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ON MAXIMAL *ot*-SUBSETS OF THE EUCLIDEAN PLANE

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Abstract. We say that a subset X of the plane $\mathbf{R}^2$ is an *ot*-set if any three points of X form an obtuse triangle. Some properties of *ot*-sets are investigated. It is shown that no finite *ot*-subset of $\mathbf{R}^2$ is maximal, but there exists a countable maximal *ot*-subset of $\mathbf{R}^2$. Several related problems are formulated and discussed.

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There are many interesting problems and open questions in the geometry of Euclidean spaces, which have a set-theoretic flavour and are closely connected with point set theory and convexity (see, e.g., [1]–[6]). A number of such problems was raised by P. Erdös (cf. [1]). Similar questions frequently arise in combinatorial geometry, classical measure theory and equidecomposability theory.

Here we would like to discuss one problem of this kind for the Euclidean plane $\mathbf{R}^2$. We begin with some definitions.

Let X be a subset of the plane. We shall say that X is an *ot*-set if any three-element subset of X forms an obtuse triangle.

We shall say that an *ot*-set $X \subset \mathbf{R}^2$ is maximal if there is no *ot*-set on the plane properly containing X.

Each *ot*-set in $\mathbf{R}^2$ is contained in some maximal *ot*-subset of the plane (this fact follows directly from the Kuratowski–Zorn lemma).

As far as we know, the following problem remains unsolved.

Problem. Give a characterization of all maximal *ot*-subsets of the plane.

Here are some simple examples of maximal *ot*-sets in $\mathbf{R}^2$.

Example 1. Let X be a semi-circumference on the plane without one of its end-points. It can be easily verified that X is a maximal *ot*-set in $\mathbf{R}^2$.

Example 2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function, let X_f denote the graph of f and suppose that the following conditions are satisfied:

(1) f is increasing and continuous;

(2) no three points of X_f belong to a straight line, i.e. all points of X_f are in general position.

Then it is not difficult to show that X_f is a maximal *ot*-set in $\mathbf{R}^2$. Note also that conditions (1) and (2) imply at once that the given function f is strictly increasing.

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Let x and y be any two distinct points in $\mathbf{R}^2$. We denote by $l(x, y)$ the straight line passing through x and y.

Let l and l' be any two distinct parallel straight lines on $\mathbf{R}^2$. We recall that all points of $\mathbf{R}^2$ lying between l and l' form the strip determined by these two lines. This strip is denoted by $S(l, l')$. In the sequel, only closed strips are under consideration, i.e. we assume throughout the paper that $S(l, l')$ contains both lines l and l'. The width of $S(l, l')$ is equal (by definition) to the distance between l and l'.

A single straight line may be regarded as a strip whose width is equal to zero.

Let x and x' be any two distinct points of $\mathbf{R}^2$, let l be the straight line passing through x and perpendicular to the segment $[x, x']$, let l' be the straight line passing through x' and perpendicular to the same segment. We shall say in the sequel that the strip $S(l, l')$ is determined by the points x and x'. In this case we also denote $S(x, x') = S(l, l')$.

Let $\{S_i : i \in I\}$ be a family of strips in $\mathbf{R}^2$. We shall say that these strips are in general position if, for any two distinct strips S_i and S_j, a boundary line of S_i is not parallel to a boundary line of S_j.

It is not hard to show that the plane cannot be covered by finitely many strips and straight lines. From this circumstance it readily follows that no finite *ot*-subset of $\mathbf{R}^2$ can be maximal.

Note that the maximal *ot*-sets described in Examples 1 and 2 above are of cardinality continuum. In this connection, it is natural to ask whether there exists a countable maximal *ot*-set in $\mathbf{R}^2$.

The main goal of this paper is to demonstrate that there are countable locally finite maximal *ot*-subsets of $\mathbf{R}^2$. In order to establish this fact, we need some auxiliary notions and statements.

Let $\{S_i : i \in I\}$ and $\{S_j : j \in J\}$ be two families of strips in $\mathbf{R}^2$. We shall say that these two families are in general position if all strips of the extended family $\{S_i : i \in I\} \cup \{S_j : j \in J\}$ are in general position.

Let X be a subset of $\mathbf{R}^2$ and let $\{S_i : i \in I\}$ be a family of strips. We shall say that X and $\{S_i : i \in I\}$ are in general position if the families $\{S_i : i \in I\}$ and $\{S(x, x') : x \in X,\ x' \in X,\ x \neq x'\}$ are in general position.

If Z is a subset of $\mathbf{R}^2$ and $\varepsilon > 0$, then the symbol $V(Z, \varepsilon)$ denotes, as usual, the ε-neighbourhood of Z. Accordingly, we utilize the notation $V(z, \varepsilon)$ for a point $z \in \mathbf{R}^2$ and $\varepsilon > 0$.

We shall say that a set C is an open circle in $\mathbf{R}^2$ if

$$C = \{x \in \mathbf{R}^2 \,:\, x_1^2 + x_2^2 < r\},$$

where r is a strictly positive real number.

Lemma 1. *Let X be a finite set of points in $\mathbf{R}^2$, such that all the strips $S(x, x')$ ($x \in X$, $x' \in X$, $x \neq x'$) are in general position, and let $C \subset \mathbf{R}^2$ be an open circle containing X. Then, for every $\varepsilon > 0$, there exists a finite family of strips $\{S_i : i \in I\}$ satisfying the relations:*

1) *for each $i \in I$, the width of S_i is less than ε;*

2) *X and $\{S_i : i \in I\}$ are in general position;*

3) $X \subset C \setminus \cup\{S_i : i \in I\} \subset V(X, \varepsilon)$.

We omit an easy proof of Lemma 1.

Lemma 2. *Let C be an open circle in $\mathbf{R}^2$, let X be a finite ot-subset of $\mathbf{R}^2$ and let $\{S_1, S_2, \ldots, S_m\}$ be a finite family of strips in $\mathbf{R}^2$ such that X and $\{S_1, S_2, \ldots, S_m\}$ are in general position. Then there exist pairwise distinct points $y_1, z_1, y_2, z_2, \ldots, y_m, z_m$ satisfying the relations:*

1) $\{y_1, z_1, y_2, z_2, \ldots, y_m, z_m\} \subset \mathbf{R}^2 \setminus C$;

2) $X^* = X \cup \{y_1, z_1, y_2, z_2, \ldots, y_m, z_m\}$ is an ot-subset of $\mathbf{R}^2$;

3) for each natural number $i \in [1, m]$, we have $S_i = S(y_i, z_i)$;

4) the family of strips $\{S(x, x') : x \in X^*,\ x' \in X^*,\ x \neq x'\}$ is in general position.

The proof of Lemma 2 can be obtained by induction on m. We omit the corresponding technical details (which are not difficult).

Theorem 1. *There exists a countable locally finite maximal ot-subset of $\mathbf{R}^2$.*

Proof. Fix a decreasing sequence $\{\varepsilon_k : k \geq 1\}$ of strictly positive real numbers, such that $\lim_{k\to\infty} \varepsilon_k = 0$.

Define, by induction on k, two increasing (with respect to the inclusion relation) families $\{X_k : k \geq 1\}$ and $\{C_k : k \geq 1\}$ of subsets of $\mathbf{R}^2$.

For $k = 1$, take any finite *ot*-set $X_1 \subset \mathbf{R}^2$ with the property that all the strips from $\{S(x, x') : x \in X_1,\ x' \in X_1,\ x \neq x'\}$ are in general position, and take also an open circle $C_1 \supset X_1$ with radius greater than 1.

Suppose that the sets X_k and C_k are already defined for a natural number $k \geq 1$.

Applying Lemma 1 to X_k, C_k and ε_k, we can find a finite family of strips $\{S_i : i \in I_k\}$ satisfying the following relations:

1) for each $i \in I_k$, the width of S_i is less than ε_k;

2) X_k and $\{S_i : i \in I_k\}$ are in general position;

3) $X_k \subset C_k \setminus \cup\{S_i : i \in I_k\} \subset V(X_k, \varepsilon_k)$.

Now, we apply Lemma 2 to the circle C_k, the set X_k and the family of strips $\{S_i : i \in I_k\}$. According to the above-mentioned lemma, there exists a finite *ot*-set $X_k^* \subset \mathbf{R}^2$ such that:

a) $X_k \subset X_k^*$;

b) $X_k^* \setminus X_k \subset \mathbf{R}^2 \setminus C_k$;

c) the family of strips $S(x, x')$ $(x \in X_k^*,\ x' \in X_k^*,\ x \neq x')$ is in general position and contains the family $\{S_i : i \in I_k\}$.

We put $X_{k+1} = X_k^*$. Besides, let C_{k+1} be an open circle in $\mathbf{R}^2$ containing $X_{k+1} \cup C_k$, whose radius is greater than $k + 1$.

Proceeding in this manner, we are able to construct the desired families of sets $\{X_k : k \geq 1\}$ and $\{C_k : k \geq 1\}$.

Note that, in virtue of our construction, we have the equality

$$\mathbf{R}^2 = \cup\{C_k : k \geq 1\}.$$

Finally, we define

$$X = \cup\{X_k : k \geq 1\}.$$

A straightforward verification shows that X is a countable locally finite *ot*-subset of the plane, and we are going to demonstrate that X is maximal. For this purpose, take any point $t \in \mathbf{R}^2 \setminus X$ and check that $X \cup \{t\}$ cannot be an *ot*-set in $\mathbf{R}^2$. Since $\lim_{k\to\infty} \varepsilon_k = 0$, there exists a natural number k_0 such that, for all natural numbers $k > k_0$, we have the relation

$$t \in C_k \setminus \cup\{V(x, \varepsilon_k) : x \in X_k\}$$

and, consequently, $t \in \cup\{S_i : i \in I_k\}$ (see relation 3) above). Hence $t \in S_i$ for some $i \in I_k$. But, in view of our construction, $S_i = S(y, z)$ where $\{y, z\} \subset X_{k+1} \setminus X_k \subset X$, and we also remember that the width of S_i is less than ε_k. Now, taking this circumstance into account, it can easily be seen that if k is large enough, then the values of all three angles in the triangle $[t, y, z]$ do not exceed $\pi/2$, whence it follows that $X \cup \{t\}$ is not an *ot*-subset of $\mathbf{R}^2$.

The theorem has thus been proved. □

A statement analogous to Theorem 1 can be established for the n-dimensional Euclidean space $\mathbf{R}^n$ where $n > 2$. In other words, we have the following

Theorem 2. *If $n > 2$, then there exists a countable locally finite maximal ot-set in the space $\mathbf{R}^n$.*

The proof of this statement can be carried out in the same manner as for the Euclidean plane $\mathbf{R}^2$ (some additional purely technical details occur, but they do not represent essential difficulties).

It would be interesting to find a characterization of all maximal *ot*-subsets of the space $\mathbf{R}^n$, where $n > 2$. In this context, the following example is relevant.

Example 3. Consider the three-dimensional Euclidean space $\mathbf{R}^3$ and its two-dimensional subspace $\mathbf{R}^2 \times \{0\}$. Let S be a closed semi-circumference in $\mathbf{R}^2 \times \{0\}$ with end-points y and z. Let $l(y, z)$ denote the straight line passing through y and z. Take any point x on $l(y, z)$ not belonging to the linear segment $[y, z]$ and put

$$X = (S \setminus \{y\}) \cup \{(x, 1)\}.$$

It is not hard to check that X is an *ot*-set in $\mathbf{R}^3$. Also, as we know, $S \setminus \{y\}$ is a maximal *ot*-subset of the plane $\mathbf{R}^2 \times \{0\}$ (see Example 1). Since X properly contains $S \setminus \{y\}$, we conclude that $S \setminus \{y\}$ is not a maximal *ot*-set in the space $\mathbf{R}^3$.

Moreover, we cannot even assert that X is a maximal *ot*-subset of $\mathbf{R}^3$. For instance, if the semi-circumference S is such that

$$(\forall t \in S)(||t - x|| < 1),$$

then the set

$$X' = (S \setminus \{y\}) \cup \{(x, 1)\} \cup \{(x, -1)\} = X \cup \{(x, -1)\}$$

turns out to be an *ot*-subset of $\mathbf{R}^3$ and properly contains X.

Remark 1. In connection with Theorem 1, the following question arises naturally: does there exist a countable bounded maximal *ot*-subset of $\mathbf{R}^2$? This question remains open.

Remark 2. Let H be a pre-Hilbert space (over the field $\mathbf{R}$) and let X be a subset of H. We say that X is an rt-set if each three-element subset of X forms a rectangular triangle. In [5] a characterization of all rt-sets was given (for more details, see [6]). In particular, it was demonstrated there that $card(X) \leq \mathbf{c}$ for any rt-set X, where $\mathbf{c}$ denotes the cardinality of the continuum. In the case $H = \mathbf{R}^2$, it can easily be shown that X is a maximal rt-subset of H if and only if X coincides with the set of vertices of a rectangle in H.

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