect to the real axis,* $\Phi(0) = 1$ *and* $\Phi'(0) > 0$.*A function $f \in \mathcal{A}$ is in the class $M_{\alpha,\beta}^{n,\lambda}(\Phi)$ if*

$$\left(\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} \right)^\alpha \left(\frac{D_\lambda^{n+2} f(z)}{D_\lambda^{n+1} f(z)} \right)^\beta \prec \Phi(z), \tag{1.3}$$

$0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, \lambda > 0.$

It follows that

$$M_{0,1}^{0,1}(\Phi) \equiv C(\Phi) \text{ and } M_{1,0}^{0,1}(\Phi) \equiv S^*(\Phi).$$

When $n = 0$ and $\lambda = 1$ we obtain the class $M_{\alpha,\beta}(\Phi)$ studied by Ravichadran et.al. [3].

In this paper we obtain the Fekete-Szegö inequality for functions in the class $M_{\alpha,\beta}^{n,\lambda}(\Phi)$.

To prove oue results we shall need the following lemmas.

Lemma 1.1. *[1] If $p_1(z) = 1 + c_1z + c_2z^2 + \ldots$ is an analytic function with positive real part in U , then*

$$\left|c_2 - vc_1^2\right| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0 \\ 2, & \text{if } 0 \leq v \leq 1 \\ 4v - 2, & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations.If $0 < v < 1$,then the equality holds if and only if $p_1(z)$ is $(1 + z^2)/(1 - z^2)$ or one of its rotations.If $v = 0$,the equality holds if and only if

$$p_1(z) = \left(\frac{1+a}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-a}{2}\right)\frac{1-z}{1+z} \;,\; 0 \leq a \leq 1$$

or one of its rotations.If $v = 1$,the equality holds if and only if p_1 is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Also the above upper bound is sharp and it can be improved as follows when $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1^2| \leq 2 \;,\; 0 < v \leq \frac{1}{2}$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1^2| \leq 2 \;,\; \frac{1}{2} < v \leq 1.$$

Lemma 1.2. *[4] If $p_1(z) = 1 + c_1z + c_2z^2 + \dots$ is an analytic function with positive real part in U , then*

$$|c_2 - vc_1^2| \leq 2\max\{1; |2v-1|\}.$$

The result is sharp for the function

$$p_1(z) = \frac{1+z^2}{1-z^2} \;\; or \;\; p_1(z) = \frac{1+z}{1-z}.$$

2 Fekete-Szegö problem

We prove our main result by making use of Lemma 1.1.

Theorem 2.1. *Let $\Phi(z) = 1 + B_1z + B_2z^2 + \dots$.If $f(z)$ given by (1.1) is in the class $M_{\alpha,\beta}^{n,\lambda}(\Phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq$$

$$\leq \begin{cases} \frac{1}{4\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]}\left[2B_2 - \frac{B_1^2}{\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\gamma\right], & if\ \mu \leq \sigma_1 \\ \frac{B_1}{2\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]}, & if\ \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{4\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]}\left[-2B_2 + \frac{B_1^2}{\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\gamma\right], & if\ \mu \geq \sigma_2. \end{cases}$$

Further,if $\sigma_1 < \mu \leq \sigma_3$,then

$$|a_3 - \mu a_2^2| +$$

$$+\frac{\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}{2(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1}\left[1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\right]|a_2|^2$$

$$\leq \frac{B_1}{2\lambda(1+2\lambda)^{2n}[\alpha+\beta(1+2\lambda)]}.$$

If $\sigma_3 < \mu \leq \sigma_2$,then

$$|a_3 - \mu a_2^2| +$$

$$+\frac{\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}{2(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1}\left[1 + \frac{B_2}{B_1} - \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\right]|a_2|^2$$

$$\leq \frac{B_1}{2\lambda(1+2\lambda)^{2n}[\alpha+\beta(1+2\lambda)]},$$

where

$$\sigma_1 := \frac{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2(B_2-B_1)}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2} -$$

$$-\frac{B_1^2(1+\lambda)^{2n}[\lambda[\alpha+\beta(1+\lambda)]^2-(\lambda+2)[\alpha+\beta(1+\lambda)^2]]}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2}$$

$$\sigma_2 := \frac{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2(B_2+B_1)}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2} -$$

$$-\frac{B_1^2(1+\lambda)^{2n}[\lambda[\alpha+\beta(1+\lambda)]^2-(\lambda+2)[\alpha+\beta(1+\lambda)^2]]}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2}$$

$$\sigma_3 := \frac{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2 B_2}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2} -$$

$$-\frac{B_1^2(1+\lambda)^{2n}[\lambda[\alpha+\beta(1+\lambda)]^2-(\lambda+2)[\alpha+\beta(1+\lambda)^2]]}{4(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]B_1^2}$$

and

$$\gamma := \lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2 -$$

$$-(\lambda+2)(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)^2] + 4\mu(1+2\lambda)^n[\alpha+\beta(1+2\lambda)].$$

These results are sharp.

Proof. Let $f \in M_{\alpha,\beta}^{n,\lambda}(\Phi)$ and let

$$p(z) := \left(\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\right)^\alpha \left(\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)}\right)^\beta = 1 + b_1 z + b_2 z^2 + \dots \tag{2.1}$$

Since the function $\Phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ is univalent and $p \prec \Phi$ then the function

$$p_1(z) = \frac{1+\Phi^{-1}(p(z))}{1-\Phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 \dots$$

is ananlytic and has positive real part in U.We also have

$$p(z) = \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2\right] z^2 + \dots$$

From (2.1) we obtain

$$b_1 = \frac{1}{2}B_1c_1 \ \text{ and } \ b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2c_1^2.$$

By making use of (1.1) and (1.2) we obtain

$$\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)} = 1 + \lambda(1+\lambda)^n a_2 z + [2\lambda(1+2\lambda)^n a_3 - \lambda(1+\lambda)^{2n}a_2^2]z^2 + \dots$$

and therefore we have

$$\left(\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\right)^\alpha =$$

$$= 1+\alpha\lambda(1+\lambda)^n a_2 z+\lambda\left[2\alpha(1+2\lambda)^n a_3 + \frac{\lambda\alpha^2 - \alpha(\lambda+2)}{2}(1+\lambda)^{2n}a_2^2\right]z^2+\dots$$

Similarly we obtain

$$\left(\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1} f(z)}\right)^\beta = 1 + \beta\lambda(1+\lambda)^{n+1}a_2 z +$$

$$+\lambda\left[2\beta(1+2\lambda)^{n+1}a_3 + \frac{\lambda\beta^2 - \beta(\lambda+2)}{2}(1+\lambda)^{2n+2}a_2^2\right]z^2 + \dots$$

Thus we have

$$\left(\frac{D_\lambda^{n+1}f(z)}{D_\lambda^n f(z)}\right)^\alpha \left(\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1} f(z)}\right)^\beta = 1 + \lambda(1+\lambda)^n[\alpha + \beta(1+\lambda)]a_2 z +$$

$$+\lambda\left\{2(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]a_3 + \right.$$

$$\left. + \frac{\lambda[\alpha+\beta(1+\lambda)]^2 - (\lambda+2)[\alpha+\beta(1+\lambda)^2]}{2}(1+\lambda)^{2n}a_2^2\right\} z^2 + \dots$$

In view of (2.1) it results

$$b_1 = \lambda(1+\lambda)^n[\alpha+\beta(1+\lambda)]a_2 \tag{2.2}$$

and

$$b_2 = 2\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]a_3 +$$

$$+\frac{\lambda^2[\alpha+\beta(1+\lambda)]^2 - \lambda(\lambda+2)[\alpha+\beta(1+\lambda)^2]}{2}(1+\lambda)^{2n}a_2^2. \tag{2.3}$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{4\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]}[c_2 - vc_1^2] \tag{2.4}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\right].$$

Our result follows now by an application of Lemma 1.1.To show that the bounds are sharp,we consider the functions $K_{\Phi,m}(m = 2, 3, \ldots)$ defined by

$$\left(\frac{D_\lambda^{n+1}K_{\Phi,m}(z)}{D_\lambda^n K_{\Phi,m}(z)}\right)^\alpha \left(\frac{D_\lambda^{n+2}K_{\Phi,m}(z)}{D_\lambda^{n+1}K_{\Phi,m}(z)}\right)^\beta = \Phi(z^{m-1}),$$

$$K_{\Phi,m}(0) = [K_{\Phi,m}]'(0) - 1 = 0$$

and the functions F_δ , G_δ $(0 \le \delta \le 1)$ defined by

$$\left(\frac{D_\lambda^{n+1}F_\delta(z)}{D_\lambda^n F_\delta(z)}\right)^\alpha \left(\frac{D_\lambda^{n+2}F_\delta(z)}{D_\lambda^{n+1}F_\delta(z)}\right)^\beta = \Phi\left(\frac{z(z+\delta)}{1+\delta z}\right), F_\delta(0) = F'_\delta(0) - 1 = 0$$

and

$$\left(\frac{D_\lambda^{n+1}G_\delta(z)}{D_\lambda^n G_\delta(z)}\right)^\alpha \left(\frac{D_\lambda^{n+2}G_\delta(z)}{D_\lambda^{n+1}G_\delta(z)}\right)^\beta = \Phi\left(-\frac{z(z+\delta)}{1+\delta z}\right), G_\delta(0) = G'_\delta(0) - 1 = 0.$$

It is clear that the functions $K_{\Phi,m}$, F_δ and G_δ belong to the class $M_{\alpha,\beta}^{n,\lambda}(\Phi)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is $K_{\Phi,2}$ or one of its rotations.When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\Phi,3}$ or one of its rotations.If $\mu = \sigma_1$, then the equality holds if and only if f is F_δ or one of its rotations.If $\mu = \sigma_2$,then the equality holds if and only if f is G_δ or one of its rotations.

By making use of Lemma 1.2. we easely obtain the next theorem.

Theorem 2.2. *Let $\Phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ and let $f(z)$ be in the class $M_{\alpha,\beta}^{n,\lambda}(\Phi)$ .For a complex number μ we have:*

$$|a_3 - \mu a_2^2| \le$$

$$\leq \frac{B_1}{2\lambda(1+2\lambda)^n[\alpha+\beta(1+2\lambda)]} \max\left\{1, \left|-\frac{B_2}{B_1} + \frac{\gamma B_1}{2\lambda(1+\lambda)^{2n}[\alpha+\beta(1+\lambda)]^2}\right|\right\}.$$

The result is sharp.

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Department of Mathematics and Computer Science
"Transilvania" University of Braşov
50091,Iuliu Maniu,50,Braşov
Romania
E-mail: dorinaraducanu@yahoo.com