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Hankel determinant for p-valently starlike and convex functions of order α

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Abstract

For p-valently starlike and convex functions $f(z)$ in the open unit disk $\mathbb{U}$, the upper bounds of the functional $|a_{p+2}-\mu a_{p+1}^2|$, defined by using the second Hankel determinant $H_2(n)$ due to J. W. Noonan and D. K. Thomas (Trans. Amer. Math. Soc. **223**(2) (1976), 337-346), are discussed.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \quad (p\in\mathbb{N}=\{1,2,3,\cdots\})$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.
Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in $\mathbb{U}$ and satisfy

$$\mathrm{Re}\ p(z) > 0 \qquad (z \in \mathbb{U}).$$

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}_p$ satisfies the following condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < p)$, then $f(z)$ is said to be p-valently starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}_p^*(\alpha)$ the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which are p-valently starlike of order α in $\mathbb{U}$. Similarly, we say that $f(z)$ belongs to the class $\mathcal{K}_p(\alpha)$ of p-valently convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}_p$ satisfies the following inequality

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < p)$.
As usual, in the present investigation, we write

$$\mathcal{S}_p^* = \mathcal{S}_p^*(0), \quad \mathcal{K}_p = \mathcal{K}_p(0), \quad \mathcal{S}^*(\alpha) = \mathcal{S}_1^*(\alpha) \quad \text{and} \quad \mathcal{K}(\alpha) = \mathcal{K}_1(\alpha).$$

Remark 1. For a function $f(z) \in \mathcal{A}_p$, it follows that

$$f(z) \in \mathcal{K}_p(\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha)$$

and

$$f(z) \in \mathcal{S}_p^*(\alpha) \quad \text{if and only if} \quad \int_0^z \frac{pf(\zeta)}{\zeta} d\zeta \in \mathcal{K}_p(\alpha).$$

Example 1.

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \in \mathcal{S}_p^*(\alpha)$$

and

$$f(z) = z^p {}_2F_1\left(2(p-\alpha), p; p+1; z\right) \in \mathcal{K}_p(\alpha)$$

where ${}_2F_1(a, b; c; z)$ *represents the hypergeometric function.*

In [7], Noonan and Thomas stated the q–th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \qquad (n, q \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

This determinant is discussed by several authors. For example, we can know that the Fekete and Szegö functional $|a_3 - a_2^2| = |H_2(1)|$ and they consider the further generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number (see, [2]). Moreover, we also know that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $|H_2(2)|$.

Janteng, Halim and Darus [4] have shown the following theorems.

Theorem 1. *Let $f(z) \in \mathcal{S}^*$. Then*

$$|a_2a_4 - a_3^2| \leqq 1.$$

Equality is attained for functions

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

and

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \cdots .$$

Theorem 2. *Let $f(z) \in \mathcal{K}$. Then*

$$|a_2a_4 - a_3^2| \leqq \frac{1}{8}.$$

The present paper is motivated by these results and the purpose of this investigation is to find the upper bounds of the generalized functional $|a_{p+2} - \mu a_{p+1}^2|$, defined by the second Hankel determinant, for functions $f(z)$ in the class $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$, respectively.

2 Preliminary results

In order to discuss our problems, we need some lemmas. The following lemma can be found in [1] or [8].

Lemma 1. *If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then*

$$|c_k| \leqq 2 \quad (k = 1, 2, 3, \cdots).$$

The result is sharp for

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} 2z^k.$$

Using the above, we derive

Lemma 2. *If a function* $p(z) = p + \sum_{k=1}^{\infty} c_k z^k$ *satisfies the following inequality*

$$\text{Re } p(z) > \alpha \quad (z \in \mathbb{U})$$

for some α $(0 \leqq \alpha < p)$, *then*

$$|c_k| \leqq 2(p-\alpha) \quad (k = 1, 2, 3, \cdots). \tag{1}$$

The result is sharp for

$$p(z) = \frac{p + (p-2\alpha)z}{1-z} = p + \sum_{k=1}^{\infty} 2(p-\alpha)z^k.$$

Proof. Let $q(z) = \dfrac{p(z)-\alpha}{p-\alpha} = 1 + \sum_{k=1}^{\infty} \dfrac{c_k}{p-\alpha} z^k$. Noting that $q(z) \in \mathcal{P}$ and using Lemma 1, we see that

$$\left| \frac{c_k}{p-\alpha} \right| \leqq 2 \quad (k = 1, 2, 3, \cdots)$$

which implies

$$|c_k| \leqq 2(p-\alpha) \quad (k = 1, 2, 3, \cdots).$$

Lemma 3. *The power series for $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ converges in $\mathbb{U}$ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \qquad (n = 1, 2, 3, \cdots),$$

where $c_{-k} = \overline{c_k}$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$ for $k \neq j$, where $p_0(z) = \dfrac{1+z}{1-z}$; in this case $D_n > 0$ for $n < m-1$ and $D_n = 0$ for $n \geqq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz, and it can be found in [3]. And then, Libera and Złotkiewicz [5] (see, also [6]) have given the following result by using this lemma with $n = 2, 3$.

Lemma 4. *If a function $p(z) \in \mathcal{P}$, then the representations*

$$\begin{cases} 2c_2 = c_1^2 + (4 - c_1^2)\zeta \\ 4c_3 = c_1^3 + 2(4 - c_1^2)c_1\zeta - (4 - c_1^2)c_1\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \end{cases}$$

for some complex numbers ζ and η $(|\zeta| \leqq 1, |\eta| \leqq 1)$, are obtained.

By virtue of Lemma 4, we have

Lemma 5. *If a function* $p(z) = p + \sum_{k=1}^{\infty} c_k z^k$ *satisfies* $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$ *for some* α $(0 \leqq \alpha < p)$, *then*

$$
\begin{aligned}
(2) \qquad 2(p-\alpha)c_2 &= c_1^2 + \{4(p-\alpha)^2 - c_1^2\}\zeta \\
4(p-\alpha)^2 c_3 &= c_1^3 + 2\{4(p-\alpha)^2 - c_1^2\}c_1\zeta - \{4(p-\alpha)^2 - c_1^2\}c_1\zeta^2 \\
&\quad + 2(p-\alpha)\{4(p-\alpha)^2 - c_1^2\}(1-|\zeta|^2)\eta
\end{aligned}
$$

for some complex numbers ζ *and* η $(|\zeta| \leqq 1, |\eta| \leqq 1)$.

Proof. Since $q(z) = \dfrac{p(z)-\alpha}{p-\alpha} = 1 + \sum_{k=1}^{\infty} \dfrac{c_k}{p-\alpha} z^k \in \mathcal{P}$, replacing c_2 and c_3 by $\dfrac{c_2}{p-\alpha}$ and $\dfrac{c_3}{p-\alpha}$ in Lemma 4, respectively, we immediately have the relations of the lemma.

We also need the next remark.

Remark 2. If $f(z) \in \mathcal{S}_p^*(\alpha)$, then there exists a function $p(z) = p + \sum_{k=1}^{\infty} c_k z^k$ such that $\operatorname{Re} p(z) > \alpha$ $(z \in \mathbb{U})$ and

$$zf'(z) = f(z)p(z)$$

which implies that

$$p + \sum_{n=p+1}^{\infty} n a_n z^{n-p} = p + \sum_{n=p+1}^{\infty} \left(\sum_{l=p}^{n} a_l c_{n-l} \right) z^{n-p}$$

where $a_p = 1$ and $c_0 = p$. Therefore, we have the follwing relation

$$(3) \qquad (n-p)a_n = \sum_{l=p}^{n-1} a_l c_{n-l} \quad (n \geqq p+1).$$

3 Main results

In this section, we begin with the upper bound of $|a_{p+2}-\mu a_{p+1}^2|$ for p-valently starlike functions of order α below.

Theorem 3. *If a function $f(z) \in \mathcal{S}_p^*(\alpha) \quad (0 \leqq \alpha < p)$, then*

$$|a_{p+2}-\mu a_{p+1}^2| \leqq \begin{cases} (p-\alpha)\left\{(2(p-\alpha)+1)-4(p-\alpha)\mu\right\} & \left(\mu \leqq \dfrac{1}{2}\right) \\ p-\alpha & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right) \\ (p-\alpha)\left\{4(p-\alpha)\mu-(2(p-\alpha)+1)\right\} & \left(\mu \geqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z^p}{(1-z)^{2(p-\alpha)}} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right) \\ \dfrac{z^p}{(1-z^2)^{p-\alpha}} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right). \end{cases}$$

Proof. If $f(z) \in \mathcal{S}_p^*(\alpha)$, then we have the equation (3) which means that $a_{p+1} = c_1$ and $a_{p+2} = \dfrac{c_2 + c_1^2}{2}$. Thus, by the inequality (1) and the representation (2), we can suppose that $c_1 = c$ $(0 \leqq c \leqq 2(p-\alpha))$ without loss of generality and we derive

$$\begin{aligned}
|a_{p+2} - \mu a_{p+1}^2| &= \left|\frac{c_2 + c^2}{2} - \mu c^2\right| \\
&= \frac{1}{2}\left|(1-2\mu)c^2 + \frac{c^2 + \{4(p-\alpha)^2 - c^2\}\zeta}{2(p-\alpha)}\right| \\
&= \frac{1}{4(p-\alpha)}|\{2(p-\alpha) - 4(p-\alpha)\mu + 1\}c^2 + \{4(p-\alpha)^2 - c^2\}\zeta| \\
&\equiv A(\zeta).
\end{aligned}$$

Applying the triangle inequality, we deduce

$$A(\zeta) \leqq \frac{1}{4(p-\alpha)}\left[|(2(p-\alpha)+1) - 4(p-\alpha)\mu|\, c^2 + \{4(p-\alpha)^2 - c^2\}\right]$$

$$= \begin{cases} \dfrac{1}{4(p-\alpha)}\left[2(p-\alpha)(1-2\mu)c^2 + 4(p-\alpha)^2\right] & \left(\mu \leqq \dfrac{2(p-\alpha)+1}{4(p-\alpha)}\right) \\[3ex] \dfrac{1}{4(p-\alpha)}\left[2\{2(p-\alpha)\mu - (p+1-\alpha)\}c^2 + 4(p-\alpha)^2\right] & \left(\mu \geqq \dfrac{2(p-\alpha)+1}{4(p-\alpha)}\right) \end{cases}$$

$$\leqq \begin{cases} (p-\alpha)\{(2(p-\alpha)+1) - 4(p-\alpha)\mu\} & \left(\mu \leqq \dfrac{1}{2}, c = 2(p-\alpha)\right) \\[3ex] p-\alpha & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{2(p-\alpha)+1}{4(p-\alpha)}, c = 0\right) \\[3ex] p-\alpha & \left(\dfrac{2(p-\alpha)+1}{4(p-\alpha)} \leqq \mu \leqq \dfrac{p+1-\alpha}{2(p-\alpha)}, c = 0\right) \\[3ex] (p-\alpha)\{4(p-\alpha)\mu - (2(p-\alpha)+1)\} & \left(\mu \geqq \dfrac{p+1-\alpha}{2(p-\alpha)}, c = 2(p-\alpha)\right). \end{cases}$$

Equality is attained for functions $f(z) \in \mathcal{S}_p^*(\alpha)$ defined by

$$\frac{zf'(z)}{f(z)} = p(z) = \frac{p + (p - 2\alpha)z}{1 - z}$$

for the case $c_1 = c = 2(p - \alpha)$, $\zeta = 1$ and $c_2 = 2(p - \alpha)$, or

$$\frac{zf'(z)}{f(z)} = p(z) = \frac{p + (p - 2\alpha)z^2}{1 - z^2}$$

for the case $c_1 = c = 0$, $\zeta = 1$ and $c_2 = 2(p - \alpha)$.

Taking $\alpha = 0$ or $p = 1$ in Theorem 3, we obtain the following corollaries, respectively.

Corollary 1. *If a function* $f(z) \in \mathcal{S}_p^*$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \begin{cases} p\{(2p+1) - 4p\mu\} & \left(\mu \leqq \dfrac{1}{2}\right) \\[2ex] p & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p+1}{2p}\right) \\[2ex] p\{4p\mu - (2p+1)\} & \left(\mu \geqq \dfrac{p+1}{2p}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z^p}{(1-z)^{2p}} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq \dfrac{p+1}{2p}\right) \\[2ex] \dfrac{z^p}{(1-z^2)^p} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{p+1}{2p}\right). \end{cases}$$

Corollary 2. *If a function $f(z) \in \mathcal{S}^*(\alpha)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} (1-\alpha)\left\{(3-2\alpha) - 4(1-\alpha)\mu\right\} & \left(\mu \leqq \dfrac{1}{2}\right) \\[2ex] 1-\alpha & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{2-\alpha}{2(1-\alpha)}\right) \\[2ex] (1-\alpha)\left\{4(1-\alpha)\mu - (3-2\alpha)\right\} & \left(\mu \geqq \dfrac{2-\alpha}{2(1-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z}{(1-z)^{2(1-\alpha)}} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq \dfrac{2-\alpha}{2(1-\alpha)}\right) \\[2ex] \dfrac{z}{(1-z^2)^{1-\alpha}} & \left(\dfrac{1}{2} \leqq \mu \leqq \dfrac{2-\alpha}{2(1-\alpha)}\right). \end{cases}$$

Also, by Corollary 1 and Corollary 2, we readily know

Corollary 3. *If a function $f(z) \in \mathcal{S}^*$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} 3-4\mu & \left(\mu \leqq \dfrac{1}{2}\right) \\[2ex] 1 & \left(\dfrac{1}{2} \leqq \mu \leqq 1\right) \\[2ex] 4\mu - 3 & (\mu \geqq 1) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z}{(1-z)^2} & \left(\mu \leqq \dfrac{1}{2} \text{ or } \mu \geqq 1\right) \\[2ex] \dfrac{z}{1-z^2} & \left(\dfrac{1}{2} \leqq \mu \leqq 1\right). \end{cases}$$

Next, in consideration of Remark 1, we derive the upper bounds of $|a_{p+2} - \mu a_{p+1}^2|$ for p-valently convex functions.

Theorem 4. *If a function* $f(z) \in \mathcal{K}_p(\alpha) \quad (0 \leqq \alpha < p)$, *then*
$|a_{p+2} - \mu a_{p+1}^2| \leqq$

$$\begin{cases} \dfrac{p(p-\alpha)\left\{(2(p-\alpha)+1)(p+1)^2 - 4(p-\alpha)p(p+2)\mu\right\}}{(p+1)^2(p+2)} & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)}\right) \\[2ex] \dfrac{p(p-\alpha)}{p+2} & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p+1-\alpha)}{2p(p+2)(p-\alpha)}\right) \\[2ex] \dfrac{p(p-\alpha)\left\{4(p-\alpha)p(p+2)\mu - (2(p-\alpha)+1)(p+1)^2\right\}}{(p+1)^2(p+2)} & \left(\mu \geqq \dfrac{(p+1)^2(p+1-\alpha)}{2p(p+2)(p-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p {}_2F_1(2(p-\alpha), p; p+1; z) & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)} \ \mathit{or}\ \mu \geqq \dfrac{(p+1)^2(p+1-\alpha)}{2p(p+2)(p-\alpha)}\right) \\[2ex] z^p {}_2F_1\left(\dfrac{p}{2}, p-\alpha; 1+\dfrac{p}{2}; z^2\right) & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^2(p+1-\alpha)}{2p(p+2)(p-\alpha)}\right). \end{cases}$$

Proof. Noting that $f(z) \in \mathcal{K}_p(\alpha)$ if and only if

$\dfrac{zf'(z)}{p} = z^p + \sum\limits_{n=p+1}^{\infty} \dfrac{n}{p} a_n z^n \in \mathcal{S}_p^*(\alpha)$ and using Theorem 3, we see that

$$\left| \frac{p+2}{p} a_{p+2} - \nu \frac{(p+1)^2}{p^2} a_{p+1}^2 \right| \leqq \begin{cases} (p-\alpha)\left\{(2(p-\alpha)+1) - 4(p-\alpha)\nu\right\} \\ \\ p-\alpha \\ \\ (p-\alpha)\left\{4(p-\alpha)\nu - (2(p-\alpha)+1)\right\}, \end{cases}$$

that is, that $\left| a_{p+2} - \dfrac{(p+1)^2}{p(p+2)} \nu a_{p+1}^2 \right| \leqq$

$$\begin{cases} \dfrac{p(p-\alpha)\left\{(2(p-\alpha)+1) - 4(p-\alpha)\nu\right\}}{p+2} & \left(\nu \leqq \dfrac{1}{2}\right) \\ \\ \dfrac{p(p-\alpha)}{p+2} & \left(\dfrac{1}{2} \leqq \nu \leqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right) \\ \\ \dfrac{p(p-\alpha)\left\{4(p-\alpha)\nu - (2(p-\alpha)+1)\right\}}{p+2} & \left(\nu \geqq \dfrac{p+1-\alpha}{2(p-\alpha)}\right). \end{cases}$$

Now, putting $\dfrac{(p+1)^2}{p(p+2)}\nu = \mu$, the proof of the theorem is completed.

When $\alpha = 0$ or $p = 1$ in Theorem 4, the following three corollaries are obtained.

Corollary 4. *If a function $f(z) \in \mathcal{K}_p$, then*

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \begin{cases} \dfrac{p^2\left\{(2p+1)(p+1)^2 - 4p^2(p+2)\mu\right\}}{(p+1)^2(p+2)} & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)}\right) \\[2ex] \dfrac{p^2}{p+2} & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^3}{2p^2(p+2)}\right) \\[2ex] \dfrac{p^2\left\{4p^2(p+2)\mu - (2p+1)(p+1)^2\right\}}{(p+1)^2(p+2)} & \left(\mu \geqq \dfrac{(p+1)^3}{2p^2(p+2)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} z^p {}_2F_1(2p, p; p+1; z) & \left(\mu \leqq \dfrac{(p+1)^2}{2p(p+2)} \text{ or } \mu \geqq \dfrac{(p+1)^3}{2p^2(p+2)}\right) \\[2ex] z^p {}_2F_1\left(\dfrac{p}{2}, p; 1+\dfrac{p}{2}; z^2\right) & \left(\dfrac{(p+1)^2}{2p(p+2)} \leqq \mu \leqq \dfrac{(p+1)^3}{2p^2(p+2)}\right). \end{cases}$$

Corollary 5. *If a function $f(z) \in \mathcal{K}(\alpha)$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \dfrac{(1-\alpha)}{3}\{(3-2\alpha) - 3(1-\alpha)\mu\} & \left(\mu \leqq \dfrac{2}{3}\right) \\[2ex] \dfrac{1-\alpha}{3} & \left(\dfrac{2}{3} \leqq \mu \leqq \dfrac{2(2-\alpha)}{3(1-\alpha)}\right) \\[2ex] \dfrac{(1-\alpha)}{3}\{3(1-\alpha)\mu - (3-2\alpha)\} & \left(\mu \geqq \dfrac{2(2-\alpha)}{3(1-\alpha)}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} \quad \text{and} \quad \log\left(\dfrac{1}{1-z}\right) & \left(\mu \leqq \dfrac{2}{3} \text{ or } \mu \geqq \dfrac{2(2-\alpha)}{3(1-\alpha)}\right) \\[2ex] z {}_2F_1\left(\dfrac{1}{2}, 1-\alpha; \dfrac{3}{2}; z^2\right) & \left(\dfrac{2}{3} \leqq \mu \leqq \dfrac{2(2-\alpha)}{3(1-\alpha)}\right). \end{cases}$$

Corollary 6. *If a function $f(z) \in \mathcal{K}$, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} 1-\mu & \left(\mu \leqq \dfrac{2}{3}\right) \\[2ex] \dfrac{1}{3} & \left(\dfrac{2}{3} \leqq \mu \leqq \dfrac{4}{3}\right) \\[2ex] \mu - 1 & \left(\mu \geqq \dfrac{4}{3}\right) \end{cases}$$

with equality for

$$f(z) = \begin{cases} \dfrac{z}{1-z} & \left(\mu \leqq \dfrac{2}{3} \text{ or } \mu \geqq \dfrac{4}{3}\right) \\[2ex] \dfrac{1}{2}\log\left(\dfrac{1+z}{1-z}\right) & \left(\dfrac{2}{3} \leqq \mu \leqq \dfrac{4}{3}\right). \end{cases}$$

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