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NON-STATIONARY PROBLEMS OF GENERALIZED ELASTOTHERMODIFFUSION FOR INHOMOGENEOUS MEDIA

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Abstract. The method of investigation of non-stationary boundary value problems of the theory of thermodiffusion using the Laplace integral transform is described. In the classical theory of elasticity this method was first used by V. Kupradze and the author.

The interconnection of deformation, thermal conductivity and diffusion processes in an elastic isotropic solid body is described by a system of five scalar partial differential equations of general type. In the classical case this system is hyperbolic with respect to some part of components of an unknown vector function and parabolic with respect to the rest components. A system of equations of the conjugate (connected) theory of thermoelasticity is a particular case [1–4].

In the classical theory of elastothermodiffusion it is assumed that propagation velocity of heat and of diffusing substance is infinitely large.

In particular, however, it is often necessary to take into account the fact that heat propagates not with an infinitely large but with a finite velocity. The heat flux does not occur in the body instantly but is characterized by the finite relaxation time.

The consideration of these physical factors makes the main system of differential equations very complicated. There exist various generalizations of this theory. Three-dimensional non-stationary problems of non-classical (generalized) thermodiffusion are treated in [5–8].

In this paper the Green–Lindsay theory is generalized to problems of elastothermodiffusion. Initial boundary value problems are investigated for the considered physical system of differential equations in piecewise-homogeneous media with boundary and contact conditions; a substantiation of the Riesz–Fischer–Kupradze method is given and approximate solutions are considered.

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Let us consider a three-dimensional homogeneous isotropic elastic medium in which a thermodiffusion process takes place. The deformed state is described by the displacement vector $v(x,t)=(v_1,v_2,v_3)=\|v_k\|_{3\times 1}$ (one-column matrix), the temperature change $v_4(x,t)$ and the "chemical potential" of the medium $v_5(x,t)$; $C(x,t)=\gamma_2 \operatorname{div} v(x,t)+a_{12}v_4(x,t)+a_2v_5(x,t)$, where $C(x,t)$ is the diffusing substance concentration; $x=(x_1,x_2,x_3)$ is a point in the Euclidean space $\mathbb{R}^3$, $t\geq 0$ is the time and $X=(X_1,X_2,X_3)$, X_4, X_5 are the given functions. We consider a system of partial differential equations of the generalized elastothermodiffusion theory written in the form

$$\begin{gathered}
A\big(\frac{\partial}{\partial x}\big)v-\sum_{k=1}^{2}\gamma_k \operatorname{grad} v_{3+k}+X=\rho\frac{\partial^2 v}{\partial t^2}+ \\
+\tau^1\sum_{k=1}^{2}\gamma_k\frac{\partial}{\partial t}\operatorname{grad} v_{3+k}, \\
\delta_1\Delta v_4+X_4=a_1\big(1+\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_4}{\partial t}+\gamma_1\frac{\partial}{\partial t}\operatorname{div} v+ \\
+a_{12}\big(1+\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_5}{\partial t}, \\
\delta_2\Delta v_5+X_5=a_2\big(1+\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_5}{\partial t}+\gamma_2\frac{\partial}{\partial t}\operatorname{div} v+ \\
+a_{12}\big(1+\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_4}{\partial t},
\end{gathered}\tag{1}$$

where $A(\frac{\partial}{\partial x})\equiv\|\mu\delta_{jk}\Delta+(\lambda+\mu)\frac{\partial^2}{\partial x_j\partial x_k}\|_{3\times 3}$ is the statical operator of Lamé [8], δ_{jk} being the Kroneker symbol. The elastic, thermal, diffusion and relaxation constants satisfy the natural restrictions

$$\begin{gathered}
\mu>0,\ \ 3\lambda+2\mu>0,\ \ \rho>0,\ \ a_k>0,\ \ \delta_k>0,\ \ \gamma_k>0,\ \ k=1,2, \\
a_1a_2-a_{12}^2>0,\ \ \tau^1\geq\tau^0>0.
\end{gathered}\tag{2}$$

In particular, for relaxation constants $\tau^1=\tau^0=0$ we have the classical case.

Let $D_1\subset\mathbb{R}^3$ be a finite domain bounded by the closed Liapunov surface S and $D_2=\mathbb{R}^3\backslash\bar{D}_1$ be an infinite domain, $n=(n_1,n_2,n_3)$ is the unit normal on S. Elastothermodiffusion constants of the domain D_j will be denoted by the left-hand subscripts ${}_j\lambda$, ${}_j\mu$, ${}_j\rho$, ${}_j\tau^0$, ${}_j\tau^1$, …, $j=1,2$.

Problem A^t. Define in the infinite cylinder $Z_\infty=\{(x,t):x\in D_1\cup D_2,\ t\in]0,\infty[\}$ the regular vetor $V=(v,v_4,v_5)\in C^1(\bar{Z}_\infty)\cap C^2(Z_\infty)$ from the conditions

$$\forall(x,t)\in Z_\infty:\ {}_j\mu\Delta v(x,t)+({}_j\lambda+{}_j\mu)\operatorname{grad}\operatorname{div} v-$$

$$
\begin{aligned}
&-\sum_{k=1}^{2} {}_j\gamma_k \operatorname{grad} v_{3+k} + {}_jX_j = \rho\frac{\partial^2 v}{\partial t^2} + {}_j\tau^1 \sum_{k=1}^{2} {}_j\gamma_k \frac{\partial}{\partial t}\operatorname{grad} v_{3+k},\\
&{}_j\delta_1 \Delta v_4(x,t) + {}_jX_4 = {}_ja_1\big(1 + {}_j\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_4}{\partial t} +\\
&\quad + {}_j\gamma_1\frac{\partial}{\partial t}\operatorname{div} v + {}_ja_{12}\big(1 + {}_j\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_5}{\partial t},\\
&{}_j\delta_2 \Delta v_5(x,t) + {}_jX_5 = {}_ja_2\big(1 + {}_j\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_5}{\partial t} +\\
&\quad + {}_j\gamma_2\frac{\partial}{\partial t}\operatorname{div} v + {}_ja_{12}\big(1 + {}_j\tau^0\frac{\partial}{\partial t}\big)\frac{\partial v_4}{\partial t},\\
&\qquad\qquad x \in D_j, \quad j = 1,2, \qquad\qquad (3)\\
&\forall x \in D_j:\ \lim_{t\to+0} V(x,t) = {}_j\varphi^{(0)}(x),\ \ j = 1,2,\\
&\quad \lim_{t\to+0}\frac{\partial V(x,t)}{\partial t} = {}_j\varphi^{(1)}(x),\ \ j=1,2,\\
&\forall (y,t) \in S^\infty \equiv \{(y,t) : y \in S, t\in [0,\infty[\} :\\
&\quad [V]_S^{\pm} \equiv V^+(y,t) - V^-(y,t) = f(y,t),\\
&\quad [RV]_S^{\pm} \equiv [{}_1R(\frac{\partial}{\partial y}, n)V(y,t)]^+ -\\
&\qquad - [{}_2R(\frac{\partial}{\partial y}, n)V(y,t)]^- = F(y,t),
\end{aligned}
$$

for large values of t and $x \in D_2$:

$$
|D^\alpha_{x,t}V(x,t)| \le \frac{const}{1+|x|^{1+|\alpha|}}\, e^{\sigma_0 t}, \quad |\alpha| = \overline{0,2}, \quad \sigma_0 \ge 0,
$$

$$
D^\alpha_{x,t} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\partial t^{\alpha_4}}, \quad |\alpha| = \sum_{k=1}^{4}\alpha_k,
$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a multi-index;

$$
\begin{aligned}
&{}_j\varphi^{(0)}(x) = ({}_j\varphi_1^{(0)}, {}_j\varphi_2^{(0)}, {}_j\varphi_3^{(0)}, {}_j\varphi_4^{(0)}, {}_j\varphi_5^{(0)}),\\
&{}_j\varphi^{(1)}(x) = ({}_j\varphi_1^{(1)}, {}_j\varphi_2^{(1)}, {}_j\varphi_3^{(1)}, {}_j\varphi_4^{(1)}, {}_j\varphi_5^{(1)}),\\
&f(y,t) = (f_1, f_2, f_3, f_4, f_5),\\
&F(y,t) = (F_1, F_2, F_3, F_4, F_5), \quad t \ge 0, \quad y \in S,
\end{aligned}
$$

are the given real functions; ${}_jR(\frac{\partial}{\partial x}, n)$ is a stress operator in the thermodiffusion theory for the medium D_j (5×5 matrix):

$$
{}_jR\big(\frac{\partial}{\partial x}, n\big) = \|{}_jR_{kl}\|_{k,l=\overline{1,5}},
$$

where

$$
\begin{aligned}
{}_jR_{kl} &= {}_j\mu\delta_{lk}\frac{\partial}{\partial n(x)} + {}_j\lambda n_l(x)\frac{\partial}{\partial x_k} + {}_j\mu n_k(x)\frac{\partial}{\partial x_l}, \quad k,l=\overline{1,3},\\
{}_jR_{kl} &= -{}_j\gamma_{l-3}n_k(1+{}_j\tau^{l-3}\frac{\partial}{\partial t}), \quad k=\overline{1,3}, \ \ l=4,5,\\
{}_jR_{kl} &= {}_j\delta_{k-3}\delta_{kl}\frac{\partial}{\partial n(x)}, \quad k=4,5, \ \ l=\overline{1,5},
\end{aligned}
$$

here $n(x)$ is C^∞-extention of n onto $\mathbb{R}^3$;

$$
\begin{aligned}
&V^+(y,t) = \lim_{D_1\ni x\to y\in S} V(x,t), \quad V^-(y,t) = \lim_{D_2\ni x\to y\in S} V(y,t),\\
&\big[{}_1R\big(\frac{\partial}{\partial y}, n(y)\big)V(y,t)\big]^+ = \lim_{D_1\ni x\to y\in S} {}_1R\big(\frac{\partial}{\partial x}, n(y)\big)V(x,t),\\
&\big[{}_2R\big(\frac{\partial}{\partial y}, n(y)\big)V(y,t)\big]^- = \lim_{D_2\ni x\to y\in S} {}_2R\big(\frac{\partial}{\partial x}, n(y)\big)V(x,t).
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
R\big(\frac{\partial}{\partial x}, n\big)V = \Big(Tv - \gamma_1\big(1+\tau^1\frac{\partial}{\partial t}\big)nv_4 - \\
\gamma_2\big(1+\tau^1\frac{\partial}{\partial t}\big)nv_5, \delta_1\frac{\partial v_4}{\partial n}, \delta_2\frac{\partial v_5}{\partial n}\Big),
\end{aligned}
$$

where T is the "classical" stress operator.

For a classical (regular) solution to exist, it is necessary that the conditions of "natural compatibility" of initial data be fulfilled. These conditions have the form

$$
\begin{aligned}
\forall y\in S\ :\ &{}_1\varphi^{(0)}(y) - {}_2\varphi^{(0)}(y) = f(y,0),\\
&{}_1\varphi^{(1)}(y) - {}_2\varphi^{(1)}(y) = \lim_{t\to+0}\frac{\partial f(y,t)}{\partial t},\\
&{}_1R\big(\frac{\partial}{\partial y}, n\big){}_1\varphi^{(0)}(y) - {}_2R\big(\frac{\partial}{\partial y}, n\big){}_2\varphi^{(1)}(y) = F(y,0),\\
&{}_1R\big(\frac{\partial}{\partial y}, n\big){}_1\varphi^{(1)}(y) - {}_2R\big(\frac{\partial}{\partial y}, n\big){}_2\varphi^{(1)}(y) = \lim_{t\to+0}\frac{\partial F(y,t)}{\partial t}.
\end{aligned}
$$

The dynamic Problem A^t is investigated by the Laplace transform method. However, the "natural compatibility" conditions of this method are not sufficient for our purpose and should therefore be complemented with "higher order compatibility" conditions. The latter have the form

$$\left.\frac{\partial^m f(y,t)}{\partial t^m}\right|_{t=0} = {}_1\varphi^{(m)}(y) - {}_2\varphi^{(m)}(y),$$
$$\left.\frac{\partial^m F(y,t)}{\partial t^m}\right|_{t=0} = {}_1R_1\varphi^{(m)}(y) - {}_2R_2\varphi^{(m)}(y), \qquad m=\overline{2,7},$$

where

$$\begin{gathered}
{}_j\varphi^{(m)}(x) \equiv ({}_j\varphi_1^{(m)}(x), {}_j\varphi_2^{(m)}(x), {}_j\varphi_3^{(m)}(x)) = \\
= {}_j\rho^{-1}\Big[{}_j\mu\Delta({}_j\varphi_1^{(m-2)}, {}_j\varphi_2^{(m-2)}, {}_j\varphi_3^{(m-2)}) + \\
+({}_j\lambda + {}_j\mu)\operatorname{grad}\operatorname{div}({}_j\varphi_1^{(m-2)}, {}_j\varphi_2^{(m-2)}, {}_j\varphi_3^{(m-2)}) - \\
-{}_j\gamma_1 \operatorname{grad}{}_j\varphi_4^{(m-2)} - {}_j\gamma_{1j}\tau^1 \operatorname{grad}{}_j\varphi_4^{(m-1)} - \\
-{}_j\gamma_2 \operatorname{grad}{}_j\varphi_5^{(m-2)} - {}_j\gamma_{2j}\tau^1 \operatorname{grad}{}_j\varphi_5^{(m-1)} + \left.\frac{\partial^{m-2}{}_jX}{\partial t^{m-2}}\right|_{t=0}\Big], \\
{}_ja_{1j}\tau^0{}_j\varphi_4^{(m)}(x) + {}_ja_{12j}\tau^0{}_j\varphi_5^{(m)}(x) = {}_j\delta_1\Delta_j\varphi_4^{(m-2)} - {}_ja_{1j}\varphi_4^{(m-1)} - \\
-{}_ja_{12j}\varphi_5^{(m-1)} - {}_j\gamma_1\operatorname{div}({}_j\varphi_1^{(m-1)}, {}_j\varphi_2^{(m-1)}, {}_j\varphi_3^{(m-1)}) + \left.\frac{\partial^{m-2}{}_jX_4}{\partial t^{m-2}}\right|_{t=0}, \\
{}_ja_{12j}\tau^0{}_j\varphi_4^{(m)}(x) + {}_ja_{2j}\tau^0{}_j\varphi_5^{(m)}(x) = {}_j\delta_2\Delta_j\varphi_5^{(m-2)} - {}_ja_{2j}\varphi_5^{(m-1)} - \\
-{}_ja_{12j}\varphi_4^{(m-1)} - {}_j\gamma_2\operatorname{div}({}_j\varphi_1^{(m-1)}, {}_j\varphi_2^{(m-1)}, {}_j\varphi_3^{(m-1)}) + \left.\frac{\partial^{m-2}{}_jX_5}{\partial t^{m-2}}\right|_{t=0}.
\end{gathered}$$

These conditions of "quantitative nature" are sufficient for the existence of the classical solution. We will not dwell on this here but proceed to the construction of approximate solutions by the Riesz–Fischer–Kupradze method.

Theorem. *If the initial data of Problem A^t satisfy the above-given "higher order compatibility" conditions, then Problem A^t has the unique classical solution which is represented by the Laplace–Mellin integral*

$$V(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\zeta t}\widehat{V}(x,\zeta)d\zeta,$$

where $\widehat{V}(x,\zeta)$ is the solution of the corresponding problem for elliptic system represented in the form

$$\widehat{V}(x,\zeta) = \sum_{k=0}^{\infty} a_k(\zeta)\overset{k}{\Omega}(x,\zeta) + \Omega(x,\zeta).$$

The series converges uniformly; $a_k(\zeta)$, $\overset{k}{\Omega}(x,\zeta)$, $\Omega(x,\zeta)$ *are the given vector-functions (constructed explicitly) and* $\zeta=\sigma+iq$, *where* $\sigma\geq\sigma_0^*>\sigma_0$; σ_0^* *is the defined constant.*

Consider the Laplace transform

$$\widehat{V}(x,\zeta)=\int\limits_0^\infty e^{-\zeta t}V(x,t)dt, \tag{4}$$

where $\zeta=\sigma+iq$ is a complex parameter.

Using formally transform (4), the dynamic problem A^t is reduced to the corresponding problem with the complex parameter ζ (spectral problem) for $\widehat{V}(x,\zeta)$.

Problem $A(\zeta)$**.** Define for each $\zeta\in\Pi_{\sigma_0^*}\equiv\{\zeta: Re\,\zeta>\sigma_0^*>\sigma_0\}$ in $D=D_1\cup D_2$ the regular vector $\widehat{V}=(\widehat{v},\widehat{v}_4,\widehat{V}_5))=\widehat{V}(\cdot,\zeta)\in C^1(\overline{D_1\cup D_2})\cap C^2(D_1\cup D_2)$ from the conditions

$$\begin{aligned}
&\forall x\in D_j,\quad j=1,2:\\
&{}_j\mu\Delta\widehat{v}+({}_j\lambda+{}_j\mu)\operatorname{grad}\operatorname{div}v-\\
&\qquad-\sum_{k=1}^2 {}_j\gamma_k(1+{}_j\tau^1\zeta)\operatorname{grad}\widehat{v}_{3+k}-{}_j\rho\zeta^2\widehat{v}={}_j\widetilde{X},\\
&{}_j\delta_1\Delta\widehat{v}_4-{}_ja_1(1+{}_j\tau^0\zeta)\widehat{v}_4-{}_ja_{12}\zeta(1+{}_j\tau^0\zeta)\widehat{v}_5-\\
&\qquad-{}_j\gamma_1\zeta\operatorname{div}\widehat{v}={}_j\widetilde{X}_4,\\
&{}_j\delta_2\Delta\widehat{v}_5-{}_ja_2(1+{}_j\tau^0\zeta)\widehat{v}_5-{}_ja_{12}\zeta(1+{}_j\tau^0\zeta)\widehat{v}_4-\\
&\qquad-{}_j\gamma_2\zeta\operatorname{div}\widehat{v}={}_j\widetilde{X}_5,\\
&|D_x^\beta\widehat{V}(x,\zeta)|\leq\frac{const}{1+|x|^{1+|\beta|}},\quad |\beta|=\overline{0,2},
\end{aligned} \tag{5}$$

where $\beta=(\beta_1,\beta_2\beta_3)$ is a multi-index,

$$\begin{aligned}
{}_j\widetilde{X}&=-{}_j\widehat{X}-{}_j\rho({}_j\varphi_1^{(1)},{}_j\varphi_2^{(1)},{}_j\varphi_3^{(1)})-\\
&\quad-{}_j\rho\zeta({}_j\varphi_1^{(0)},{}_j\varphi_2^{(0)},{}_j\varphi_3^{(0)})-\sum_{k=1}^2 {}_j\gamma_{kj}\operatorname{grad}{}_j\varphi_{3+k}^{(0)},\\
{}_j\widetilde{X}_4&=-{}_j\widehat{X}_4-{}_ja_{1j}\varphi_4^{(0)}-{}_ja_{1j}\tau^0({}_j\varphi_4^{(1)}+\zeta_j\varphi_4^{(0)})-\\
&\quad-{}_ja_{12j}\varphi_5^{(0)}-{}_ja_{12j}\tau^0({}_j\varphi_5^{(1)}+\zeta_j\varphi_5^{(0)})-\\
&\quad-\gamma_1\operatorname{div}({}_j\varphi_1^{(0)},{}_j\varphi_2^{(0)},{}_j\varphi_3^{(0)}),\\
{}_j\widetilde{X}_5&=-{}_j\widehat{X}_5-{}_ja_{2j}\varphi_5^{(0)}-{}_ja_{2j}\tau^0({}_j\varphi_5^{(1)}+\zeta_j\varphi_5^{(0)})-
\end{aligned}$$

$$-\,{}_ja_{12j}\varphi_4^{(0)} - {}_ja_{12j}\tau^0\big({}_j\varphi_4^{(1)} + \zeta_j\varphi_4^{(0)}\big) -$$
$$-\,{}_j\gamma_2\,\mathrm{div}\big({}_j\varphi_1^{(0)}, {}_j\varphi_2^{(0)}, {}_j\varphi_3^{(0)}\big);$$
$$\forall y \in S\ : \widehat{V}^+(y,\zeta) - \widehat{V}^-(y,\zeta) = \widehat{f}(y,\zeta),$$
$$\big[{}_1R\big(\frac{\partial}{\partial y}, n\big)\widehat{V}(y,\zeta)\big]^+ - \big[{}_2R\big(\frac{\partial}{\partial y}, n\big)\widehat{V}(y,\zeta)\big]^- = \widetilde{F}(y,\zeta),$$
$$\widetilde{F}(y,\zeta) = \widehat{F}(y,\zeta) - {}_1\gamma_1\,{}_1\tau^1 n(y)\,{}_1\varphi_4^{(0)} + {}_2\gamma_1\,{}_2\tau^1 n(y)\,{}_2\varphi_4^{(0)} -$$
$$-\,{}_1\gamma_2\,{}_1\tau^1 n(y)\,{}_1\varphi_5^{(0)} + {}_2\gamma_2\,{}_2\tau^1 n(y)\,{}_2\varphi_5^{(0)},$$
$${}_j\widehat{R}\big(\frac{\partial}{\partial y}, n\big)\widehat{V} = \Big(T\widehat{v} - n(y)\sum_{k=1}^{2} {}_j\gamma_k(1 + {}_j\tau^1\zeta)\widehat{v}_{3+k}, {}_j\delta_1\frac{\partial \widehat{v}_4}{\partial n}, {}_j\delta_2\frac{\partial \widehat{v}_5}{\partial n}\Big).$$

Let $L(\frac{\partial}{\partial x},\zeta)$ be a matrix differential operator of Problem $A(\zeta)$ and $\Phi(x,\zeta) = \|\Phi_{jk}(x,\zeta)\|_{5\times5} = \|\overset{1}{\Phi}, \overset{2}{\Phi}, \dots, \overset{5}{\Phi}\|_{5\times5}$ be a matrix of fundamental solutions of this operator $L(\frac{\partial}{\partial x},\zeta)$, $\overset{k}{\Phi}(x,\zeta) = (\Phi_{1k}, \Phi_{2k}, \dots, \Phi_{5k})$, $k = \overline{1,5}$, be column vectors. The matrix $\Phi(x,\zeta)$ is constructed explicitly in terms of elementary functions [8]. Namely:

$$\Phi(x,\zeta) \equiv \widehat{L}\big(\frac{\partial}{\partial x},\zeta\big)\varphi(x,\zeta) \equiv$$
$$\equiv \widehat{L}_0\big(\frac{\partial}{\partial x},\zeta\big)\big(\Delta + \lambda_4^2\big)\varphi(x,\zeta) \equiv \widehat{L}_0\big(\frac{\partial}{\partial x},\zeta\big)\widehat{\varphi}(x,\zeta),$$
$$\widehat{\varphi}(x,\zeta) = \sum_{k=1}^{4} c_k\,\frac{\exp(i\lambda_k|x|)}{|x|},$$

where λ_k, c_k, $k = \overline{1,4}$ are constants, $\widehat{L}(\frac{\partial}{\partial x},\zeta)$ is a matrix connected with $L(\frac{\partial}{\partial x},\zeta) : \widehat{L}L \equiv L\widehat{L} \equiv I \cdot \det L$, I is the unit 5×5 matrix.

In the above assumptions the sense of the notations ${}_jL(\frac{\partial}{\partial x},\zeta)$ and ${}_j\Phi(x,\zeta)$ becomes quite clear.

Thus we have to construct the solution of

Problem $A(\zeta)$**.**

$$\widehat{V} = (\widehat{v}, \widehat{v}_4, \widehat{v}_5) \in C^1(\bar{D}) \cap C^2(D),$$
$$\forall x \in D_j\ :\ {}_jL\big(\frac{\partial}{\partial x},\zeta\big)\widehat{V}(x,\zeta) = {}_j\chi(x), \quad j = 1,2, \tag{6}$$
$$\forall y \in S\ :\ [\widehat{V}]_S^{\pm} \equiv \widehat{V}^+(y,\zeta) - \widehat{V}^-(y,\zeta) = \widehat{f}(y,\zeta),$$
$$[\widehat{R}\widehat{V}]_S^{\pm} \equiv \big[{}_1R\big(\frac{\partial}{\partial y}, n\big)\widehat{V}(y,\zeta)\big]^+ - \big[{}_2\widehat{R}\big(\frac{\partial}{\partial y}, n\big)\widehat{V}(y,\zeta)\big]^- = \widetilde{F}(y,\zeta), \tag{7}$$

$$|D_x^\beta \widehat{V}(x,\zeta)| \leq \frac{const}{1+|x|^{1+|\beta|}}, \quad |\beta| = \overline{0,2}, \tag{8}$$

where ${}_j\chi(x) = ({}_j\widetilde{X}, {}_j\widetilde{X}_4, {}_j\widetilde{X}_5)$ is the given vector.

Let $\widehat{V}(x,\zeta)$ be a regular solution of Problem $A(\zeta)$. Taking into account the contact conditions, by virtue of the formulas for general representation of the solution [8] we have

$$\begin{aligned} \forall x \in D_1 \ : \widehat{V}(x,\zeta) = & \int\limits_S {}_1\Phi(x-z,\zeta)({}_1\widehat{R}\widehat{V})^+ d_zS - \\ & - \int\limits_S ({}_1\widetilde{\widehat{R}}_1\Phi^*)^* \widehat{V}^+ d_zS - \int\limits_{D_1} {}_1\Phi_1\chi dz, \end{aligned} \tag{9}$$

$$\begin{aligned} \forall x \in D_2 \ : 0 = & \int\limits_S {}_1\Phi({}_1\widehat{R}\widehat{V})^+ d_zS - \int\limits_S ({}_1\widetilde{\widehat{R}}_1\Phi^*)^* \widehat{V}^+ d_zS - \\ & - \int\limits_{D_1} {}_1\Phi_1\chi dz, \end{aligned} \tag{10}$$

$$\begin{aligned} \forall x \in D_2 \ : \widehat{V}(x,\zeta) = & -\int\limits_S {}_2\Phi({}_2\widehat{R}\widehat{V})^+ dS + \int\limits_S ({}_2\widetilde{\widehat{R}}_2\Phi^*)^* \widehat{V}^+ dS - \\ & - \int\limits_{D_2} {}_2\Phi_2\chi dz + \int\limits_S {}_2\Phi\widetilde{F} dS - \int\limits_S ({}_2\widetilde{\widehat{R}}_2\Phi^*)^* \widehat{f} dS, \end{aligned} \tag{11}$$

$$\begin{aligned} \forall x \in D_1 \ : 0 = & -\int\limits_S {}_2\Phi({}_1\widehat{R}\widehat{V})^+ dS + \int\limits_S ({}_2\widetilde{\widehat{R}}_2\Phi^*)^* \widehat{V}^+ dS - \\ & - \int\limits_{D_2} {}_2\Phi_2\chi dz + \int\limits_S {}_2\Phi\widetilde{F} dS - \int\limits_S ({}_2\widetilde{\widehat{R}}_2\Phi^*)^* \widehat{f} dS, \end{aligned} \tag{12}$$

where the superscripts * and $^\sim$ denote transposition and Lagrange's conjugation, respectively.

It is clear that by substituting $\widehat{V}^+$ and $({}_1\widehat{R}\widehat{V})^+$ found from (10) and (12) in (9) and (11) we will solve Problem $A(\zeta)$. It appears that (10) and (12) can be used for constructing approximate values of the unknown vectors.

We introduce the following notations: $z \in S$, $x \in \mathbb{R}^3$,

$${}_1\Psi(x,z,\zeta) = \left\| \boxed{({}_1\widetilde{R}_1\Phi^*(x-z,\zeta))^*}_{5\times5}, \ \boxed{-{}_1\Phi(x-z,\zeta)}_{5\times5} \right\|_{5\times10}, \tag{13}$$

$${}_2\Psi(x,z,\zeta) = \left\| \boxed{({}_2\widetilde{R}_2\Phi^*(x-z,\zeta))^*}_{5\times5}, \ \boxed{-{}_2\Phi(x-z,\zeta)}_{5\times5} \right\|_{5\times10}, \tag{14}$$

$\psi(x,\zeta) = \|\psi_k\|_{10\times 1} = (\widehat{V}^+, ({}_1\widehat{R}\widetilde{V})^+)$ is the sought for vector. Now relations (10) and (12) can be rewritten in the form

$$\forall x \in D_2 \ : \ \int\limits_S {}_1\Psi(x,z,\zeta)\psi(z,\zeta)d_zS = {}_1F(x), \tag{15}$$

$$\forall x \in D_1 \ : \ \int\limits_S {}_2\Psi(x,z,\zeta)\psi(z,\zeta)d_zS = {}_2F(x), \tag{16}$$

where

$${}_1F(x) = -\int\limits_{D_1} {}_1\Phi_1\chi\, dz,$$

$${}_2F(x) = \int\limits_{D_2} {}_2\Phi_2\chi\, dz - \int\limits_S {}_2\Phi\widetilde{F}\, dS + \int\limits_S ({}_2\widetilde{\widehat{R}}_2\Phi^*)^*\widehat{f}\, dS$$

are the given vectors.

Let us construct auxiliary domains and surfaces in the following manner: $\widetilde{D}_1$ is a domain bounded by $\widetilde{S}_1$ located strictly in D_1, i.e, $\overline{\widetilde{D}}_1 \subset D_1$; $\widetilde{D}_2$ is an infinite domain bounded by $\widetilde{S}_2$ located strictly in D_2. It is clear that $\widetilde{S}_1 \cap S = \varnothing$, $\widetilde{S}_2 \cap S = \varnothing$.

Let $\{{}_jx^k\}_{k=1}^{\infty}$, $j = 1, 2$, be a countable, dense everywhere, set of points on the auxiliary surface $\widetilde{S}_j$, $j = 1, 2$. From (15) and (16) we have

$$\int\limits_S {}_1\Psi({}_2x^k,z,\zeta)\psi(z,\zeta)d_zS = {}_1F({}_2x^k), \quad k = \overline{1,\infty}, \tag{17}$$

$$\int\limits_S {}_2\Psi({}_1x^k,z,\zeta)\psi(z,\zeta)d_zS = {}_2F({}_1x^k), \quad k = \overline{1,\infty}. \tag{18}$$

We denote the rows of the matrix ${}_j\Psi$ considered as ten-component vectors by ${}_j\Psi^1, {}_j\Psi^2, {}_j\Psi^3, {}_j\Psi^4, {}_j\Psi^5$ and consider the countably infinite set of vectors

$$\left\{{}_1\Psi^l({}_2x^k,z,\zeta)\right\}_{k=1,\, l=1}^{\infty,\, 5} \bigcup \left\{{}_2\Psi^l({}_1x^k,z,\zeta)\right\}_{k=1,\, l=1}^{\infty,\, 5}. \tag{19}$$

It is proved that (19) is linearly independent and complete in the space $L_2(S)$; i.e., forms the basis in this space.

Let us enumerate set (19) arbitrarily and denote the resulting countable set by

$$\{\psi^k(z)\}_{k=1}^{\infty}. \tag{20}$$

We have, for example, performed enumeration like this:

$$\psi^k(z) \equiv {}_{a_k}\Psi^{l_k}({}_{b_k}x^{q_k},z,\zeta), \quad k = \overline{1,\infty},$$

where

$$a_k = k - 2\Big[\frac{k-1}{2}\Big], \quad b_k = 2\Big[\frac{k+1}{2}\Big] - k + 1,$$
$$l_k = \Big[\frac{k+1}{2}\Big] - 5\left[\frac{[\frac{k+1}{2}]-1}{5}\right], \quad q_k = \left[\frac{[\frac{k+1}{2}]+4}{5}\right];$$

$[k]$ is the integer part of the number k. It is clear that by virtue of (17) and (18) the scalar product

$$(\psi^k, \bar{\psi}) = \int\limits_S \psi^k \psi \, dS = (\psi, \bar{\psi}^k)$$

is known for any k. Using our notations, we have

$$\int\limits_S \psi^k \psi \, dS = {}_{a_k}F_{l_k}({}_{b_k}x^{q_k}), \quad k = \overline{1, \infty}.$$

Obviously, the complex conjugate system

$$\{\bar{\psi}^k(z)\}_{k=1}^{\infty} \tag{21}$$

is also complete.

Now we have to find coefficients α_k, $k = \overline{1, N}$ assuming that the mean-square norm

$$\Big\|\psi(z) - \sum_{k=1}^{N} \alpha_k \bar{\psi}^k(z)\Big\|_{L_2(S)}$$

is minimal. As is well-known, for this it is necessary and sufficient that

$$\Big(\psi(z) - \sum_{k=1}^{N} \alpha_k \bar{\psi}^k(z), \bar{\psi}^j(z)\Big) = 0, \quad j = \overline{1, N}.$$

Hence we arrive at an algebraic system of equations

$$\sum_{k=1}^{N} \alpha_k (\bar{\psi}^k, \bar{\psi}^j) = (\psi, \bar{\psi}^j), \quad j = \overline{1, N},$$

with the known right-hand side and Gram's determinant differing from zero, which defines coefficients α_k. Therefore, due to the property of the space $L_2(S)$, we have

$$\lim_{N\to\infty} \Big\|\psi(z) - \sum_{k=1}^{N} \alpha_k \bar{\psi}^k(z)\Big\|_{L_2(S)} = 0. \tag{22}$$

Let us introduce the notation

$$\overset{N}{\psi}(z) = \sum_{k=1}^{N} \alpha_k \bar{\psi}^k(z),$$

$${}_N\widehat{V}^+ = (\overset{N}{\psi}_1, \overset{N}{\psi}_2, \dots, \overset{N}{\psi}_5) \equiv \sum_{k=1}^{N} \alpha_k (\bar{\psi}_1^k, \bar{\psi}_2^k, \dots, \bar{\psi}_5^k),$$

$${}_N({}_1R_\tau \widehat{V})^+ = (\overset{N}{\psi}_6, \overset{N}{\psi}_7, \dots, \overset{N}{\psi}_{10}) \equiv \sum_{k=1}^{N} \alpha_k (\bar{\psi}_6^k, \bar{\psi}_7^k, \dots, \bar{\psi}_{10}^k).$$

Then we have in the sense of the metric of $L_2(S)$:

$$\psi(z) = \lim_{N\to\infty} \overset{N}{\psi}(z), \quad \widehat{V}^+ = \lim_{N\to\infty} {}_N\widehat{V}^+, \quad ({}_1R\widehat{V})^+ = \lim_{N\to\infty} {}_N({}_1R\widehat{V})^+.$$

Substituting the obtained approximate values in (9) and (11) and denoting the result of the substitution by ${}_N\widehat{V}(x,\zeta)$, we get

$$\forall x \in D_1 \;:\; {}_N\widehat{V}(x,\zeta) = \int_S {}_1\Phi\Big(\sum_{k=1}^{N} \alpha_k (\bar{\psi}_6^k, \bar{\psi}_7^k, \dots, \bar{\psi}_{10}^k)\Big) dS -$$
$$- \int_S ({}_1\widetilde{\widehat{R}}_1 \Phi^*)^* \Big(\sum_{k=1}^{N} \alpha_k (\bar{\psi}_1^k, \bar{\psi}_2^k, \dots, \bar{\psi}_5^k)\Big) dS - \int_{D_1} {}_1\Phi_1 \chi dz,$$

$$\forall x \in D_2 \;:\; {}_N\widehat{V}(x,\zeta) = -\int_S {}_2\Phi\Big(\sum_{k=1}^{N} \alpha_k (\bar{\psi}_6^k, \bar{\psi}_7^k, \dots, \bar{\psi}_{10}^k)\Big) dS +$$
$$+ \int_S ({}_2\widetilde{\widehat{R}}_2 \Phi^*)^* \Big(\sum_{k=1}^{N} \alpha_k (\bar{\psi}_1^k, \bar{\psi}_2^k, \dots, \bar{\psi}_5^k)\Big) dS -$$
$$- \int_{D_2} {}_2\Phi_2 \chi dz + \int_S {}_2\Phi \widetilde{F}\, dS - \int_S ({}_2\widetilde{\widehat{R}}_2 \Phi^*)^* \widehat{f}\, dS.$$

Now for any $\varepsilon \geq 0$ we can give a positive number $N(\varepsilon)$ such that for $N > N(\varepsilon)$ we will have

$$\big|\widehat{V}(x,\zeta) - {}_N\widehat{V}(x,\zeta)\big| < \varepsilon,$$

$x \in \bar{D}' \subset D$; $\widehat{V}(x,\zeta)$ is the exact solution of the problem, i.e.,

$$\widehat{V}(x,\zeta) = \lim_{N\to\infty} {}_N\widehat{V}(x,\zeta), \quad x \in \bar{D}';$$

the convergence to the limit is uniform in $\bar{D}'$.

The method presented here can also be generalized for other more complicated problems.

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