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A SECOND-ORDER NONLINEAR PROBLEM WITH TWO-POINT AND INTEGRAL BOUNDARY CONDITIONS

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Abstract. The paper gives sufficient conditions for the existence and nonuniqueness of monotone solutions of a nonlinear ordinary differential equation of the second order subject to two nonlinear boundary conditions one of which is two-point and the other is integral. The proof is based on an existence result for a problem with functional boundary conditions obtained by the author in [6].

The present paper is concerned with the theory of nonlinear boundary value problems for equations with ordinary derivatives, see e.g. [1-4], and is closely related to [5, 6]. We deal here with the solvability of a certain essentially nonlinear second-order problem.

The following notation is used:

$\mathbb{R}$ is the set of all real numbers;

$[a, b]$ denotes a closed interval where a differential equation is considered, $-\infty < a < b < +\infty$;

C^0 denotes the space of all continuous functions;

C^1 is the space of all continuously differentiable functions;

L_1 denotes the space of all Lebesgue measurable functions with integrable absolute value;

AC stands for the space of all absolutely continuous functions;

CL_1^2 is the space of all $x(\cdot) \in C^1$ such that $\dot{x}(\cdot) \in AC$.

We consider the existence of monotone solutions of the boundary value problem

$$\ddot{x} = f(t, x, \dot{x}), \quad t \in [a, b], \tag{1}$$

$$\omega(x(a), x(b)) = 0, \tag{2}$$

$$\int_a^b \varphi(|\dot{x}(\tau)|) d\tau = g. \tag{3}$$

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The solution $x(\cdot) \in CL_1^2([a,b],\mathbb{R})$ should satisfy equation (1) almost everywhere. Assume that the function $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e. $f(t,x_0,x_1)$ is measurable in t for any fixed numbers x_0, x_1 and is continuous in x_0, x_1 for almost every fixed t. Assume also that $|f(t,x_0,x_1)| \le M$ for almost all t and all x_0, x_1, the constant M is positive, the number $g \in \mathbb{R}$ is fixed, the functions $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi : [0,\infty) \to \mathbb{R}$ are continuous, $\omega(s_1,s_2)$ is nondecreasing in each of the arguments s_1, s_2 and is strictly increasing at least in one of the two arguments, the set of pairs s_1, s_2 that satisfy equality $\omega(s_1,s_2) = 0$ is nonempty, the function $\varphi(z)$ strictly increases and

$$\lim_{z \to +\infty} \varphi(z) = +\infty.$$

For example, if $\varphi(z) = z$ then the boundary condition (3) fixes L_1-norm of the derivative of the unknown function. And in the case $\varphi(z) = \sqrt{1+z^2}$ the equality (3) fixes the length of the curve which is the graph of the solution $x(t), t \in [a,b]$. Let us note also that the equality $x(a) = g_0$, where $g_0 \in \mathbb{R}$ is a number, can be considered as the simplest special case of (2). Thus, the boundary conditions (2), (3) can describe, in particular, a curve with a fixed length emanating from a given initial point.

Denote

$$A_\varphi = \int_0^{b-a} \varphi(M\tau)d\tau.$$

Theorem. *If* $g \ge A_\varphi$ *then every solution of boundary value problem* (1)-(3) *is strictly monotone, and there exist at least one increasing and at least one decreasing solutions.*

The above theorem was previously announced by the author, cf. Proposition 2 in [5]. The proof will be given below.

Remark 1. If $g < A_\varphi$ then problem (1)-(3) may have no monotone solutions. This is the case, for example, if $f \equiv M > 0$.

Remark 2. A similar theorem is valid also for an equation with deviating arguments and for condition (3) where $g = g(x(\cdot))$ is a nonlinear functional.

The proof of Theorem employs (and illustrates) the following existence result for a boundary value problem of the form

$$\ddot{x}(t) = F(x(\cdot))(t), \quad t \in [a,b], \tag{4}$$

$$B_0(x(\cdot)) = B_1(\dot{x}(\cdot)) = 0. \tag{5}$$

The solution $x(\cdot) \in CL_1^2 = CL_1^2([a,b],\mathbb{R})$ satisfies equation (4) almost everywhere. The mappings

$$F : C^1 \to L_1, \quad B_0 : CL_1^2 \to \mathbb{R}, \quad B_1 : AC \to \mathbb{R}$$

are assumed to be continuous. Let us fix a closed set of functions $A \subset CL_1^2$. Denote $A^{(k)} = \{x^{(k)}(\cdot) : \ x(\cdot) \in A\}$. Assume that the family of functions A satisfies $A^{(2)} = L_1$.

Proposition 1. *Let M, N be fixed numbers. Consider the following conditions:*

a) *if $\|x(\cdot)\|_{C^1} \leq N$ then $|F(x(\cdot))(t)| \leq M$ for almost all t,*

b) *if $x(\cdot) \in A$ satisfies* (5) *and $|\ddot{x}(t)| \leq M$ almost everywhere then $\|x(\cdot)\|_{C^1} \leq N$,*

c) *if $x(\cdot) \in A$ and almost everywhere $|\ddot{x}(t)| \leq M$ then there exist a unique number $c_0 \in \mathbb{R}$ such that*

$$B_0(x(\cdot) + c_0) = 0, \quad x(\cdot) + c_0 \in A,$$

and a unique number $c_1 \in \mathbb{R}$ that satisfies

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(\cdot) + c_1 \in A^{(1)}.$$

Conditions a), b), c) *imply that problem* (4), (5) *has at least one solution in A.*

A more general version of Proposition 1 was proven by the author in [6]. We need also the following simple auxiliary result.

Proposition 2. *Let the above given assumptions on φ hold. If a function $u : [a,b] \to \mathbb{R}$ satisfies the Lipschitz condition with the coefficient M and vanishes at least at one point then*

$$\int_a^b \varphi(|u(\tau)|)d\tau \leq \int_0^{b-a} \varphi(M\tau)d\tau.$$

Here the equality holds only for the following four functions

$$u = \pm M(t-a), \quad u = \pm M(b-t).$$

Proof of Proposition 2. Let $u(s) = 0$ for some $s \in [a,b]$. Then $|u(t)| \leq M|t-s|$. Thus

$$\int_a^b \varphi(|u(\tau)|)d\tau \leq \int_a^b \varphi(M|\tau - s|)d\tau. \tag{6}$$

Denote the right-hand side of (6) by $\Psi(s)$. We have

$$\Psi(s) = \int_a^s \varphi(M(s-\tau))d\tau + \int_s^b \varphi(M(\tau - s))d\tau = \\ = \int_0^{s-a} \varphi(M\tau)d\tau + \int_0^{b-s} \varphi(M\tau)d\tau.$$

And so, the derivative

$$\frac{d\Psi}{ds} = \varphi(M(s-a)) - \varphi(M(b-s))$$

is negative for $a \le s < \frac{1}{2}(a+b)$ and positive for $\frac{1}{2}(a+b) < s \le b$. Consequently, the value $\Psi(a) = \Psi(b) = \int_0^{b-a} \varphi(M\tau)d\tau$ is the maximum of $\Psi(s)$ for $s \in [a,b]$, which is attained only at the ends of the interval. The desired inequality is proven. If the right- and left-hand sides of this inequality are equal then s equals either a or b, and besides that (6) turns to equality. Taking into account strict monotonicity of φ we come to the conclusion that $|u(t)| \le M|t-s|$ also turns to equality. Thus, either $u = \pm M(t-a)$, or $u = \pm M(b-t)$. ■

Proof of Theorem. The boundary value problem (1)-(3) is a special case of problem (4), (5). Really, it suffices to assume

$$F(x(\cdot))(t) = f(t, x(t), \dot{x}(t)),$$
$$B_0(x(\cdot)) = \omega(x(a), x(b)),$$
$$B_1(u(\cdot)) = \int_a^b \varphi(|u(\tau)|)d\tau - g.$$

The mappings $F: C^1 \to L_1$, $B_0 : C^0 \to \mathbb{R}$, $B_1 : C^0 \to \mathbb{R}$ are continuous. Denote by A_+ the set of all monotone nondecreasing functions in CL_1^2 and by A_- the set of all nonincreasing ones. The sets A_+, A_- are closed in CL_1^2, and $A_+^{(2)} = A_-^{(2)} = L_1$. Condition a) of Proposition 1 holds obviously. Let us verify condition b). Let $x(\cdot) \in CL_1^2$ satisfy (2), (3), and $|\ddot{x}(t)| \le M$ be true almost everywhere. Since $\varphi(z) \to +\infty$ as $z \to +\infty$ there exists a number r that satisfies $\varphi(r) > (b-a)^{-1}g$. If we suppose that $|\dot{x}(t)| \ge r$ for all t then

$$\int_a^b \varphi(|\dot{x}(\tau)|)d\tau \ge (b-a)\varphi(r) > g,$$

which contradicts (3). Consequently, $|\dot{x}(s)| < r$ for at least one s. Consider some l_1, l_2 such that $\omega(l_1, l_2) = 0$. Let us show that

$$\min\{l_1, l_2\} \le x(\sigma) \le \max\{l_1, l_2\} \tag{7}$$

for some σ. Really, if (7) does not hold for any $\sigma \in [a,b]$ then due to continuity of $x(t)$ two cases are possible. Either $x(t) > \max\{l_1, l_2\}$ for all t, or $x(t) < \min\{l_1, l_2\}$ for all t. Monotonicity of ω implies that $\omega(x(a), x(b)) > \omega(l_1, l_2) = 0$ in the first case, and $\omega(x(a), x(b)) < 0$ in the second case. It follows from (2) that neither of the two cases can take place. The existence of the numbers r, s, σ named above and the inequality $|\ddot{x}(t)| \le M$ imply boundedness of $\|x(\cdot)\|_{C^1}$. Let us verify now condition c) for $A = A_+$ and for $A = A_-$. The function $x(\cdot)$ being fixed, the number c_0 is defined uniquely by the equality $B_0(x(\cdot) + c_0) = 0$ due to the properties of the real function

$$B_0(x(\cdot) + c) = \omega(x(a) + c, x(b) + c)$$

of the argument c. Really, the function is continuous and strictly increasing. It suffices to show that this function takes both positive and negative values. As above, we fix l_1, l_2 for which $\omega(l_1, l_2) = 0$. Then for $c > \max\{l_1 - x(a), l_2 - x(b)\}$ we have $\omega(x(a) + c, x(b) + c) > \omega(l_1, l_2) = 0$, and for $c < \min\{l_1 - x(a), l_2 - x(b)\}$ we obtain $\omega(x(a) + c, x(b) + c) < \omega(l_1, l_2) = 0$. So, the desired properties of the function $B_0(x(\cdot) + c)$ are established. We have only to note that $x(\cdot) + c$ for a fixed c is monotone in the same sense as $x(\cdot)$. Consider now c_1. Assume that $x(\cdot) \in CL_1^2$ and almost everywhere $|\ddot{x}(t)| \le M$. Continuity of φ implies that the function

$$\Phi(c) = \int_a^b \varphi(|\dot{x}(\tau) + c|)d\tau$$

is also continuous. With the help of Proposition 2 we obtain the following. For $c \in (-\infty, -\max_t \dot{x}(t)]$ we have $\dot{x}(t) + c \le 0$, and the function $\Phi(c)$ strictly decreases taking values from $+\infty$ to a number not larger than A_φ; and if $c \in [-\min_t \dot{x}(t), +\infty)$ then $\dot{x}(t) + c \ge 0$, and the function $\Phi(c)$ strictly increases taking values from a number not larger than A_φ to $+\infty$. A conclusion follows that if $x(\cdot) \in CL_1^2$ and almost everywhere $|\ddot{x}(t)| \le M$ then there exists a unique c_1 that satisfies

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(t) + c_1 \ge 0.$$

Similarly, conditions

$$B_1(\dot{x}(\cdot) + c_1) = 0, \quad \dot{x}(t) + c_1 \le 0$$

also define a unique c_1. Thus, condition c) is valid. It follows from Proposition 1 that boundary value problem (1)-(3) is solvable in A_+ and in A_-. Now we have to show that every solution $x(t)$ of this problem is strictly monotone. If the derivative $\dot{x}(t)$ does not vanish then all its values have the same sign, and so $x(t)$ is obviously monotone. Let now the derivative $\dot{x}(t)$

vanish at least at one point. Using Proposition 2 we obtain

$$\int_a^b \varphi(|\dot{x}(\tau)|)d\tau \leq A_\varphi. \tag{8}$$

Taking into account the inequality $A_\varphi \leq g$ and boundary condition (3) we see that the two values in (8) are equal. Employing again Proposition 2 we conclude that either $\dot{x}(t) = \pm M(t-a)$, or $\dot{x}(t) = \pm M(b-t)$. And since $M \neq 0$ the function $x(t)$ is strictly monotone. Theorem is proven. Let us note that Theorem can be proven also basing on results of [7]. ∎

In conclusion we verify Remark 1. Assume $f \equiv M > 0, g < A_\varphi$. We need to show that problem (1)-(3) has no monotone solutions. The equation (1) takes the form $\ddot{x} = M$. And since $M \neq 0$ we obtain $\dot{x}(t) = M(t-\gamma)$. Thus, $x(t)$ can be monotone only if $\gamma \leq a$ or $\gamma \geq b$. Let us consider these two cases separately. If $\gamma \leq a$ then $|\dot{x}(\tau)| = M(\tau-\gamma) \geq M(\tau-a)$ for $\tau \in [a,b]$, and (3) implies $g \geq \int_a^b \varphi(M(\tau-a))d\tau = A_\varphi > g$. This contradiction shows that the inequality $\gamma \leq a$ does not hold. Similarly, if $\gamma \geq b$ then $|\dot{x}(\tau)| = M(\gamma-\tau) \geq M(b-\tau)$ for $\tau \in [a,b]$ and thus $g \geq \int_a^b \varphi(M(b-\tau))d\tau = A_\varphi > g$. And so, the case $\gamma \geq b$ is not possible either. Remark 1 is verified.

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