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ORTHOGONAL RANDOM VECTORS AND THE HURWITZ-RADON-ECKMANN THEOREM

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Abstract. In several different aspects of algebra and topology the following problem is of interest: find the maximal number of unitary antisymmetric operators U_i in $H = \mathbb{R}^n$ with the property $U_iU_j = -U_jU_i \quad (i \neq j)$. The solution of this problem is given by the Hurwitz-Radon-Eckmann formula. We generalize this formula in two directions: all the operators U_i must commute with a given arbitrary self-adjoint operator and H can be infinite-dimensional. Our second main result deals with the conditions for almost sure orthogonality of two random vectors taking values in a finite or infinite-dimensional Hilbert space H. Finally, both results are used to get the formula for the maximal number of pairwise almost surely orthogonal random vectors in H with the same covariance operator and each pair having a linear support in $H \oplus H$.

The paper is based on the results obtained jointly with N.P.Kandelaki (see [1,2,3]).

1. Introduction. Two kinds of results will be given in this paper. One is of stochastic nature and deals with random vectors taking values in a finite- or infinite- dimensional real Hilbert space H. The other is algebraic or functional-analytic, and deals with unitary operators in H. Our initial problem was to find conditions for almost sure orthogonality of random vectors with values in H. Then the question arose: what is the maximal number of pairwise almost surely orthogonal random vectors in H. The analysis of this question led us to a problem which is a natural extension of an old problem in linear algebra, finally solved in 1942. It can be called the Hurwitz-Radon-Eckmann (HRE) problem in recognition of the authors who made the crucial contribution in obtaining the final solution during the different stages of the investigation.

In Section 2 we give the formulation of this problem, provide its solution, and also give a brief enumeration of areas in which this problem is of primary interest. In Section 3 we give the solution of the generalized HRE problem.

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Section 4 is for the conditions of almost sure orthogonality of two random vectors in H. In Section 5 we give an analysis of these conditions. In Section 6 our initial problem of determining the maximal number of pairwise orthogonal random vectors is solved under some restrictions. These restrictions simplify the problem, so that the generalized HRE formula can provide the solution. Finally, in Section 7 we give the proofs of the theorems formulated in previous sections.

2. The Hurwitz-Radon-Eckmann Theorem. In this section we deal only with the finite-dimensional case: $H = \mathbb{R}^n$. To begin the formulation of the problem, we first recall that a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ is called unitary (or orthogonal) if $U^* = U^{-1}$ (and hence it preserves the distances).

HRE Problem. Find the maximal number of unitary operators $U_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions (I is the identity operator):

$$U_i^2 = -I, \quad U_i U_j = -U_j U_i, \quad i \neq j. \tag{1}$$

The solution of this problem is the number $\rho(n)-1$, where $\rho(n)$ is defined as follows: represent the number n as a product of an odd number and a power of two, $n = (2a(n)+1)2^{b(n)}$, and divide $b(n)$ by 4, $b(n) = c(n)+4d(n)$, where $0 \leq c(n) \leq 3$. Then

$$\rho(n) = 2^{c\,(n)} + 8d(n). \tag{2}$$

The HRE problem is directly connected with the problem of orthogonal multiplication in vector spaces. A bilinear mapping $p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is called an orthogonal multiplication if $\|p(x, y)\| = \|x\| \cdot \|y\|$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$. An orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ exists if $k \leq \rho(n)$ and it can easily be constructed if we have $k-1$ unitary operators satisfying conditions (1). Conversely, if we have an orthogonal multiplication $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ then we can easily construct $k-1$ unitary operators with the properties (1). Of course, there can be different sets of orthogonal operators satisfying (1), and correspondingly, there can be different orthogonal multiplications $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$.

Formula (2) shows that we always have $\rho(n) \leq n$. The equality $\rho(n) = n$ holds only for $n = 1, 2, 4, 8$ and so there exists an inner multiplication in $\mathbb{R}^n$ only for those values of the dimension (and $\mathbb{R}^n$ becomes an algebra for those n). For $n = 1$, the corresponding algebra is the usual algebra of real numbers. For $n = 2, 4$, and 8 we can choose the unitary operators in such a way that the corresponding algebras will, respectively, be the algebras of complex numbers, quaternions, and Kelly numbers. Properties like (1) arose also in the theory of representation of Clifford algebras.

The HRE problem first appeared in the investigation of the classical problem of computing the maximal number of linearly independent vector fields on the surface S^{n-1} of the unit ball in $\mathbb{R}^n$. The linear vector fields were considered first, and the final result for this case given by B.Eckmann (1942) represents this number as $\rho(n)-1$. As was later shown by J.Adams (1962) with the implementation of the K-theory, this number does not increase if we consider, instead of linear vector fields, general (continuous) vector fields.

The information given in this section can be found with discussions and further references among others in [4], Chapter 10.

3. The Generalized HRE Problem and Its Solution. Now and in what follows H can be infinite-dimensional, and in this case we suppose that it is separable. The statement that a continuous linear operator U is unitary means that the image $\operatorname{im} U = H$ and $U^* = U^{-1}$.

Let B be a given arbitrary continuous self-adjoint linear operator in H.

Generalized HRE Problem. Find the maximal number of unitary operators U_i in H satisfying conditions (1) along with the additional condition $U_i B = B U_i$ for all i.

Clearly, if $H = \mathbb{R}^n$ and $B = I$, this problem coincides with the HRE problem. To formulate the solution of this problem we need the following auxiliary assertion.

Theorem on Multiplicity of the Spectrum (see [5], Theorem VII.6). *For any continuous linear self-adjoint operator B there exists a decomposition $H = H_1 \oplus H_2 \oplus \cdots \oplus H_\infty$ that satisfies the following conditions:*

a) *Each H_m $(m = 1, 2, \cdots, \infty)$ is invariant with respect to B;*

b) *The restriction of B to H_m is an operator of homogeneous multiplicity m, i.e., is unitarily equivalent to the operator of multiplication by the independent variable in the product of m copies of the space $L_2(\mu_m)$;*

c) *The measures μ_m are given on the spectrum of B, are finite, and are mutually singular for different m (in fact, it is not the measures themselves that are of importance, but the collections of corresponding sets of zero measure).*

Remark. For some m the measures μ_m can be zero; the collection of the remaining m is denoted by $\mathfrak{M}$. Now let

$$\rho(B) = \min_{m \in \mathfrak{M}} \rho(m), \tag{3}$$

where $\rho(m)$ is defined by the equality (2) for m=$1, 2, \cdots$ and $\rho(\infty) = \infty$.

Note that if the operator B has purely point spectrum, then the relation (3) gives

$$\rho(B) = \min_j \rho(m_j),$$

where $m_1, m_2 \cdots$ are the multiplicities of eigenvalues $\lambda_1, \lambda_2, \cdots$ of B. In particular, $\rho(I) = \rho(n)$ if $H = \mathbb{R}^n$ and $\rho(I) = \infty$ if H is infinite-dimensional.

Now we give the formulation of one of the main results of this paper.

Theorem 1 (Solution of the Generalized HRE problem). *The maximal number of unitary operators in H satisfying conditions* (1) *and also commuting with B is equal to $\rho(B)-1$.*

Remark 1. For the case $H=\mathbb{R}^n$ and $B=I$ this theorem gives the HRE result. However, our proof of Theorem 1 is based on the HRE theorem and, of course, we do not pretend to have a new proof for it.

Remark 2. As has been noted, $\rho(I)=\infty$ if H is infinite-dimensional. Thus Theorem 1 says that in the infinite-dimensional case there exists an infinite set of unitary operators satisfying the condition (1).

As an easy consequence of Theorem 1 we get the following simple assertions.

Corollary 1. *No self-adjoint operator having an eigenvalue of an odd multiplicity can commute with a unitary antisymmetric operator.*

Corollary 2. *There does not exist a compact self-adjoint operator in H which commutes with infinitely many unitary operators satisfying condition* (1).

4. Orthogonality Conditions for Two Random Vectors. We begin this section with some preliminaries which are meant mostly for those readers who usually do not deal with probabilistic terminology. (This preliminary material can be found with detailed discussions and proofs in [6], Chapter 3). Let $(\Omega,\mathcal{B},P)$ be a basic probability space, i.e., a triple where Ω is an abstract set, $\mathcal{B}$ is some σ-algebra of its subsets and P is a normed measure on $\mathcal{B}$, $P(\Omega)=1$. Let ξ be a random vector with values in H. Because of the assumed separability of H the two main definitions of measurable sets in H coincide and a random vector means nothing but a Borel measurable function $\Omega\to H$. We will assume for simplicity that ξ is centered (has zero mean):

$$E(\xi|h)=0 \quad \text{for all} \quad h\in H,$$

where E stands for the integral over Ω with respect to the measure P and $(\cdot|\cdot)$ denotes the scalar product in H. We will consider only random vectors having weak second order:

$$E(\xi|h)^2<+\infty \quad \text{for all} \quad h\in H.$$

This restriction is less than the demand of strong second order ($E\|\xi\|^2<+\infty$) and coincides with it only if H is finite-dimensional.

For any random vector ξ having weak second order we can define an analogue of covariance matrix which will be a continuous linear operator $B : H \to H$ defined by the relation

$$(Bh|g) = E(\xi|h)(\xi|g), \quad h, g \in H \tag{4}$$

(we recall that ξ is assumed to be centered).

Any covariance operator is self-adjoint and positive: $(Bh|h) \geq 0, h \in H$.

If we have two random vectors ξ_1 and ξ_2 we can define also the cross-covariance (or mutual covariance) operator $T = T_{\xi_1\xi_2}$ as follows (we assume again, for simplicity, that ξ_1 and ξ_2 are centered):

$$(Th|g) = E(\xi_1|h)(\xi_2|g). \quad h, g \in H.$$

The cross-covariance operator T is also a continuous linear operator and satisfies the condition

$$(Th|g)^2 \leq (B_1h|h)(B_2g, g), \quad h, g \in H, \tag{5}$$

where B_i is the covariance operator for $\xi_i (i = 1, 2)$.

The pair (ξ_1, ξ_2) can be regarded as a random vector with values in the Hilbert space $H \oplus H$ and the usual definition, like (4), of covariance operator can be applied. Then, using the fact that the inner product in the Hilbert direct sum $H \oplus H$ is given as the sum of the inner products of the components, we easily get that the covariance operator K of the pair (ξ_1, ξ_2) is determined by a 2×2 matrix with operator-valued elements:

$$K = \begin{pmatrix} B_1, & T^* \\ T, & B_2 \end{pmatrix},$$

where T^* is the operator adjoint to T (in fact it is equal to $T_{\xi_2\xi_1}$).

Now we can formulate the main result of this section, which gives sufficient conditions for almost sure (P-almost everywhere) orthogonality of random vectors ξ_1 and ξ_2. It may seem somewhat surprising that the conditions can be expressed only in terms of the covariance operator K (second moment characteristics) and more specific properties of the distribution have no effect.

Theorem 2. *If the covariance operator K satisfies the conditions*

$$T^*B_1 = -B_1T, \quad TB_2 = -B_2T^*, \quad T^2 = -B_2B_1, \tag{6}$$

then any (centered) random vector (ξ_1, ξ_2) with this covariance operator has almost surely orthogonal components, i.e., $P\{(\xi_1|\xi_2) = 0\} = 1$.

Generally speaking, condition (6) is not necessary. Here is a simple example: $\xi_1 = \epsilon_1\zeta$, $\xi_2 = \epsilon_2 U\zeta$, where ϵ_1 and ϵ_2 are independent Bernoulli random variables ($P\{\epsilon_i = 1\} = P\{\epsilon_i = -1\} = 1/2; \quad i = 1,2$), U is a continuous linear antisymmetric operator in $H(U^* = -U)$, and ζ is any nondegenerate random vector in H.

However, the necessity holds for a wide class of distributions, containing Gaussian ones.

Theorem 3. *If the support of random vector $\xi = (\xi_1, \xi_2)$ is a linear subspace of $H \oplus H$, then conditions* (6) *are also necessary for ξ_1 and ξ_2 to be almost surely orthogonal.*

5. Analysis of the Orthogonality Conditions. The orthogonality conditions (6) are in fact an operator equation with triples (B_1, B_2, T) as its solutions. For the special case $H = R^2$ the general solution of this equation can easily be given. For the case $H = R^n$ with $n > 2$ and, especially, for the infinite-dimensional case we cannot expect to have the same simple picture. However some basic properties of solutions can be described.

We begin with simple properties.

Theorem 4. *If conditions* (6) *hold for the operators B_1, B_2, T, then the following assertions are true:*

a) $\ker B_1 \subset \ker T, \ \ker B_2 \subset \ker T^*$; (7)

$\overline{\mathrm{im}} T^* \subset \overline{\mathrm{im}} B_1, \ \overline{\mathrm{im}} T \subset \overline{\mathrm{im}} B_2$; (8)

b) *T commutes with B_1 if and only if it commutes with B_2 and this happens if and only if $T^* = -T$. In this case we have also $B_1B_2 = B_2B_1$;*

c) *$T = T^*$ only if $T = 0$;*

d) *TT^* is not necessarily equal to T^*T (so T is not necessarily a normal operator);*

e) *$TB_2B_1 = B_2B_1T, \ T^*B_1B_2 = B_1B_2T^*$;*

f) *If $B_1 = B_2$, then $TB = BT$ and $T^* = -T$;*

g) *In the finite-dimensional case $H = R^n$ with an odd n, either* $\ker B_1 \neq 0$ *or* $\ker B_2 \neq 0$*;*

h) *In any finite-dimensional case the trace of T is zero;*

i) *In the two-dimensional case $H = R^2$ we always have $B_1B_2 = B_2B_1$.*

The main part of the next theorem shows that conditions (6) for the covariance operator of a random vector (ξ_1, ξ_2) are essentially equivalent to the existence of a linear antisymmetric connection between the components. To avoid the word "essentially," we assume that one of the covariance operators B_1 or B_2 is nonsingular. Suppose for definiteness that $\ker B_1 = 0$ ($B_1 h = 0 \Rightarrow h = 0$). This assumption implies that the inverse operator B_1^{-1} exists; in general it is unbounded, and not defined on the whole of H but only on the range $\mathrm{im}\, B_1$. Consider the operator TB_1^{-1} on this dense linear manifold. It is easy to verify that under conditions (6)

TB_1^{-1} is always closable. Denote the closure by U and its domain by $\mathcal{D}(U)$. Clearly, $\operatorname{im} B_1 \subset \mathcal{D}(U) \subset H$. In some cases we can have $\mathcal{D}(U) = H$ and then U is continuous. Finally, denote by Γ the graph of U, i.e.,

$$\Gamma = \{(x, Ux),\ x \in \mathcal{D}(U)\}$$

and let also

$$\Gamma' = \{(Ux, x),\ x \in \mathcal{D}(U)\}.$$

Theorem 5. *Suppose that the covariance operator K of the random vector (ξ_1, ξ_2) satisfies conditions* (6) *and* $\ker B_1 = 0$. *Then the following assertions are true:*

a) $\overline{\operatorname{im} K} = \Gamma,\ \ \ker K = \Gamma'$;

b) $\mathcal{D}(U^*) \supset \mathcal{D}(U)$ *and* $U^* = -U$ *on* $\mathcal{D}(U)$;

c) $B_2 = -UB_1U$, *and, moreover,* $\mathcal{D}(U)$ *is a dense Bortel set in* $H, P\{\xi_1 \in \mathcal{D}(U)\} = 1$ *and* $P\{\xi_2 = U\xi_1\} = 1$.

Remark 1. If instead of $\ker B_1 = 0$ we assume $\ker B_2 = 0$, then we can introduce the operator V, which is the closure of $T^* B_2^{-1}$. Of course, for V the theorem is again true (with natural slight alterations). If both B_1 and B_2 are nonsingular, we can introduce both U and V; they will be convertible and we will have $U^{-1} = V$.

Remark 2. Let both B_1 and B_2 be nonsingular. Then we have both U and V. Generally, neither U nor V is necessarily extended to a continuous operator in H. The example below shows that in fact all four possibilities can be realized.

Example 1. Let some basis in H be fixed and B_1 be given as a diagonal matrix with positive numbers $\lambda_1, \lambda_2, \dots$ on the diagonal. Let B_2 be also diagonal with the positive numbers $a_1^2\lambda_2$, $a_1^2\lambda_1$, $a_2^2\lambda_4$, $a_2^2\lambda_3$, $a_3^2\lambda_6$, $a_3^2\lambda_5, \dots$ on the diagonal. Finally let T be quasi-diagonal with the following two-dimensional blocks on the diagonal:

$$\begin{pmatrix} 0, & a_1\lambda_2 \\ -a_1\lambda_1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & a_2\lambda_4 \\ -a_2\lambda_3, & 0 \end{pmatrix}, \begin{pmatrix} 0, & a_3\lambda_6 \\ -a_3\lambda_5, & 0 \end{pmatrix}, \dots$$

Remark 3. Let A denote the linear operator determined in $H \oplus H$ by the matrix $\|A_{ij}\|$, where $A_{11} = A_{22} = O$ and $A_{12} = A_{21} = I$, and let $d = AK$. Conditions (6) can be written as $d^2 = O$ ("differentiality" of d). According to assertion a in Theorem 5 we have $\overline{\operatorname{im} d} = \ker d$ (and get zero homology), provided B_1 or B_2 is nonsingular. If this is not the case, then the inclusion $\overline{\operatorname{im} d} \subset \ker d$ (which is a consequence of $d^2 = O$) can be strict. Here is a simple example.

Example 2. $H = R^4$, B_1 and B_2 are given by diagonal matrices with the numbers $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 = \lambda_4 = 0$ and $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 > 0$ respectively, and $T = 0$.

Remark 4. According to assertion c of the theorem, any triple (B_1, B_2, T) with nonsingular B_1, satisfying conditions (6) can be given with the pair of operators (B, U) by the following relations: $B_1 = B$, $B_2 = -UBU$, $T = UB$. In a finite-dimensional case the converse is also true: any pair of operators (B, U), where B is an arbitrary nonsingular self-adjoint positive operator and U is an arbitrary antisymmetric operator, gives the triple satisfying conditions (6). In the infinite-dimensional case, as shown by Example 1, unbounded operators can arise in the direct assertion. Therefore, if we want to obtain the general solution of the system (6) we should either confine the collection of possible pairs (B, U) or extend the triples (B_1, B_2, T) to admit unbounded operators. However, when $B_1 = B_2$, unbounded operators do not occur and the problem of the description of the general solution of (6) can be solved using only continuous operators.

Theorem 6. *If* $B_1 = B_2 = B$ $(\ker B = 0)$*, then* $\mathcal{D}(U) = H$ *and hence* U *is continuous. Furthermore, we have* $UB = BU$, $U^2 = -I$ *and so the antisymmetric operator* U *is also unitary* $(U^* = U^{-1})$.

6. Systems of Pairwise Orthogonal Random Vectors. Now we consider systems of $k > 2$ random vectors and look for conditions of pairwise almost sure orthogonality. A direct application of the conditions for two random vectors to each pair of the system might yield a solution, but the latter would be too complicated to be of interest. Therefore, to ease the problem we impose some restrictions on the systems under consideration. Namely, we assume that all random vectors $\xi_1, \xi_2, \dots, \xi_k$ have the same covariance operator B, and that each pair (ξ_i, ξ_j) has a linear support in $H \oplus H$. We assume also, without loss of generality, that $\ker B = 0$ and all ξ_i are centered. Such systems of random vectors we call $S(H, B)$–systems. An $S(H, B)$–system $\xi_1, \xi_2, \dots, \xi_k$ is said to be an $SO(H, B)$–system if $P\{(\xi_i|\xi_j) = 0\} = 1$ for $i, j = 1, 2, \dots, k$; $i \neq j$.

Let now $(\xi_1, \xi_2, \dots, \xi_k)$ be any $SO(H, B)$–system. Fix one of ξ_i's, say ξ_1, and consider the pairs (ξ_1, ξ_2), $(\xi_1, \xi_3), \dots, (\xi_1, \xi_k)$. Denote by T_i the cross-covariance operator $T_{\xi_1\xi_i}$ and let $U_i = T_i B^{-1} (i = 2, 3, \dots, k)$. According to Theorem 3 and Theorem 6 we have $\xi_i = U_i \xi_1$ almost surely, and each of the U_i's is unitary and commutes with B, and also we have $U_i^2 = -I$. It is easy to show that orthogonality of ξ_i and ξ_j gives the condition $U_i U_j = -U_j U_i$, and if we apply Theorem 1, we get $k \leq \rho(B)$. Conversely, let now $U_2, U_3, \dots, U_{\rho(B)}$ be $\rho(B) - 1$ unitary operators from the generalized HRE problem which exist again by Theorem 1. Let also ξ_1 be any centered random vector with a linear support in H and with covariance operator B.

It is easy to verify that $(\xi_1, U_2\xi_1, U_3\xi_1, \dots, U_{\rho(B)}\xi_1)$ is an $SO(H,B)$–system. Therefore, we have derived the following result.

Theorem 7. *For any covariance operator B there exists an $SO(H,B)$-system containing $\rho(B)$ random vectors and this is the maximal number of random vectors forming any $SO(H,B)$-system.*

Finally we give some corollaries of this theorem concerning Gaussian random vectors.

Corollary 1. *For any natural number k there exists an $SO(\mathbb{R}^n, B)$–system consisting of k Gaussian random vectors (n and B should be chosen properly).*

Corollary 2. *For any natural number k there exists an $SO(H,B)$–system consisting of k Gaussian random vectors such that H is infinite-dimensional and the Gaussian random vectors are also infinite-dimensional.*

Corollary 3. *There does not exist an infinite $SO(H,B)$-system consisting of Gaussian random vectors.*

Remark. Corollary 3 means that an infinite system of centered Gaussian random vectors which are pairwise almost surely orthogonal does not exist if: a) all pairs of the system have linear supports in $H \oplus H$; b) all vectors of the system have the same covariance operator. In connection with this we note that such kind of system does exist if we drop either one of these two restrictions.

7. Proofs of the Results.

Proof of Theorem 1. The proof is performed in two steps. First we consider the case of the operator B having a homogeneous multiplicity.

Lemma 1. *Let B be a linear bounded operator of homogeneous multiplicity m $(1 \le m \le \infty)$. There exist k $(0 \le k \le \infty)$ unitary operators satisfying conditions* (1) *and commuting with B if and only if $k \le \rho(m) - 1$ (we recall that $\rho(\infty)$ is defined as ∞).*

Proof. According to the condition on B there exists a linear isometry v from H onto $L_2^m(\mu)$ such that $B = v^{-1}\bar{B}v$, where $L_2^m(\mu)$ is the Hilbert direct sum of m copies of $L_2(\mu)$ with some finite Borel measure μ supported by a compact set $M \subset R^1$ and $\bar{B}$ is the operator from $L_2^m(\mu)$ to itself defined by the equality

$$(\bar{B}f)(\lambda) = \lambda f(\lambda), \quad \lambda \in M.$$

Here $f = (f_1, f_2, \dots, f_m)$ with $f_i \in L_2(\mu)$ $(i = 1, 2, \dots, m)$ for the case of finite m and $f = (f_1, f_2, \dots)$ with the additional assumption $\sum \|f_i\|^2 < \infty$ if $m = \infty$.

Consider first the case $m < \infty$. To prove the sufficiency part of the lemma we construct $\rho(m)-1$ unitary operators $\bar{U}_i$ in $L_2^m(\mu)$ that satisfy conditions (1) and commute with $\bar{B}$; the operators $U_i = v^{-1}\bar{U}_i v$ will solve the problem in H. By virtue of classical HRE theorem there exist $\rho(m)-1$ orthogonal operators $\widetilde{U}_i$ in R^m satisfying conditions (1). Let $\|\widetilde{U}_i(p,q)\|$ $(p,q=1,2,\dots,m)$ be the matrix of $\widetilde{U}_i$ in the natural basis of R^m, and define the operator $\bar{U}_i : L_2^m(\mu) \to L_2^m(\mu)$ as $\bar{U}_i f = g$ $(i=1,2,\dots,\rho(m)-1)$, where

$$g_p(\lambda) = \sum_{q=1}^{m} \widetilde{U}_i(p,q) f_q(\lambda), \quad p=1,2,\dots,m. \tag{9}$$

It is easy to check that the operators $\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{\rho(m)-1}$ have all the needed properties.

To prove the necessity part of the lemma we have to show that $k \le \rho(m)-1$ if $U_1, U_2, \dots, U_k$ is any system of unitary operators satisfying (1) and commuting with B. Let $\bar{U}_i = vU_iv^{-1}$ and $\bar{B} = vBv^{-1}$ be the corresponding isometric images of U_i and B. These are the operators acting from $L_2^m(\mu)$ to itself. Any linear operator $L_2^m(\mu) \to L_2^m(\mu)$ can be written in a standard way as an $m \times m$ matrix with entries that are $L_2(\mu) \to L_2(\mu)$ operators. Let $\|\bar{U}_i(p,q)\|$ be the matrix of the operator $\bar{U}_i$ $(i=1,2,\dots,k;$ $p,q=1,2,\dots,m)$. Since $\bar{U}_i\bar{B} = \bar{B}U_i$, we have $\bar{U}_i f_0(\bar{B}) = f_0(\bar{B})\bar{U}_i$ for any continuous function $f_0 : R^1 \to R^1$. Clearly, the operator $f_0(\bar{B})$ is the multiplication by the function f_0. Therefore for all $i=1,2,\dots,k$ and $f \in L_2^m(\mu)$ we have the following m relations $(p=1,2,\dots,m)$:

$$\sum_{q=1}^{m} \big[\bar{U}_i(p,q) f_0 f_q\big](\lambda) = f_0(\lambda) \sum_{q=1}^{m} \big[\bar{U}_i(p,q) f_q\big](\lambda).$$

If we take now $f = (f_1, f_2, \dots, f_m)$ with $f_q \equiv 0$ for $q \ne s$ and $f_s \equiv 1$ $(s=1,2,\dots,m)$, we get

$$\big[\bar{U}_i(p,s) f_0\big](\lambda) = \bar{V}_i(p,s;\lambda) f_0(\lambda) \tag{10}$$

where

$$\bar{V}_i(p,s;\lambda) = \big[\bar{U}_i(p,s)1\big](\lambda).$$

Since the operators $\bar{U}_i(p,s)$ are bounded, the relations (10) hold not only for continuous f_0 but also for all $f_0 \in L_2(\mu)$. Now it can be shown by elementary reasonings that for almost all fixed values of λ the $R^m \to R^m$ operators corresponding to the k matrices $\|\bar{V}_i(p,q)\|$ are unitary and satisfy conditions (1). Therefore, $k \le \rho(m)-1$ by the classical HRE theorem.

To finish the proof of the lemma, we consider the case $m=\infty$. In this case only the sufficiency part is to be proved. The existence of an infinite system of unitary operators in H satisfying conditions (1) was proved in

[7]. The proof that we give here (see also [3]) is based on the same idea, although the use of block matrices simplifies the technique of the proof.

Let Δ_i be a quadratic matrix of order 2^i with the second (nonprincipal) diagonal consisting of $+1$'s in the upper half and -1's in the lower half and all other entries equal to zero. Denote by $\widetilde{U}_i$ $(i = 1, 2, \dots)$ the infinite diagonal block matrix with the matrices Δ_i on the (principal) diagonal. Clearly, Δ_i are unitary and $\Delta_i^2 = -I$. Hence $\widetilde{U}_i$ are unitary and $\widetilde{U}_i^2 = -I$. To prove the property $\widetilde{U}_i\widetilde{U}_j = -\widetilde{U}_j\widetilde{U}_i$ $(i \neq j)$, it is convenient to consider $\widetilde{U}_j$ (if $j < i$) as a diagonal block matrix with the matrices (blocks) of the same order 2^i as in the case of $\widetilde{U}_i$. This can be achieved by combining 2^{i-j} diagonal blocks of $\widetilde{U}_j$ in one with zeros as other entries. Denote this matrix (block) of order 2^i by $\Delta_{j,i}$. Now $\widetilde{U}_j$ is a diagonal block matrix with the diagonal blocks $\Delta_{j.i}$ of the same order 2^i, and it is enough to show that $\Delta_i\Delta_{j,i} = -\Delta_{j,i}\Delta_i$. For this, recall that $\Delta_{j,i}$ is a diagonal block matrix with 2^{i-j} diagonal blocks of order 2^j each, and represent Δ_i also as a block matrix with the blocks of order 2^j each. This way we get a block matrix with blocks that are all zero matrices except those situated on the second (nonprincipal) diagonal which are δ in the uppeer half and $-\delta$ in the lower one. Here δ is the matrix of order 2^j with $+1$'s on the second (nonprincipal) diagonal and all other entries equal to zero. Now it is quite easy to show that the needed equality $\Delta_i\Delta_{j,i} = -\Delta_{j,i}\Delta_i$ is a consequence of the elementary one: $\Delta_j\delta = -\delta\Delta_j$, and this completes the proof of $\rho(\infty) = \infty$. The proof of the lemma is also finished now: define the operators $\bar{U}_i$ $(i = 1, 2, \dots)$in $L_2^m(\mu)$, $m = \infty$, by the relations (9) with $m = \infty$; it can easily be checked that the operators $\bar{U}_i$ $(i = 1, 2, \dots)$ satisfy conditions (1) and commute with $\bar{B}$.

Now we can finish the proof of Theorem 1. Clearly, $\rho(B) = \rho(m)$ if B is of homogeneous multiplicity m, and Lemma 1 coincides with Theorem 1 for this case. For the general case we will write B as the diagonal matrix with the restrictions of B to H_1, $H_2, \dots$ on the diagonal (this is possible because every H_m is invariant for B). The restriction of B to H_m, $m \in \mathfrak{M}$, is of homogeneous multiplicity m, and by Lemma 1 there exist $\rho(B) - 1$ unitary operators U_i^m $(i = 1, 2, \dots, \rho(B) - 1)$ in each H_m $(m \in \mathfrak{M})$ that satisfy conditions (1) and commute with $B|_{H_m}$. The $\rho(B) - 1$ unitary operators corresponding to diagonal matrices with the operators U_i^1, $U_i^2, \dots$ on the diagonal satisfy conditions (1) and commute with B.

Finally, let U_i $(i = 1, 2, \dots, k)$ be unitary operators in H satisfying conditions (1) and commuting with B, and let H_m $(m \in \mathfrak{M})$ be the invariant subspaces corresponding to B. Because of commutativity, the subspaces H_m are invariant also for all U_i and Lemma 1 easily gives that $k \leq \rho(B) - 1$. □

Proof of Theorem 2. Let A denote the 2×2 matrix with the operator-valued elements $A_{11} = A_{22} = O$ and $A_{12} = A_{21} = \frac{1}{2}I$, where O and I

denote, as before, zero and identity operators. It is obvious that $(\xi_1|\xi_2) = (A\xi|\xi)$, $\xi = (\xi_1, \xi_2)$, and hence the problem is transformed to the problem of orthogonality of ξ and $A\xi$ in the Hilbert space $H_1 \oplus H_2$. It is easily seen, using the definition, that the covariance operator of $A\xi$ is AKA. We have the equalities

$$(AKA)K = (AK)^2 = \frac{1}{4}\left[\begin{pmatrix} O & I \\ I & O \end{pmatrix}\begin{pmatrix} B_1, & T^* \\ T, & B_2 \end{pmatrix}\right]^2 =$$
$$= \frac{1}{4}\begin{pmatrix} T^2 + B_2B_1, & TB_2 + B_2T^* \\ B_1T + T^*B_1, & B_1B_2 + T^{*2} \end{pmatrix} = O$$

and the use of the following lemma ends the proof.

Lemma 2. *Let ζ and η be centered random vectors in a Hilbert space H, with covariance operators B_ζ and B_η, respectively. If $B_\zeta B_\eta = O$, then $P\{(\zeta|\eta) = 0\} = 1$.*

Proof. The topological support S_ζ of a random vector ζ with values in a (separable) Hilbert space H is defined as the minimal closed set in H having probability 1, i.e. as the intersection of all closed sets $F \subset H$ such that $P\{\zeta \in F\} = 1$. Denote by $l(S_\zeta)$ the minimal closed subspace in H containing S_ζ. If $h \perp l(S_\zeta)$, then $h \perp \operatorname{im} B_\zeta$ and $(B_\zeta h, h) = O$; hence $(B_\zeta g, h) = 0$ for all $g \in H$, and $h \perp \operatorname{im} B_\zeta$. Conversely, if $h \perp \operatorname{im} B_\zeta$ then $P\{\zeta \perp h\} = 1$ and so the closed subspace orthogonal to h has probability 1; hence it contains $l(S_\zeta)$ and we get $h \perp l(S_\zeta)$. Therefore we have

$$l(S_\zeta) = \overline{\operatorname{im} B_\zeta}. \tag{11}$$

The condition of the lemma gives that $(B_\zeta h|B_\eta g)! =!0$ for all $h, g \in H$. So, $\overline{\operatorname{im} B_\zeta} \perp \overline{\operatorname{im} B_\eta}$ and the application of relation (11) together with the following obvious relation

$$P\{\zeta \in l(S_\zeta),\ \ \eta \in l(S_\eta)\} = 1$$

completes the proof. □

Proof of Theorem 3. We have $S_\xi = \overline{\operatorname{im} K}$ because of the linearity of S_ξ and the relation (11) written for ξ (recall that $B_\xi = K$). On the other hand, the condition $\xi_1 \perp \xi_2$ a.s. gives that $S_\xi \subset L$ where $L = \{(u,v),\ u, v \in H,\ (u|v) = 0\}$. Therefore, $\operatorname{im} K \subset L$ and so, for all $x, y \in H$, we have the relation

$$(B_1x + T^*y|Tx + B_2y) = 0. \tag{12}$$

If we take $y = 0$, then we get the relation $(T^*B_1x, x) = 0$ for all $x \in H$, which shows that the operator T^*B_1 is antisymmetric and thus proves the first equality in (6). The second one is proved in the same way by taking $x = 0$ in (12). Now the relation (12) gives that $(T^*y|Tx) + (B_1x|B_2y) =$

$((T^2 + B_2B_1)x|y) = 0$ for all $x, y \in H$, and the third equality in (6) is also proved. $\square$

Remark. The necessity of conditions (6) was originally proved for the case of Gaussian random vectors. The possibility of extension to this more general case was noticed later by S.A.Chobanyan.

Proof of Theorem 4. a) Relations (7) are an easy consequence of inequality (5). Relations (8) follow from (7) because $\ker A + \overline{\operatorname{im} A^*} = H$ for any linear bounded operator A.

b) It is enough to show that $TB_1 = B_1T$ gives $T^* = -T$ (the implication $TB_2 = B_2T \Rightarrow T^* = -T$ can be shown analogously). The condition $T^*B_1 = -B_1T$ gives $T^*x = -Tx$ for $x \in \overline{\operatorname{im} B_1}$. Let now $x \in \ker B_1$. Then, according to (7), $Tx = 0$ and it suffices to show that $T^*x = 0$ too, or $(T^*x|y) = 0$ for all $y \in H$. If $y \in \ker B_1$, this is clear; if $y \in \operatorname{im} B_1$, then $y = B_1z$ for some $z \in H$ and $(T^*x|y) = (x|TB_1z) = (B_1x|Tz) = 0$ since $x \in \ker B_1$. The last assertion is an easy consequence of $T^* = -T$ (which gives $T^{*2} = T^2$).

c) The last equality in (6) shows that if $T^* = T$, then $B_1B_2 = B_2B_1$. Therefore, B_1B_2 is a positive operator and $(T^2x, x) = -(B_2B_1x, x) \leq 0$. On the other hand, $(T^2x, x) = (Tx, Tx) \geq 0)$. Consequently, $Tx = 0$ for all $x \in H$.

d) The counterexample can be given even in R^2. Let

$$B_1 = \begin{pmatrix} a, & 0 \\ 0, & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0, & 0 \\ 0, & b \end{pmatrix}, \quad T = \begin{pmatrix} 0, & 0 \\ t, & 0 \end{pmatrix},$$

where $a > 0,\ b > 0,\ t \in \mathbb{R}^1$.

e) These equalities immediately follow from the first and second equalities in (6).

f) If $B_1 = B_2 = B$, then T commutes with B^2 by virtue of the previous statement and hence it commutes with B too by virtue of lemma of the square root (see [5], Theorem VI.9). It is enough now to apply statement b of this theorem.

g) If $n = 2m+1$, then $\det(T^2) = \det(-B_2B_1) = (-1)^{2m+1} \det B_1 \times \det B_2 \leq 0$. On the other hand, $\det(T^2) = (\det T)^2 \geq 0$. Therefore $\det(T^2) = 0$ and hence $\det B_1 \det B_2 = 0$.

h) The first condition in (6) gives the relation $(Tx|B_1x) = 0$ which shows that $(Tf|f) = 0$ if $Bf = \lambda f$ and $\lambda \neq 0$. The last condition $\lambda \neq 0$ can be omitted due to the first relation in (7). Therefore, $(Tf|f) = 0$ for any eigenvector of B_1, and it is enough to note that in a finite-dimensional space normed eigenvectors of any self-adjoint operator constitute a basis and the trace does not depend on the choice of the basis.

i) It can be checked directly that any linear operator T in $\mathbb{R}^2$ with $\operatorname{tr} T = 0$ has the property $T^2 = T^{*2}$. So, this statement is an immediate consequence of statement h and of the last equality in (6). $\square$

Proof of Theorem 5. According to relations (6), we have the equality $TB_1^{-1}T^* = B_2$ on $\operatorname{im} B_1$, and hence if $B_1x + T^*y$ is denoted by h, then $Tx + B_2y$ will be equal to $TB^{-1}h$. Therefore the following equality in $H \oplus H$ is true:

$$\{(B_1x + T^*y, Tx + B_2y) : x, y \in \operatorname{im} B_1\} =$$
$$= \{(h, TB^{-1}h) : h \in \operatorname{im} B_1\},$$

which gives the first equality in statement a. To prove the second one, note that $\ker K$ is the collection of pairs (x, y) satisfying the system of equations $B_1x + T^*y = 0,\ Tx + B_2y = 0$. The first equaition gives, because of the equality $B_1^{-1}T^* = -TB_1^{-1}$ on $\operatorname{im} B_1$, the relation $B_1(x - U_y) = 0$ and hence $x = Uy$. It is also easy to show that the pair (Uy, y) satisfies the second equation for all $y \in \mathcal{D}(U)$ as well.

Using again the equality $B_1^{-1}T^* = -TB_1^{-1}$ on $\operatorname{im} B_1$, we get for any fixed $y \in \operatorname{im} B_1$ the equality

$$(TB_1^{-1}x|y) = -(x|TB_1^{-1}y) \quad \text{for all} \ \ x \in \mathcal{D}(U)$$

which just means the validity of statement b.

Finally we prove statement c. The equality $B_2 = -UB_1U$ can be verified directly. It is clear that $\mathcal{D}(U)$ is the projection of Γ on H that is a continuous one-to-one mapping of the closed set $\Gamma \subset H \oplus H$ into H. Therefore $\mathcal{D}(U)$ is a Borel set (this can be shown, for example, by the Kuratowski theorem, [6], p.5). Furthermore, any random vector belongs a.s. to its topological support, and the support of the random vector (ξ_1, ξ_2) is included in $\overline{\operatorname{im} K}$ (see relation (11)), which is equal to Γ according to statement a. Therefore, $(\xi_1, \xi_2) \in \Gamma$ a.s., which means that $\xi_1 \in \mathcal{D}(U)$ and $\xi_2 = U\xi_1$ a.s. $\square$

Proof of Theorem 6. The continuity of the everywhere defined closed operators is well known. We show that $\mathcal{D}(U) = H$. Since $\ker B = 0$, $\overline{\operatorname{im} B} = H$ and it is enough to show that convergence of Bz_n implies convergence of $TB^{-1}(Bz_n),\ z_n \in H,\ n = 1, 2 \dots$ which is easily checked by using $T^* = -T$ (Theorem 4) and $T^2 = -B^2$ (relation 6). Finally, since $BT = TB$ (Theorem 4), $UB = TB^{-1}B = BTB^{-1} = BU$, and $-B^2 = T^2 = (UB)^2 = U^2B^2$, which gives that $U^2 = -I$. $\square$

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