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# ON SOME CONVEXITY PROPERTIES OF GENERALIZED CESÁRO SEQUENCE SPACES

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**Abstract.** We define a generalized Cesáro sequence space and consider it equipped with the Luxemburg norm under which it is a Banach space, and we show that it is locally uniformly rotund.

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## 1. Preliminaries

For a Banach space $X$, we denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of $X$, respectively. A point $x_0 \in S(X)$ is called

a) an *extreme point* if for every $x, y \in S(X)$ the equality $2x_0 = x + y$ implies $x = y$;

b) a *locally uniformly rotund point* (LUR-point for short) if for any sequence $(x_n)$ in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$ there holds $\|x_n - x\| \to 0$ as $n \to \infty$;

c) an *$H$-point* if for any sequence $(x_n)$ in $X$ such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of $(x_n)$ to $x_0$ (write $x_n \xrightarrow{w} x_0$) implies that $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space $X$ is said to be *rotund* (R) if every point of $S(X)$ is an extreme point.

If every $x \in S(X)$ is a LUR-point, then $X$ is said to be *locally uniformly rotund* ($LUR$).

$X$ is said to possess property (H) provided every point of $S(X)$ is an $H$-point.

For these geometric notions and their role in Mathematics we refer to the monographs [1], [6], [12] and [13]. Some of them were studied for Orlicz spaces in [1], [7], [8], [12] and [14].

Let $X$ be a real vector space. A functional $\varrho : X \to [0, \infty]$ is called a *modular* if it satisfies the conditions

(i) $\varrho(x) = 0$ if and only if $x = 0$;
(ii) $\varrho(\alpha x) = \varrho(x)$ for all scalar $\alpha$ with $|\alpha| = 1$;
(iii) $\varrho(\alpha x+\beta y) \le \varrho(x)+\varrho(y)$ for all $x, y \in X$ and all $\alpha$, $\beta \ge 0$ with $\alpha+\beta = 1$.
The modular $\varrho$ is called *convex* if
(iv) $\varrho(\alpha x + \beta y) \le \alpha\varrho(x) + \beta\varrho(y)$ for all $x, y \in X$ and all $\alpha$, $\beta \ge 0$ with $\alpha + \beta = 1$.

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If $\varrho$ is a modular in $X$, we define

$$X_\varrho = \Big\{x \in X : \lim_{\lambda \to 0^+} \varrho(\lambda x) = 0\Big\},$$
$$\text{and} \quad X_\varrho^* = \Big\{x \in X : \varrho(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\Big\}.$$

It is clear that $X_\varrho \subseteq X_\varrho^*$. If $\varrho$ is a convex modular, for $x \in X_\varrho$ we define

$$\|x\| = \inf\Big\{\lambda > 0 : \ \varrho\Big(\frac{x}{\lambda}\Big) \le 1\Big\}. \tag{1.1}$$

Orlicz [13] proved that if $\varrho$ is a convex modular in $X$, then $X_\varrho = X_\varrho^*$ and $\|.\|$ is a norm on $X_\varrho$ for which it is a Banach space. The norm $\|.\|$ defined as in (1.1) is called the Luxemburg norm.

A modular $\varrho$ on $X$ is called

(a) *right-continuous* if $\lim_{\lambda \to 1^+} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$;
(b) *left-continuous* if $\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x)$ for all $x \in X_\varrho$;
(c) *continuous* if it is both left-continuous and right-continuous.

The following known results gave some relationships between the modular $\varrho$ and the Luxemburg norm $\|.\|$ on $X_\varrho$.

**Theorem 1.1.** *Let $\varrho$ be a convex modular on $X$ and let $x \in X_\varrho$ and $(x_n)$ a sequence in $X_\varrho$. Then $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $\varrho(\lambda(x_n - x)) \to 0$ as $n \to \infty$ for every $\lambda > 0$.*

*Proof.* See [11, Theorem 1.3]. □

**Theorem 1.2.** *Let $\varrho$ be a convex modular on $X$ and $x \in X_\varrho$.*

(i) *If $\varrho$ is right-continuous, then $\|x\| < 1$ if and only if $\varrho(x) < 1$.*
(ii) *If $\varrho$ is left-continuous, then $\|x\| \le 1$ if and only if $\varrho(x) \le 1$.*
(iii) *If $\varrho$ is continuous, then $\|x\| = 1$ if and only if $\varrho(x) = 1$.*

*Proof.* See [11, Theorem 1.4]. □

Let us denote by $l^0$ the space of all real sequences. For $1 \le p < \infty$, the Cesáro sequence space ($ces_p$, for short) is defined by

$$ces_p = \Big\{x \in l^0 : \sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p < \infty\Big\}$$

equipped with the norm

$$\|x\| = \Big(\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{i=1}^{n}|x(i)|\Big)^p\Big)^{\frac{1}{p}}.$$

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operators and others (see [9] and [10]). Some geometric properties of the Cesáro sequence space $ces_p$ were studied by many mathematicians. It is known that $ces_p$ is LUR and possesses property (H) (see [10] ). Y. A. Cui and H. Hudzik [2] proved that $ces_p$ has the Banach-Saks property, and it was shown in [5] that $ces_p$ has property $(\beta)$.

Now let $p = (p_k)$ be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. The Nakano sequence space $l(p)$ is defined by

$$l(p) = \{x \in l^0 : \sigma(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\},$$

where $\sigma(x) = \sum_{i=1}^{\infty} |x(i)|^{p_i}$. We consider the space $l(p)$ equipped with the norm

$$\|x\| = \inf\left\{\lambda > 0 : \sigma\Big(\frac{x}{\lambda}\Big) \leq 1\right\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$l(p) = \left\{x \in l^0 : \sum_{i=1}^{\infty} |x(i)|^{p_i} < \infty\right\}.$$

Several geometric properties of $l(p)$ were studied in [1] and [4].

The generalized Cesáro sequence space $ces(p)$ is defined by

$$ces(p) = \{x \in l^0 : \varrho(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\},$$

where $\varrho(x) = \sum_{n=1}^{\infty} (\frac{1}{n} \sum_{i=1}^{n} |x(i)|)^{p_n}$. We consider this space equipped with the so-called Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \varrho\Big(\frac{x}{\lambda}\Big) \leq 1\right\}$$

under which it is a Banach space. If $p = (p_k)$ is bounded, we have

$$ces(p) = \left\{x = x(i) : \sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{i=1}^{n} |x(i)|\Big)^{p_n} < \infty\right\}.$$

W. Sanhan [15] proved that $ces(p)$ is nonsquare when $p_k > 1$ for all $k \in \mathbb{N}$. In this paper, we show that the Cesáro sequence space $ces(p)$ equipped with the Luxemburg norm is $LUR$ and has property $(H)$ when $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$.

Throughout this paper we assume that $p = (p_k)$ is bounded with $p_k > 1$ for all $k \in \mathbb{N}$, and $M = \sup_k p_k$.

## 2. Main Results

We begin by giving some basic properties of the modular $\varrho$ on the space $ces(p)$. By the convexity of the function $t \to |t|^{p_k}$, for every $k \in \mathbb{N}$ we have that $\varrho$ is a convex modular. So we have the following proposition.

**Proposition 2.1.** *The functional $\varrho$ on the Cesáro sequence space $ces(p)$ is a convex modular.*

**Proposition 2.2.** *For $x \in ces(p)$, the modular $\varrho$ on $ces(p)$ satisfies the following properties:*

(i) *if $0 < a < 1$, then $a^M \varrho\Big(\dfrac{x}{a}\Big) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$,*

(ii) *if $a \geq 1$, then $\varrho(x) \leq a^M \varrho\Big(\dfrac{x}{a}\Big)$,*

(iii) *if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.*

*Proof.* All assertions are clearly obtained by the definition of $\varrho$. $\square$

**Proposition 2.3.** *The modular $\varrho$ on $ces(p)$ is continuous.*

*Proof.* For $\lambda > 1$, by Proposition 2.2 (ii) and (iii), we have

$$\varrho(x) \leq \lambda\varrho(x) \leq \varrho(\lambda x) \leq \lambda^M \varrho(x). \tag{2.1}$$

By taking $\lambda \to 1^+$ in (2.1), we have $\lim_{\lambda\to 1^+} \varrho(\lambda x) = \varrho(x)$. Thus $\varrho$ is right-continuous. If $0 < \lambda < 1$, by Proposition 2.2 (i), we have

$$\lambda^M \varrho(x) \leq \varrho(\lambda x) \leq \lambda\varrho(x) \tag{2.2}$$

By taking $\lambda \to 1^-$ in (2.2), we have that $\lim_{\lambda\to 1^-} \varrho(\lambda x) = \varrho(x)$, hence $\varrho$ is left-continuous. Thus $\varrho$ is continuous. $\square$

Next, we give some relationships between the modular $\varrho$ and the Luxemburg norm on $ces(p)$.

**Proposition 2.4.** *For any $x \in ces(p)$, we have*

(i) *if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$,*
(ii) *if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$,*
(iii) *$\|x\| = 1$ if and only if $\varrho(x) = 1$,*
(iv) *$\|x\| < 1$ if and only if $\varrho(x) < 1$,*
(v) *$\|x\| > 1$ if and only if $\varrho(x) > 1$,*
(vi) *if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and*
(vii) *if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.*

*Proof.* If $\|x\| \leq 1$ , it follows by the convexity and continuity of $\varrho$ that $\varrho(x) = \varrho\Big(\|x\|\dfrac{x}{\|x\|}\Big) \leq \|x\|\varrho\Big(\dfrac{x}{\|x\|}\Big) \leq \|x\|$. So (i) is obtained. If $\|x\| > 1$, then there is $\varepsilon_0 > 0$ such that $\|x\| - \varepsilon > 1$ for all $\varepsilon \in (0, \varepsilon_0)$. Consequently, $\varrho(x) = \varrho\Big((\|x\| - \varepsilon)\dfrac{x}{\|x\| - \varepsilon}\Big) \geq (\|x\| - \varepsilon)\varrho\Big(\dfrac{x}{\|x\| - \varepsilon}\Big) > \|x\| - \varepsilon$, so (ii) is satisfied. It is clear that (iii), (iv) and (v) follow by Theorem 1.2, and properties (vi) and (vii) follow by Proposition 2.2. $\square$

**Proposition 2.5.** *Let $(x_n)$ be a sequence in $ces(p)$.*

(i) *If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.*
(ii) *$\|x_n\| \to 0$ as $n \to \infty$ if and only if $\varrho(x_n) \to 0$ as $n \to \infty$.*

*Proof.* (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\epsilon \in (0, 1)$. Then there exists $N \in \mathbb{N}$ such that $1 - \epsilon < \|x_n\| < 1 + \epsilon$ for all $n \geq N$. By Proposition 2.4 (vi) and (vii), we have $(1 - \epsilon)^M < \varrho(x_n) < (1 + \epsilon)^M$ for all $n \geq N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) It follows from Theorem 1.1 that if $\|x_n\| \to 0$ as $n \to \infty$, then $\varrho(x_n) \to 0$ as $n \to \infty$. Conversely, suppose $\|x_n\| \not\to 0$ as $n \to \infty$. Then there is $\epsilon \in (0, 1)$ and a subsequence $(x_{n_k})$ of $(x_n)$ such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Proprosition 2.4 (vi), we have $\varrho(x_{n_k}) > \epsilon^M$ for all $k \in \mathbb{N}$. This implies $\varrho(x_n) \not\to 0$ as $n \to \infty$. $\square$

**Proposition 2.6.** *Let* $(x_n) \subseteq B(l(p))$ *and* $(y_n) \subseteq B(l(p))$. *If* $\sigma\Big(\dfrac{x_n+y_n}{2}\Big) \to 1$, *then* $x_n(i) - y_n(i) \to 0$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$.

*Proof.* We first note that if $x \in B(\ell(p)$, then $\sigma(x) \leq 1$. Supose that $x_n(i) - y(i) \not\to 0$ as $n \to \infty$ for some $i \in \mathbb{N}$. Without loss of generality we may assume that $i = 1$, and then assume without loss of generality (passing to a subsequence if necessary) that, for some $\epsilon > 0$,

$$\big|x_n(1) - y_n(1)\big|^{p_1} \geq \epsilon \quad \forall\, n \in \mathbb{N}.$$

Thus

$$2^{p_1}(|x_n(1)|^{p_1} + |y_n(1)|^{p_1}) \geq \epsilon \quad \forall\, n \in \mathbb{N}. \tag{2.3}$$

Since the function $t \to |t|^{p_1}$ is uniformly convex, there exists $\delta > 0$ such that

$$\Big|\frac{x_n(1)+y_n(1)}{2}\Big|^{p_1} \leq (1-\delta)\Big(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2}\Big) \;\; \forall\, n \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) and (2.4) that for each $n \in \mathbb{N}$,

$$\begin{aligned}
\sigma\Big(\frac{x_n+y_n}{2}\Big) &= \sum_{i=1}^{\infty}\Big|\frac{x_n(i)+y_n(i)}{2}\Big|^{p_i} \\
&= \Big|\frac{x_n(1)+y_n(1)}{2}\Big|^{p_1} + \sum_{i=2}^{\infty}\Big|\frac{x_n(i)+y_n(i)}{2}\Big|^{p_i} \\
&\leq (1-\delta)\Big(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2}\Big) + \frac{1}{2}\sum_{i=2}^{\infty}|x_n(i)|^{p_i} + \frac{1}{2}\sum_{i=2}^{\infty}|y_n(i)|^{p_i} \\
&= \frac{1}{2}\,\sigma(x_n) + \frac{1}{2}\,\sigma(y_n) - \delta\Big(\frac{|x_n(1)|^{p_1} + |y_n(1)|^{p_1}}{2}\Big) \\
&\leq \frac{1}{2} + \frac{1}{2} - \delta\frac{\epsilon}{2^{p_1+1}} = 1 - \delta\frac{\epsilon}{2^{p_1+1}}.
\end{aligned}$$

This implies that $\sigma\Big(\dfrac{x_n+y_n}{2}\Big) \not\to 1$ as $n \to \infty$, a contradiction, which finishes the proof. $\square$

**Proposition 2.7.** *Let* $(x_n) \subseteq B(ces(p))$ *and* $x \in S(ces(p))$. *If* $\varrho\Big(\dfrac{x_n+x}{2}\Big) \to 1$ *as* $n \to \infty$, *then* $x_n(i) \to x(i)$ *as* $n \to \infty$ *for all* $i \in \mathbb{N}$.

*Proof.* For each $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$s_n(i) = \begin{cases} sgn(x_n(i)+x(i)) & \text{if } \; x_n(i)+x(i) \neq 0, \\ 1 & \text{if } \; x_n(i)+x(i) = 0. \end{cases}$$

Hence we have

$$1 \leftarrow \varrho\Big(\frac{x_n + x}{2}\Big) = \sum_{k=1}^{\infty}\Big(\frac{1}{k}\sum_{i=1}^{k}\Big|\frac{x_n(i) + x(i)}{2}\Big|\Big)^{p_k}$$
$$= \sum_{k=1}^{\infty}\Big(\frac{1}{k}\sum_{i=1}^{k} s_n(i)\frac{x_n(i)}{2} + \frac{1}{k}\sum_{i=1}^{k} s_n(i)\frac{x(i)}{2}\Big)^{p_k}. \tag{2.5}$$

Let $a_n(k) = \frac{1}{k}\sum_{i=1}^{k} s_n(i)x_n(i)$ and $b_n(k) = \frac{1}{k}\sum_{i=1}^{k} s_n(i)x(i)$ for all $n, k \in \mathbb{N}$. Then $(a_n) \in l(p)$ and $(b_n) \in l(p)$, and from (2.5) we have

$$\sigma\Big(\frac{a_n + b_n}{2}\Big) \to 1 \quad \text{as} \quad n \to \infty.$$

Form Proposition 2.6 we have

$$a_n(i) - b_n(i) \to 0 \quad \text{as} \quad n \to \infty \tag{2.6}$$

for all $i \in \mathbb{N}$. Now we shall show that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$. From (2.6) we have

$$s_n(1)x_n(1) - s_n(1)x(1) \to 0 \quad \text{as} \quad n \to \infty.$$

This implies $x_n(1) \to x(1)$ as $n \to \infty$. Assume that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \le k - 1$. Then we have

$$s_n(i)(x_n(i) - x(i)) \to 0 \quad \text{as} \quad n \to \infty \tag{2.7}$$

for all $i \le k-1$. Since $s_n(k)(x_n(k)-x(k)) = k(a_n(k)-b_n(k))-\sum_{i=1}^{k-1} s_n(i)(x_n(i)-x(i))$, it follows from (2.6) and (2.7) that $s_n(k)(x_n(k) - x(k)) \to 0$ as $n \to \infty$. This implies $x_n(k) \to x(k)$ as $n \to \infty$. So we have by induction that $x_n(k) \to x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$. □

**Theorem 2.8.** *The space $ces(p)$ is LUR.*

*Proof.* Let $(x_n) \subseteq B(ces(p))$ and $x \in S(ces(p))$ be such that $\|x_n + x\| \to 2$ as $n \to \infty$. Then $\Big\|\frac{x_n + x}{2}\Big\| \to 1$ as $n \to \infty$. By Proposition 2.5 (i) we have $\varrho\Big(\frac{x_n + x}{2}\Big) \to 1$ as $n \to \infty$. By Proposition 2.7 we have $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$.

Now let $\epsilon > 0$ be given. Then there exist $k_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty}\Big(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\Big)^{p_k} < \frac{\epsilon}{3}\frac{1}{2^{M+1}}, \tag{2.8}$$

$$\sum_{k=1}^{k_0}\Big(\frac{1}{k}\sum_{i=1}^{k}|x_n(i) - x(i)|\Big)^{p_k} < \frac{\epsilon}{3} \quad \text{for all} \quad n \ge n_0, \tag{2.9}$$

$$\sum_{k=1}^{k_0}\Big(\frac{1}{k}\sum_{i=1}^{k}|x_n(i)|\Big)^{p_k} > \sum_{k=1}^{k_0}\Big(\frac{1}{k}\sum_{i=1}^{k}|x(i)|\Big)^{p_k} - \frac{\epsilon}{3}\frac{1}{2^M} \quad \text{for all} \quad n \ge n_0. \tag{2.10}$$

By Proposition 2.4 (i) and (iii) we have $\varrho(x_n) \leq 1$ for all $n \in \mathbb{N}$ and $\varrho(x) = 1$. From these together with (2.8), (2.9), (2.10) and the fact that $(a+b)^{p_k} \leq 2^{p_k}(a^{p_k} + b^{p_k})$ for $a, b \geq 0$ we have that for all $n \geq n_0$,

$$\begin{aligned}
\varrho(x_n - x) &= \sum_{k=1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i) - x(i)|\Big)^{p_k} \\
&= \sum_{k=1}^{k_0} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i) - x(i)|\Big)^{p_k} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i) - x(i)|\Big)^{p_k} \\
&< \frac{\epsilon}{3} + 2^M \Bigg( \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i)|\Big)^{p_k} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&= \frac{\epsilon}{3} + 2^M \Bigg( \varrho(x_n) - \sum_{k=1}^{k_0} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i)|\Big)^{p_k} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&\leq \frac{\epsilon}{3} + 2^M \Bigg( 1 - \sum_{k=1}^{k_0} \Big(\frac{1}{k}\sum_{i=1}^{k} |x_n(i)|\Big)^{p_k} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&< \frac{\epsilon}{3} + 2^M \Bigg( 1 - \sum_{k=1}^{k_0} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&= \frac{\epsilon}{3} + 2^M \Bigg( \varrho(x) - \sum_{k=1}^{k_0} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&= \frac{\epsilon}{3} + 2^M \Bigg( \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} + \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} \Bigg) \\
&= \frac{\epsilon}{3} + 2^M \Bigg( 2\sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} + \frac{\epsilon}{3}\frac{1}{2^M} \Bigg) \\
&= \frac{\epsilon}{3} + 2^{M+1} \sum_{k=k_0+1}^{\infty} \Big(\frac{1}{k}\sum_{i=1}^{k} |x(i)|\Big)^{p_k} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This shows that $\varrho(x_n - x) \to 0$ as $n \to \infty$. By Proposition 2.5(ii) we have $\|x_n - x\| \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

It is known in general that a locally uniformly rotund space has property (H). So we have the following result.

**Corollary 2.9.** *The space $ces(p)$ possesses property (H).*

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