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COMBINATORIAL HOMOLOGY IN A PERSPECTIVE OF IMAGE ANALYSIS

MARCO GRANDIS

Dedicated to Professor Hvedri Inassaridze,
on the occasion of his seventieth birthday

Abstract. This is the sequel of a paper where we introduced an intrinsic homotopy theory and homotopy groups for simplicial complexes. We study here the relations of this homotopy theory with the well-known homology theory of simplicial complexes. Also, our investigation is aimed at applications in image analysis. A metric space X, representing an image, has a structure of simplicial complex *at each resolution* $\varepsilon > 0$, and the corresponding *combinatorial homology groups* $H_n^\varepsilon(X)$ give information on the image. Combining the methods developed here with programs for automatic computation of combinatorial homology might open the way to realistic applications.

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Introduction

This paper is devoted to the homology of simplicial complexes, its connections with the intrinsic homotopy theory developed in a previous work ([7], cited as Part I) and some methods of direct computation of combinatorial homology. As in Part I, the applications are aimed at image analysis in metric spaces, and connected with digital topology and mathematical morphology. In fact, a metric space X has a structure $t_\varepsilon X$ of simplicial complex *at each resolution* $\varepsilon > 0$, and the corresponding homotopy and homology groups $\pi_n^\varepsilon(X)$, $H_n^\varepsilon(X)$ detecting singularities which can be captured by an n-dimensional grid, with edges bound by ε; this works equally well for continuous regions of $\mathbf{R}^n$ or discrete ones; in the latter case, our results are closely related with analyses of 0- or 1-connection in "digital topology" (cf. [11, 12, 2]). Such methods, combined with the use of recent computer programs for the homology of simplicial complexes, should be of use in the analysis of complicated 2- or 3-dimensional images, as produced by scanning a geographical region or an object; this work is under progress, and it would be difficult to say now whether it will produce results of practical interest: "realistic images" tend to produce big simplicial complexes, on which the existing programs for computing homology do not terminate in a reasonable time; but algorithms can be developed to reduce the size of the simplicial complex without changing its homology.

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To give a first idea of these applications, *in an elementary case*, consider the subset $X \subset \mathbf{R}^2$ of Fig. (a), representing a planar image we want to analyse, for instance the map of a region

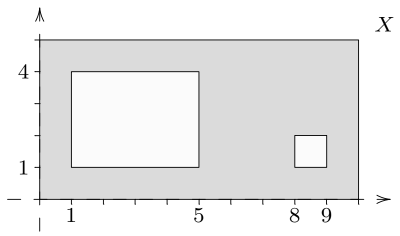


Fig. (a)

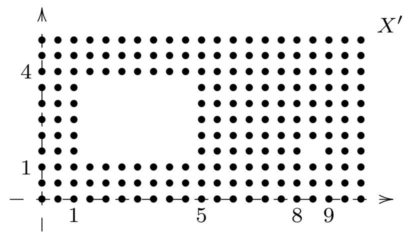


Fig. (b)

Viewing X as a topological space, we keep some relevant information which can be detected by the usual tools of algebraic topology; e. g. the fact that X is path-connected, with two "holes". However, we miss all metric information and are not able to distinguish a lake from a puddle. Further, if this "continuous" subspace X is replaced by a discrete trace $X' = X \cap (\rho\mathbf{Z} \times \rho\mathbf{Z})$ scanned at resolution $\rho = 1/2$, as in Fig. (b), we miss any topological information: X' is a discrete space.

It is more useful to view X and X' as *metric* spaces (we shall generally use the l_∞-metric of the plane, $d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$, essentially because this is *the* metric of the categorical product $\mathbf{R} \times \mathbf{R}$, cf. 1.2), and explore them at a variable resolution ε ($0 \leqslant \varepsilon \leqslant \infty$). This will mean to associate to any metric space X a simplicial complex $t_\varepsilon X$ at resolution ε, whose distinguished parts are the finite subsets $\xi \subset X$ with $\mathrm{diam}(\xi) \leqslant \varepsilon$, and study *this complex* by combinatorial homology.

Thus, the homology group $H_1^\varepsilon(X) = H_1(t_\varepsilon X)$ of the metric space X, at resolution ε, allows us to distinguish, in Fig. (a): two basins at fine resolution ($0 < \varepsilon < 1$); then, one basin for $1 \leqslant \varepsilon < 3$; and finally no relevant basin at coarse resolution ($\varepsilon \geqslant 3$). The finite model X' gives the same results, *as soon as* $\varepsilon \geqslant \rho$; of course, if $\varepsilon < \rho$, i. e. if the resolution of the analysis is finer than the scanner's, we have a totally disconnected object, in accord with a general principle: a "very fine" analysis resolution is too affected by the plotting procedure or by errors, and unreliable. Rather, the whole analysis is of interest, and can be expressed – as above – by some *critical values* (detecting metric characters of the image) together with the value of our invariant within the intervals they produce.

One can note that Shape Theory has also been proposed as a theoretical support for pattern recognition and perception. Geometric shape theory introduced by Borsuk [1] and then developed in various versions (e. g. Mardešić–Segal [13]), aims at approaching a (compact) metric or topological space by polyhedra, and defines its *shape* as a weak homotopy type detected by such approximations; one can find a description and comparison of various such theories in Cordier–Porter's text [3]. In this sense, our proposal is quite different: the metric space in Fig. (b) is already a compact polyhedron, and of course we do not want to view it in this trivial form, as a collection of points. However, the categorical setting in [3] is much more general, and our methods might perhaps be presented in some form of that type.

Now, for a general overview of the present approach, let us recall that a simplicial complex, also called here a *combinatorial space*, is a set X equipped with a family of distinguished finite subsets, the *linked parts*, so that every subset of a linked part is linked and so are all singletons. Part I introduced intrinsic homotopies and homotopy groups for simplicial complexes, based on the standard (integral) line $\mathbf{Z}$ (1.3), the set of integers with linked parts contained in contiguous pairs $\{i, i+1\}$. A path in the simplicial complex X is precisely a map $a : \mathbf{Z} \to X$ which is eventually constant on the left and the right. The set of paths PX inherits the simplicial structure from the hom-object $X^{\mathbf{Z}} = \mathrm{Hom}(\mathbf{Z}, X)$ (the category of simplicial complexes being cartesian closed). Then, *combinatorial homotopies* are defined as maps $\alpha : X \to PY$; this is more general than the classical contiguity relation (based on simplicial maps $a : \{0, 1\} \to X$), in an effective way: for instance, the integral line is contractible with respect to the present notion, while it is not so with respect to the equivalence relation spanned by contiguity (2.3).

Homology of simplicial complexes is a well known tool, intrinsically defined via oriented or ordered simplicial chains ([17], Ch. 4; [10], Ch. 2). Here, in Section 1, we prefer to use *cubical* chains, generated by the cubical set $T_*X = (T_nX)_{n\geq 0}$ of *links*, the maps $a : \mathbf{2}^n \to X$ defined over a power of the *elementary integral interval* $\mathbf{2}$ (the object on two linked points, 0 and 1). Section 2 deals with the interaction with combinatorial homotopies; the present homotopical invariance theorem for homology (2.4) is stronger than the classical one, in as much as our homotopy relation is wider. For metric spaces, the derived *metric combinatorial homology* $H_n^\varepsilon(X) = H_n(t_\varepsilon X)$ satisfies, at a fixed resolution, the axioms of Eilenberg–Steenrod *in an adapted form depending on* ε (2.6); it might be interesting to compare such groups with the Vietoris construction for compact metric spaces [18].

In Section 3, we consider various "comparisons": the *combinatorial* Hurewicz homomorphism, from homotopy to homology of simplicial complexes (3.1); the well-known canonical isomorphism from combinatorial homology to singular homology of the geometric realisation (3.2); and, in a particular case and up to degree 1, a canonical isomorphism from combinatorial homology to the singular homology of the *open-spot dilation* (3.4–5); the latter is a particular "dilation operator" considered in mathematical morphology (cf. [8]).

In Section 4, direct computations of combinatorial homology groups are given, using the Mayer–Vietoris sequence (1.6) and *telescopic homotopies*, a tool introduced in Part I to reduce combinatorial subspaces of $t_\varepsilon\mathbf{R}^n$ to simpler ones (cf. 2.3). In particular, the examples of 4.3–4 should be sufficient to show how the study of the combinatorial homology groups $H_n^\varepsilon(X)$ of a metric subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$, *at variable resolution*, can be used for image analysis. Of course, the planar cases examined here can also be analysed via the fundamental group, as in Part I; but, for 3-dimensional spaces (or complicated planar cases), homology groups are far easier to compute. It is also relevant to note that the geometric realisation of these spaces is huge and of little help (cf. I.1.9). Finally, *critical values* for the family $H^{\varepsilon}_n(X)$ are briefly considered (4.4-5).

Notation. We use the same notation as in Part I. A homotopy α between the maps $f, g : X \to Y$ is written as $\alpha : f \to g : X \to Y$. The usual bracket notation for intervals refers to the *real* or *integral* line, according to context. The letter κ denotes an element of $\mathbf{2} = \{0, 1\}$ or $\mathbf{S}^0 = \{-1, 1\}$, according to convenience; it is always written $-$, $+$ in superscripts. The reference I.m, or I.$m.n$, or I.$m.n.p$, applies to Part I, and precisely to its section m, or subsection $m.n$, or formula (p) in the latter.

1. On the Homology of Simplicial Complexes

After recalling the basic properties of simplicial complexes, we review their homology; this is constructed by means of cubical chains, based on the *elementary integral interval* $\mathbf{2} = \{0, 1\}$.

1.1. **Simplicial complexes.** A *simplicial complex*, also called here a *combinatorial space* (c-space for short), is a set X equipped with a set $!X \subset \mathscr{P}_{\mathrm{f}}X$ of finite subsets of X, called *linked parts*, which contains the empty subset, contains all singletons and is down closed: if ξ is linked, any $\xi' \subset \xi$ is so. A *morphism* of simplicial complexes, or *map*, or *combinatorial mapping* $f : X \to Y$ is a mapping between the underlying sets which preserves the linked sets: if ξ is linked in X, then $f(\xi)$ is linked in Y. (Note that linked parts, here, are meant to express a notion of "proximity", possibly derived from a metric; we shall generally avoid their usual name of *simplices*, as associated with a geometric realisation which is often inadequate for the present applications.)

As easily seen, the category $\mathbf{Cs}$ of combinatorial spaces is complete, cocomplete and cartesian closed (I.1). The linked parts of a cartesian product $X_1 \times X_2$ are the subsets of all products $\xi_1 \times \xi_2$ of linked parts; the *exponential* $\mathrm{Hom}(A, Y) = Y^A$, characterised by the exponential law $\mathbf{Cs}(X \times A, Y) = \mathbf{Cs}(X, Y^A)$, is given by the set of maps $\mathbf{Cs}(A, Y)$ equipped with the structure where a finite subset φ of maps $A \to Y$ is linked whenever, for all ξ linked in A, $\varphi(\xi) = \bigcup_{f\in\varphi} f(\xi)$ is linked in Y.

The forgetful functor $|-| : \mathbf{Cs} \to \mathbf{Set}$ has left adjoint D and right adjoint C: the *discrete* structure DS is the finest (i. e., smallest) on the set S (only the empty subset and the singletons are linked), while the *chaotic* or *codiscrete* structure CS is the coarsest (all finite parts are linked). Also D has a left adjoint, the functor

$$\pi_0 : \mathbf{Cs} \to \mathbf{Set}, \qquad\qquad \pi_0(X) = |X|/\sim, \tag{1}$$

produced by the equivalence relation $\sim$ spanned by $\{x, y\} \in !X$. A non-empty c-space X is said to be *connected* (or *path-connected*) if π_0X is a point; π_0 is called the functor of *connected components* (or *path-components*). Any object is the sum of its connected components.

A *subobject* $X' \prec X$ is a subset equipped with a combinatorial structure making the inclusion $i : X' \to X$ a map; equivalently, $!X' \subset !X$ (this is the usual notion of *simplicial subcomplex* [17]). The subobjects of X form a complete lattice: $\bigcap X_i$ (resp. $\bigcup X_i$) is the intersection (resp. union) of the underlying subsets, with structure $\bigcap !X_i$ (resp. $\bigcup !X_i$). More particularly, a (combinatorial) *subspace*, or *regular subobject* $X' \subset X$, is a subobject with the induced structure: a part of X' is linked iff it is so in X (the initial structure for $i : X' \to X$, i. e. the coarsest one making i a map); any intersection or union of subspaces is a subspace. An equivalence relation R in X produces a quotient X/R, equipped with the finest structure making the projection $X \to X/R$ a map: a subset of the quotient is linked iff it is the image of some linked part of X.

The category $\mathbf{Cs}^*$ of *pointed* combinatorial spaces is also complete and co-complete.

1.2. **Tolerance sets and metric spaces.** **Tol** denotes the category of *tolerance sets*, equipped with a reflexive and symmetric relation $x!x'$; the maps preserve such relations. Equivalently, one can consider a simple reflexive unoriented graph (as more used in combinatorics), or an *adjacency* relation, symmetric and *anti*-reflexive (as used in digital topology, cf. [11, 12]). The forgetful functor $t : \mathbf{Cs} \to \mathbf{Tol}$ takes the c-space X to the tolerance set over $|X|$, with $x!y$ iff $\{x, y\} \in !X$; it has a left adjoint d and a right adjoint $c : \mathbf{Tol} \to \mathbf{Cs}$. We are more interested in the latter: for a tolerance set A, cA is the coarsest combinatorial space over A inducing the relation ! (*a finite subset is linked iff all its pairs are* !*-related*); we shall always *identify a tolerance set A with the combinatorial space cA*. Thus, **Tol** becomes a full reflective subcategory of **Cs**, consisting of those c-spaces where a finite subset is linked iff all its parts of two elements are so. The embedding c preserves all limits and is closed under subobjects; in particular, a product of tolerance sets, in **Cs**, is a tolerance set.

A metric space X has a family of canonical combinatorial structures $t_\varepsilon X$, *at resolution* $\varepsilon \in [0, \infty]$, where a finite subset ξ is linked iff its diameter is $\leqslant \varepsilon$. Each of them is a tolerance set, defined by $x!x'$ iff $d(x, x') \leqslant \varepsilon$. The category **Mtr** of *metric spaces and weak contractions* has thus a family of forgetful functors $t_\varepsilon : \mathbf{Mtr} \to \mathbf{Cs}$, trivial for $\varepsilon = 0$ (giving the discrete structure) and $\varepsilon = \infty$ (the chaotic one). Marginally, and for $\varepsilon > 0$, we also consider the "open" tolerance structure $t^-_\varepsilon X$ defined by $d(x, x') < \varepsilon$; but the family $t_\varepsilon X$ yields finer results (cf. 3.4).

Unless otherwise stated, the real line $\mathbf{R}$ has the standard metric and the combinatorial structure $t_1\mathbf{R}$, with $x!x'$ iff $|x - x'| \leqslant 1$. Beware of the fact that, in **Mtr**, *a product has the l_∞-metric*, given by the least upper bound $d(x, y) = \sup_i d_i(x_i, y_i)$; this precise metric has to be used if we want to "assess" a map into a product by its components: a mapping $f : Z \to \prod X_i$ is a weak contraction if and only if all its components f_i are so. Unless differently stated, the real n-space $\mathbf{R}^n$ will be endowed with the l_∞-metric $\max_i |x_i - y_i|$ and the derived structure $t_1\mathbf{R}^n$: its linked parts are the finite subsets of all elementary real cubes $\prod_i [x_i, x_i + 1]$.

1.3. **Combinatorial line and spheres.** The set of integers $\mathbf{Z}$, equipped *with the combinatorial structure of contiguity*, generated by all contiguous pairs $\{i, i+1\}$, is called the *standard* (integral) *line* and plays a crucial role in our homotopy theory. It is a combinatorial subspace of $\mathbf{R}$ and a tolerance set, with $i!j$ whenever i, j are equal or contiguous; all its powers and subobjects of powers are tolerance sets. An integral interval has the induced structure, unless otherwise stated.

The structure of the *standard* (integral) *n-space* $\mathbf{Z}^n \subset \mathbf{R}^n$ is generated by the "elementary cubes" $\prod_k \{i_k, i_k+1\}$. As a crucial fact, the join and meet operations $\vee, \wedge : \mathbf{Z}^2 \to \mathbf{Z}$ are combinatorial mappings (while sum and product are not), as well as $- : \mathbf{Z} \to \mathbf{Z}$; thus, $\mathbf{Z}$ is an involutive lattice in $\mathbf{Cs}$. The standard elementary interval $\mathbf{2} = [0,1] \in \mathbf{Z}$ is the chaotic c-space on two points, $\mathrm{C}\{0,1\}$. The standard elementary cube $\mathbf{2}^n \subset \mathbf{Z}^n$ is also chaotic, as well as the standard elementary simplex $\mathbf{e}^n = \mathrm{C}\{e_0, \ldots, e_n\} \subset \mathbf{Z}^{n+1}$, consisting of the $n+1$ unit points of the axes (the canonical basis).

The discrete $\mathbf{S}^0 = \{-1, 1\} \subset \mathbf{Z}$ is the *standard* 0-*sphere* (pointed at 1, when viewed in $\mathbf{Cs}^*$). There is no standard circle (I.6.6). But, for every integer $k \geq 3$, there is a *k-point circle*, the quotient $C_k = \mathbf{Z}/\equiv_k = \{[0], [1], \ldots, [k-1]\}$, with respect to congruence modulo k; the structure is generated by the contiguous pairs $\{[i], [i+1]\}$; the base point is $[0]$; the homology groups are the ones of the circle (4.1; while the *c*-spaces similarly obtained for $k = 1, 2$ are chaotic, hence contractible). Such circles are not homotopically equivalent, but related by the following maps, identifying two points

$$p_k : C_{k+1} \to C_k, \qquad p_k([i]) = [i] \qquad (i = 0, \ldots, k), \tag{1}$$

which are weak homotopy equivalences.

More generally, there is no standard n-sphere for $n > 0$. The *simplicial* (or tetrahedral) *n-sphere* $\Delta S^n \prec \mathbf{Z}^{n+2}$ has the same $n+2$ points of $\mathbf{e}^{n+1} = C(e_0, \ldots, e_{n+1}) \subset \mathbf{Z}^{n+2}$, but a subset is linked iff it is not total; the base point is e_0. The *cubical* n-sphere $\Box S^n \prec \mathbf{Z}^{n+1}$ has the same 2^{n+1} points of the cube $\mathbf{2}^{n+1} = C\{0,1\}^{n+1} \subset \mathbf{Z}^{n+1}$, but the linked parts are the sets of vertices contained in some face of the cube, i.e. in some hyperplane $t_i = 0$ or $t_i = 1$; the base point is 0. The *octahedral* n-sphere $\diamond S^n = \{\pm e_0, \ldots, \pm e_n\} \subset \mathbf{Z}^{n+1}$ has $2n+2$ points and the subspace structure: a subset is linked iff it does not contain opposed pairs $\pm e_i$; the base point is e_0. Thus, $\Delta S^0 \cong \Box S^0 \cong \diamond S^0 = \mathbf{S}^0$, $\Delta S^1 \cong C_3$, $\Box S^1 \cong \diamond S^1 \cong C_4$. All these will be seen to be *homological n-spheres* (4.2).

Works in digital topology have considered various tolerance structures on $\mathbf{Z}^n$; among the most used ones are the product structure, induced by the l_∞-metric (called 8-*adjacency* for $\mathbf{Z}^2$, because any point is linked to 8 others), and the structure $t_1(\mathbf{Z}^n, d_1)$ induced by the l_1-metric $\Sigma_i |x_i - y_i|$ (called 4-*adjacency* for $\mathbf{Z}^2$); the fundamental group of regions of the latter has been considered in I.7.4.

1.4. **Links.** The elementary interval $\mathbf{2} = [0,1] = C(0,1)$ is an involutive lattice (with minimum and maximum); its structure is formalised by the following maps in **Cs**: *faces* (∂^-, ∂^+), *degeneracy* (e), *connections* (g^-, g^+) and *symmetries* (the *reversion* r and the *interchange* s)

$$\{*\} \underset{e}{\overset{\partial^\kappa}{\rightleftarrows}} \mathbf{2} \overset{g^\kappa}{\longleftarrow} \mathbf{2}^2 \qquad r : \mathbf{2} \to \mathbf{2}, \qquad s : \mathbf{2}^2 \to \mathbf{2}^2, \tag{1}$$

$$\partial^-(*) = 0, \quad \partial^+(*) = 1, \quad g^-(i,j) = \max(i,j) = i \vee j,$$
$$g^+(i,j) = \min(i,j) = i \wedge j, \quad r(i) = 1 - i, \quad s(i,j) = (j,i).$$

As a consequence, the endofunctor of 1-links or *elementary paths* or *immediate paths* in X

$$T(X) = X^{\mathbf{2}} = \{(x, x') \mid x!x' \in X\}, \quad (x,x')!(y,y') \Leftrightarrow \{x, x', y, y'\} \in !X, \tag{2}$$

has natural transformations denoted by the same symbols and names

$$1 \underset{\mathbf{e}}{\overset{\partial^\kappa}{\leftrightarrows}} T \overset{g^\kappa}{\longrightarrow} T^2 \qquad r : T \to T, \qquad s : T^2 \to T^2, \tag{3}$$

which satisfy the axioms of a cubical comonad with symmetries ([5, 6]; or I.2.4). By cartesian closedness, the power $T^n(X) = X^{\mathbf{2}^n}$ is the functor of *n-links*, or *elementary n-paths* a: $\mathbf{2}^n \to X$ $(n \geq 0)$. Globally, such functors form a *cubical object with symmetries* $T_*(X)$ $(i = 1, \ldots, n;\ \kappa \in \mathbf{2})$

$$\begin{aligned}
&T^n(X) = X^{\mathbf{2}^n}, \\
&\partial_i^\kappa = T^{n-i}\partial^\kappa T^{i-1} : T^n \to T^{n-1}, \\
&\qquad \partial_i^\kappa(a)(t_1, \ldots, t_{n-1}) = a(t_1, \ldots, \kappa, \ldots, t_{n-1}), \\
&e_i = T^{n-i}eT^{i-1} : T^{n-1} \to T^n, \\
&\qquad e_i(a)(t_1, \ldots, t_n) = a(t_1, \ldots, \hat{t}_i, \ldots, t_n), \\
&g_i^\kappa = T^{n-i}g^\kappa T^{i-1} : T^n \to T^{n+1}, \\
&\qquad g_i^\kappa(a)(t_1, \ldots, t_{n+1}) = a(t_1, \ldots, g^\kappa(t_i, t_{i+1}), \ldots, t_{n+1}), \\
&r_i = T^{n-i}rT^{i-1} : T^n \to T^n, \\
&\qquad r_i(a)(t_1, \ldots, t_n) = a(t_1, \ldots, 1 - t_i, \ldots, t_n), \\
&s_i = T^{n-i}sT^{i-1} : T^{n+1} \to T^{n+1}, \\
&\qquad s_i(a)(t_1, \ldots, t_{n+1}) = a(t_1, \ldots, t_{i+1}, t_i, \ldots, t_{n+1}).
\end{aligned} \tag{4}$$

$T_*(X)$ is a subobject of the cubical set with symmetries $S_n|X| = |X|^{2^n}$ similarly obtained in **Set**, the cubical set *of* (elementary) *cubes* of the set $|X|$: the latter coincides with $T_*(C|X|)$. Globally, we have a functor T_* with values in the category of cubical sets with symmetries.

1.5. **Cubical combinatorial homology.** Every cubical set A determines a collection $D_n(A) = \cup_i \mathrm{Im}(e_i : A_{n-1} \to A_n)$ of subsets of *degenerate elements* (with $D_0A = \emptyset$), yielding the *normalised chain complex* $N : \mathbf{Cub} \to C_*\mathbf{Ab}$

$$N_n(A) = F(A_n)/F(D_nA) = F(A_n \setminus D_nA), \quad \partial_n\hat{a} = \Sigma_{i,\kappa}(-1)^{i+\kappa}(\partial_i^\kappa a)\hat{}, \tag{1}$$

where $a \in A_n$ and $\hat{a}$ denotes its class in the quotient.

The *cubical chain complex* of the simplicial complex X is the normalised chain complex $C_*X = NT_*X$, whose elements are the (normalised) *cubical chains* of X

$$\Sigma_i \lambda_i \hat{a}_i \in C_n(X) = F(T^n X)/F(D_n T_* X) \quad (\lambda_i \in \mathbf{Z}, \quad a_i : \mathbf{2}^n \to X); \tag{2}$$

we shall often write the normalised class $\hat{a}$ as a, identifying all degenerate links to 0. We have thus the *homology* of a combinatorial space

$$H_n : \mathbf{Cs} \to \mathbf{Ab}, \qquad H_n(X) = H_n(C_*X) = H_n(NT_*(X)) \quad (n \geqslant 0). \tag{3}$$

Relative homology is defined in the usual way. A *combinatorial pair* (X, A) consists of a subobject $A \prec X$ (1.1): the subset A has a combinatorial structure finer than the restricted one so that the inclusion $i : A \to X$ is a map. We shall write $\mathbf{Cs_2}$ their category: a map $f : (X, A) \to (Y, B)$ comes from a map $f : X \to Y$ whose restriction $A \to B$ is also a map.

The induced map on cubical sets $i_* : T_*A \to T_*X$ is injective as well as $i_* : C_*A \to C_*X$ (a link in A is degenerate in X iff it is already so in A). We obtain the *relative chains* of (X, A) by the usual short exact sequence of chain complexes

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X, A) \longrightarrow 0 \tag{4}$$

the *relative homology* as the homology of the quotient, $H_n(X, A) = H_n(C_*(X, A))$, and the natural exact sequence of the pair (X, A) from the exact homology sequence of (4), with $\Delta_n[c] = [\partial_n c]$

$$\begin{aligned} \cdots \to H_nA \to H_nX \to H_n(X, A) \xrightarrow{\Delta} H_{n-1}A \to \cdots \\ \to H_0A \to H_0X \to H_0(X, A) \to 0. \end{aligned} \tag{5}$$

Plainly, $C_*(X, \emptyset) = C_*(X)$ and $H_n(X, \emptyset) = H_n(X)$. More generally, given a *combinatorial triple* (X, A, B), consisting of subobjects $B \prec A \prec X$, the snake lemma gives a short exact sequence of chain complexes $C_*(A, B) \rightarrowtail C_*(X, B) \twoheadrightarrow C_*(X, A)$ and the exact sequence

$$\begin{aligned} \cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to \cdots \\ \to H_0(X, A) \to 0. \end{aligned} \tag{6}$$

Plainly, the homology of a sum $X = \Sigma X_i$ is a direct sum $H_nX = \oplus H_nX_i$ (and every combinatorial space is the sum of its connected components, 1.1). It is also easy to see that if X is connected (non empty), then $H_0(X) \cong \mathbf{Z}$ (via the augmentation $\widetilde{\partial}_0 : C_0X = F|X| \to \mathbf{Z}$ taking each point $x \in X$ to $1 \in \mathbf{Z}$); thus, for every c-space X, $H_0(X)$ is the free abelian group generated by $\pi_0 X$.

Finally, *combinatorial homology has finite supports*: the group $H_n(X)$ is the inductive limit of the system of groups $H_n(A)$, where A varies in the set of *finite* subspaces $A \subset X$ (ordered by inclusion). In fact, this is already true at the chain level: each chain $c = \Sigma_i \lambda_i \hat{a}_i \in C_n(X)$ belongs to $C_n(A)$, where A is the (finite) union of all images $a_i(\mathbf{2}^n)$.

1.6. **Mayer–Vietoris and excision.** Recall that, given two subobjects $U, V \prec X$, the structure of their union $U \cup V$ is $!U \cup !V$, while the structure of $U \cap V$ is $!U \cap !V$ (1.1). It follows easily that C_* takes subobjects of X to chain subcomplexes of C_*X, preserving joins and meets

$$C_*(U \cup V) = C_*U + C_*V, \qquad C_*(U \cap V) = C_*U \cap C_*V. \tag{1}$$

These facts have two important well-known consequences [17].

(a) *The Mayer–Vietoris sequence.* Let $X = U \cup V$ be a combinatorial space (we shall say that X *is covered by its subobjects* U, V). Then we have an exact sequence

$$\begin{aligned} \cdots \longrightarrow H_n(U \cap V) \xrightarrow{(i_*, j_*)} (H_nU) \oplus (H_nV) \xrightarrow{[u_*, -v_*]} H_n(X) \\ \xrightarrow{\Delta} H_{n-1}(U \cap V) \longrightarrow \cdots \end{aligned} \tag{2}$$

with the obvious meaning of round and square brackets; the maps $u : U \to X$, $v : V \to X$, $i : U \cap V \to X$, $j : U \cap V \to X$ are inclusions, and the *connective* Δ is:

$$\Delta[c] = [\partial_n a], \qquad c = a + b \qquad (a \in N_n(T_*U),\ \ b \in N_n(T_*V)). \tag{3}$$

The sequence is natural, for a map $f : X \to X' = U' \cup V'$, whose restrictions $U \to U'$, $V \to V'$ are maps. (If our subobjects are subspaces, it is sufficient to know that $fU \subset U'$ and $fV \subset V'$).

(b) *Excision.* Let a combinatorial space X be given, with subobjects $B \prec Y, A$. The inclusion map $i : (Y, B) \to (X, A)$ is said to be *excisive* whenever $!Y \backslash !B = !X \backslash !A$ (or equivalently: $Y \cup A = X$, $Y \cap A = B$, *in the lattice of subobjects of* X). Then i induces isomorphisms in homology.

The proof is similar to the topological one, simplified by the fact that here no subdivision is needed. For (a), it is sufficient to apply the algebraic theorem of the exact homology sequence to the following sequence of chain complexes

$$0 \longrightarrow C_*(U \cap V) \xrightarrow{(i_*, j_*)} (C_*U) \oplus (C_*V) \xrightarrow{[u_*, -v_*]} C_*(X) \longrightarrow 0 \tag{4}$$

whose exactness needs one non-trivial verification. Take $a \in C_nU$, $b \in C_nV$ and assume that $u_*(a) = v_*(b)$; therefore, each link really appearing in a (and b) has image in $U \cap V$, and, by hypothesis, is a link there; globally, there is (one) normalised chain $c \in C_n(U \cap V)$ such that $i_*(c) = a$, $i_*(c) = b$.

For (b), the proof reduces to a Noether isomorphism for the chain complexes

$$\begin{aligned} C(Y, B) &= (C_*Y)/(C_*(Y \cap A)) = (C_*Y)/((C_*Y) \cap (C_*A)) \\ &= (C_*Y + C_*A)/(C_*A) = (C_*(Y \cup A))/(C_*A) = C_*(X, A). \end{aligned} \tag{5}$$

2. Combinatorial Homotopy and Homology

We develop here the interaction of combinatorial homology with the intrinsic homotopy theory of simplicial complexes introduced in Part I.

2.1. **Paths.** A *line* of the combinatorial space X is a map $a : \mathbf{Z} \to X$, i.e. a sequence of points of X, written $a(i)$ or a_i, with $a_i!a_{i+1}$ for all $i \in \mathbf{Z}$. The lines of X form the combinatorial space $L(X) = X^{\mathbf{Z}}$; a finite set Λ of lines is linked iff each set $\cup_{a\in\Lambda}\{a_i, a_{i+1}\}$ is linked in X (for $i \in \mathbf{Z}$).

A *path* (I.2.2) in X is a line $a : \mathbf{Z} \to X$ eventually constant on the left and on the right: there is a finite interval $\rho = [\rho^-, \rho^+] \subset \mathbf{Z}$ ($\rho^- \leqslant \rho^+$) such that a is constant on the half-lines $]-\infty, \rho^-]$, $[\rho^+, \infty[$

$$a(i) = a(\rho^\kappa), \quad \text{for} \quad \kappa i \geqslant \kappa\rho^\kappa \quad (\kappa = \pm 1), \tag{1}$$

and determined by its values over ρ; the latter is called an (admissible) *support* of a. The end points of a, or *faces* $\partial^\kappa a = a(\rho^\kappa)$, do not depend on its choice.

The *path object* $PX \subset X^{\mathbf{Z}}$ is the combinatorial subspace of paths. The *path functor* $P : \mathbf{Cs} \to \mathbf{Cs}$ acts on a morphism $f : X \to Y$ as a subfunctor of $(-)^{\mathbf{Z}}$

$$Pf : PX \to PY, \qquad (Pf)(a) = fa; \tag{2}$$

the faces are natural transformations $\partial^\kappa : P \to 1$. The functor P is again a cubical comonad with symmetries (I.2.4), which is relevant for the study of homotopy. Here, we just need "first order properties" of homotopy, and it is sufficient to recall: the *degeneracy* $e : 1 \to P$, taking a point x to the constant path at x, $e(x) : \mathbf{Z} \to X$, and the *reversion* $r : P \to P$, taking the path a to the reversed path $r(a) = -a : i \mapsto a(-i)$.

Two points $x, x' \in X$ are linked by a path in X iff $x \sim x'$, for the equivalence relation generated by the tolerance relation ! of X (1.1): $\pi_0 X = |X|/\sim$ is indeed the set of path-components of X.

2.2. **Homotopies.** Classically (cf. [17, 3.5]), two maps $f, g : X \to Y$ are said to be *contiguous* if, for each ξ linked in X, $f(\xi) \cup g(\xi)$ is linked in Y, i.e. if $f!g$ in the simplicial complex Y^X (1.1); a *contiguity class* of maps is an equivalence class generated by the previous relation. As in Part I, we shall use a wider notion of homotopy, the one deriving from the path functor P.

A *homotopy* of simplicial complexes (I.3.1) $\alpha : f \to g : X \to Y$ is a map $\alpha : X \to PY$ such that $\partial^-\alpha = f$, $\partial^+\alpha = g$. It can also be viewed as a map $\alpha : X \to Y^{\mathbf{Z}}$, or $\alpha : \mathbf{Z} \times X \to Y$, such that every line $\alpha(x)$ admits a support $\rho(x) = [\rho^-(x), \rho^+(x)]$ and

$$\alpha(i, x) = f(x), \quad \text{for } i \leqslant \rho^-(x), \qquad \alpha(i, x) = g(x), \quad \text{for } i \geqslant \rho^+(x). \tag{1}$$

Our homotopy is said to be *bounded* if it admits a constant support $\rho(x) = \rho$; and *bounded on connected components* if this holds on every connected component of X. Similarly, in a *left bounded* (resp. *positive, immediate*) homotopy α, all paths $\alpha(x)$ admit a support $[\rho^-, \rho^+(x)]$ (resp. $[0, \rho^+(x)]$, $[0, 1]$). Immediate homotopies belong to the set $\mathbf{Cs}(X, Y^{\mathbf{2}}) = \mathbf{Cs}(\mathbf{2} \times X, Y)$: they are represented by the functor $TY = Y^{\mathbf{2}}$ of elementary paths and corepresented by the *elementary cylinder* $\mathbf{2} \times X$. Thus, *an immediate homotopy amounts to a pair of contiguous maps* $f!g$, and two maps *are bounded homotopic iff they are in the same contiguity class.* Plainly, if X is finite, each homotopy defined on it is bounded.

On the other hand, general homotopies are represented by the path functor P (as maps $X \to PY$), but cannot be corepresented (as maps $IX \to Y$, for some object IX): the path functor has no left adjoint and there is no cylinder functor (in fact, P preserves finite limits, but does not preserve infinite products, I.2.4). The category **Cs** will always be equipped with general homotopies and the operations produced by the path functor, its degeneracy and reversion:

(a) *whisker composition* of maps and homotopies (for $u : X' \to X$, $v : Y \to Y'$):

$$v \circ \alpha \circ u = vgu \qquad (v \circ \alpha \circ u = Pv.\alpha.u : X' \to PY'),$$

(b) *trivial homotopies*:

$$0_f : f \to f \qquad (0_f = ef : X \to PY),$$

(c) *reversion*:

$$-\alpha : g \to f \qquad (-\alpha = r\alpha : X \to PY).$$

Therefore, the *homotopy relation* $f \simeq g$, defined by the existence of a homotopy $f \to g$, is a reflexive and symmetric relation, "weakly' consistent with composition ($f \simeq g$ implies $vfu \simeq vgu$), but *presumably* not transitive (related congruences are discussed in I.3.2). Two objects are *homotopy equivalent* if they are linked by a *finite sequence* of homotopy equivalences. The object X is *contractible* if it is homotopy equivalent to a point.

A *deformation retract* S of a combinatorial space X is a subspace whose inclusion u has a retraction p, with $up \simeq idX$

$$u : S \rightleftarrows X : p, \qquad pu = 1, \quad \alpha : up \to 1_X, \tag{2}$$

and we speak of a *positive* (resp. *bounded, immediate*) deformation retract when the homotopy α can be so chosen. Thus, an immediate deformation retract $u : S \subset X$ has a retraction p with $(idX)!(up)$, i.e. $\xi \cup up(\xi)$ is linked for all $\xi \in !X$. An object is positively (resp. immediately) contractible if it admits a positive (resp. immediate) deformation retract reduced to a point; thus, X is immediately contractible to its point x_0 iff the latter can be added to any linked part (or is linked to any point, in a tolerance set). A non-empty chaotic space is immediately contractible to each of its points.

2.3. **Telescopic homotopies.** The simplicial complexes $\mathbf{Z}^n$ and $\mathbf{R}^n$ are contractible, but *not* bounded contractible. This can be shown by means of "telescopic" homotopies based on lattice operations, quite different from the usual topological contraction $\varphi : [0, 1] \times \mathbf{R}^n = \mathbf{R}^n$, $\varphi(t, x) = t.x$. Such homotopies can also be adapted to various regions of $\mathbf{Z}^n$ and $\mathbf{R}^n$, forming a tool of immediate use in computing homotopy or homology groups of planar or 3-dimensional images. We give here some prime examples, referring the reader to Part I (Section 3) for a more detailed study.

First, if E denotes either $\mathbf{Z}$ or $\mathbf{R}$ (with the t_1-structure) one can consider the homotopy

$$\begin{aligned} &\alpha : 0 \to id : E \to E, \\ &\alpha(i, x) = 0 \vee (i \wedge x), \qquad \alpha(i, -x) = -\alpha(i, x) \quad (x \geqslant 0), \end{aligned} \tag{1}$$

whose general path $\alpha(-, x)$ has a positive support, namely $[0, \rho^+(x)]$, with $|x| \leqslant \rho^+(x) < |x| + 1$.

We call α a telescopic homotopy because it can be pictured as a collection of "telescopic arms" which stretch down, in the diagram below (for $E = \mathbf{Z}$), at increasing $i \geqslant 0$; the arm at x stabilities at depth $\rho^+(x) = |x|$

$$\begin{array}{ccccccccccc} \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & (i=0) \\ \dots & -1 & -1 & -1 & 0 & 1 & 1 & 1 & \dots & & (i=1) \\ \dots & -2 & -2 & -1 & 0 & 1 & 2 & 2 & \dots & & (i=2) \\ \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots & & (i=3) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\ \hline \dots & -3 & \text{-2} & \text{-1} & 0 & 1 & 2 & 3 & \dots & & (x \in \mathbf{Z}) \end{array} \tag{2}$$

Note that there is *no* positive homotopy in the opposite direction $id \to 0$. In fact, in the integral case, any positite homotopy $\beta : id \to f : \mathbf{Z} \to \mathbf{Z}$ is necessarily trivial, since the map $g = \beta(1, -)$ adjacent to id

$$\begin{array}{cccccccc} \dots & -2 & -1 & 0 & 1 & 2 & \dots & (i=0) \\ \dots & g(-2) & g(-1) & g(0) & g(1) & g(2) & \dots & (i=1) \end{array} \tag{3}$$

must coincide with the former ($g(j)$ is linked with $j-1, j, j+1$, whence $g(j) = j$), and so on. It is easy to see that the same holds in the real case. It follows that *the only bounded deformation retract of the integral or real line is the line itself.*

Now, for the n-dimensional space E^n, a *telescopic homotopy* will be any product of 1-dimensional telescopic homotopies (centred at any point) and trivial homotopies. For instance, for $n = 2$, consider $\beta = \alpha \times \alpha$ (*centred at the origin*) and $\gamma = 0_{id} \times \alpha$ (*centred at the horizontal axis*)

$$\beta : 0 \to id : E^2 \to E^2, \qquad \beta(i, x_1, x_2) = (\alpha(i, x_1), \alpha(i, x_2)), \tag{4}$$

$$\gamma : p_1 \to id : E^2 \to E^2, \qquad \gamma(i, x_1, x_2) = (x_1, \alpha(i, x_2)), \tag{5}$$

which show, respectively, that the origin and the horizontal axis are homotopy retracts of E^2.

Less trivially, to prove that the simplicial complex $t_\varepsilon X \subset t_\varepsilon \mathbf{R}^2$ described in Fig. (a) of the Introduction is contractible for $\varepsilon \geqslant 3$, we need a *generalised telescopic homotopy* centred at the horizontal axis, with "variable vertical jumps" 1, 3, 1 (all $\leqslant \varepsilon$ and adjusted to jump over the two holes); for a precise definition, see I.3.6.

2.4. **Invariance Theorem.** *The homology functors $H_n : \mathbf{Cs} \to \mathbf{Ab}$ (1.5) are homotopy invariant: if $f \simeq g$, then $H_n(f) = H_n(g)$. Similarly for relative homology.*

Proof. Let us begin from an *immediate* homotopy $\alpha : f^- \to f^+$, represented by a map $\alpha : \mathbf{2} \times X \to Y$. As for *cubical* singular homology [14], yields a homotopy of cubical sets

$$\begin{aligned} &\beta : T_*f^- \to T_*f^+ : T_*X \to T_*Y, \\ &\beta_n : T^nX \to T^{n+1}Y, \qquad \beta_n(a : \mathbf{2}^n \to X) = \alpha \circ (\mathbf{2} \times a) : \mathbf{2}^{n+1} \to Y; \end{aligned} \tag{1}$$

in fact, for $a \in T^nX$, $b \in T^{n-1}X$, $1 \leqslant i \leqslant n$, and $\kappa = 0, 1$ ($-, +$ in superscripts)

$$\begin{aligned} (\partial^\kappa_{i+1}\beta_n a)(t_1, \dots, t_n) &= \alpha \circ (t_1, a(t_2, \dots, \kappa, \dots, t_n)) = (\beta_{n-1}\partial^\kappa_i a)(t_1, \dots, t_n), \\ (\partial^\kappa_1 \beta_n a)(t_1, \dots, t_n) &= \alpha \circ (\kappa, a(t_1, \dots, t_n)) = (f^\kappa a)(t_1, \dots, t_n), \\ (\beta_n e_i b)(t_1, \dots, t_{n+1}) &= \alpha \circ (t_1, b(t_2, \dots, \hat{t}_{i+1}, \dots, t_{n+1})) = \\ &= (e_{i+1}\beta_{n-1}b)(t_1, \dots, t_{n+1}). \end{aligned} \tag{2}$$

Then β produces a homotopy of the associated normalised chain complexes

$$\gamma_n : C_nX \to C_{n+1}Y, \qquad \gamma_n\left(\Sigma_i \lambda_i \hat{a}_i\right) = \Sigma_i \lambda_i (\beta_n(a_i))\hat{}. \tag{3}$$

Now, a *bounded* homotopy is a finite concatenation of immediate ones, and produces again a homotopy of chain complexes. Finally, for a general homotopy, it suffices to recall that combinatorial homology has finite supports (1.5), and that each homotopy on a finite domain is bounded (2.2). □

2.5. **Reduced homology and the elementary suspension.** Simplicial complexes have an *elementary suspension* ΣX, which plays the role of a *homological suspension*, i.e. a functor $\Sigma : \mathbf{Cs} \to \mathbf{Cs}$ with a natural isomorphism $h_n : \widetilde{H}_n(X) = \widetilde{H}_{n+1}(\Sigma X)$, in reduced homology. (Its topological analogue is McCord's *non-Hausdorff suspension*, [15, Section 8].)

The *augmented* cubical chain complex $\widetilde{C}_*X$ has a component $C_{-1}(X) = \mathbf{Z}$, with *augmentation* $\widetilde{\partial}_0 : C_0X = F|X| \to \mathbf{Z}$ taking each point $x \in X$ to $1 \in \mathbf{Z}$. Its homology is the *reduced* homology of X

$$\widetilde{H}_n : \mathbf{Cs} \to \mathbf{Ab}, \qquad \widetilde{H}_n(X) = H_n(\widetilde{C}_*X) \qquad (n \geqslant -1). \tag{1}$$

Let $Z_* = \widetilde{C}_*(\emptyset)$ be the chain complex reduced to one component $\mathbf{Z}$, in degree -1. The obvious short exact sequence $Z_* \rightarrowtail \widetilde{C}_*X \twoheadrightarrow C_*X$ produces an exact homology sequence, which reduces to a four-term exact sequence in low dimension and a sequence of identities for $n \geqslant 1$

$$0 \to \widetilde{H}_0(X) \to H_0(X) \to \mathbf{Z} \to \widetilde{H}_{-1}(X) \to 0, \qquad \widetilde{H}_n(X) = H_n(X). \tag{2}$$

Thus, also $\widetilde{H}_0(X)$ is free abelian. If X is empty, all homology and reduced homology groups vanish, except for $\widetilde{H}_{-1}(\emptyset) = \mathbf{Z}$. In the contrary, $\widetilde{\partial}_0 : F|X| \to \mathbf{Z}$ is surjective and $\widetilde{H}_{-1}(X) = 0$: the sequence in (2) splits as $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbf{Z}$. Reduced homology also has a Mayer–Vietoris sequence, which ends in degree -1, proved in the same way; but note that it does not preserve sums.

Now, ΣX will be the *c*-space $|X| \cup \{\pm\infty\}$, with two *new* points, and new linked subsets of type $\xi \cup \{-\infty\}$, $\xi \cup \{+\infty\}$ for $\xi \in !X$. Thus, the octahedral spheres are obtained as iterated suspensions $\diamond S^n = \Sigma^n \mathbf{S}^0 = \Sigma^{n+1}(\emptyset)$ (in a particular instance of ΣX, the new points are written as convenient).

ΣX is covered by two subspaces, the *lower* and *upper elementary cones*

$$\begin{aligned}
&C^-X = X \cup \{-\infty\}, \qquad u^+ : X \subset C^-X, \qquad \partial^- : \{*\} \to C^-X, \;\; * \mapsto -\infty,\\
&C^+X = X \cup \{+\infty\}, \qquad u^- : X \subset C^+X, \qquad \partial^+ : \{*\} \to C^+X, \;\; * \mapsto +\infty,
\end{aligned} \tag{3}$$

which are immediately contractible to their vertex κ^∞ (2.2). Since their intersection is X, the Mayer–Vietoris sequence of ΣX in reduced homology gives a natural isomorphism (for $n \geqslant -1$)

$$\Delta_{n+1} : \widetilde{H}_{n+1}(\Sigma X) = \widetilde{H}_n(X), \qquad \Delta_{n+1}[c] = [\partial_{n+1}a], \tag{4}$$

where $c = a + b$, $a \in C_{n+1}(C^-X)$ and $b \in C_{n+1}(C^+X)$.

The isomorphism $s_n = (-1)^{n+1}(\Delta_{n+1})^{-1} : \widetilde{H}_n(X) = \widetilde{H}_{n+1}(\Sigma X)$ has a more canonical description: it is induced by the following chain map $\bar{s}_*$ of degree 1

$$\begin{aligned}
&\bar{s}_{-1} : \widetilde{C}_{-1}X = \mathbf{Z} \to C_0(\Sigma X), \qquad 1 \mapsto (-\infty) - (+\infty),\\
&\bar{s}_n : C_nX \to C_{n+1}(\Sigma X), \qquad (a : \mathbf{2}^n \to X) \mapsto a^- - a^+ \quad (n \geqslant 0),\\
&a^\kappa : \mathbf{2}^n \times \mathbf{2} \to C^\kappa X \subset \Sigma X, \qquad a^\kappa(t, 0) = a(t), \quad a^\kappa(t, 1) = \kappa^\infty,\\
&\partial(a^- - a^+) = (\Sigma_{i,\kappa}(-1)^{i+\kappa}\partial_i^\kappa(a^- - a^+)) + (-1)^{n+1}(a - a)\\
&\qquad\qquad = (\partial a)^- - (\partial a)^+ \quad (i \leqslant n; \;\; \kappa = \pm).
\end{aligned} \tag{5}$$

Since we already know that Δ_{n+1} is iso, it is sufficient to check that $\Delta_{n+1} \cdot s_n = (-1)^{n+1} id$ on each homology class $[\Sigma\lambda_i a_i]$ $(\partial(\Sigma\lambda_i a_i) = 0)$

$$\begin{aligned}
\Delta_{n+1} \cdot s_n[\Sigma\lambda_i a_i] &= \Delta_{n+1}[\Sigma\lambda_i(a_i^- - a_i^+)] = [\Sigma\lambda_i\partial(a_i^-)]\\
&= [(\Sigma\lambda_i\partial a_i)^-] + (-1)^{n+1}[\Sigma\lambda_i a_i] = (-1)^{n+1}[\Sigma\lambda_i a_i].
\end{aligned} \tag{6}$$

We end with some formal remarks. The elementary lower cone C^-X comes with an *immediate* homotopy $\delta^- : \partial^- p \to u^+ : X \to C^-X$ from the vertex to the basis, which is universal for all *immediate homotopies* $f \to g : X \to Y$ *where f is constant* (factors through the point); symmetrically for C^+X. The elementary suspension comes with a homotopy *of support* $[-1, 1]$)

$$\begin{aligned}
\sigma : \partial^- \to \partial^+ : X \to \Sigma X, \qquad &\sigma(-1, x) = -\infty,\\
\sigma(0, x) = x, \qquad &\sigma(1, x) = +\infty,
\end{aligned} \tag{8}$$

which is universal for all *homotopies of support* $[-1, 1]$*, between constant maps* defined on X.

But there is no *standard* cone and no *standard* suspension, representing homotopies of the previous kinds *without* restrictions on supports. In fact, and loosely speaking, the cone or the suspension of the singleton would produce a standard interval $\mathbf{I}$ and a cylinder functor $\mathbf{I} \times (-)$, which we already know not to exist. (For a precise proof, one can adapt the argument of I.6.5 showing that the pointed $\mathbf{S}^0$ has no suspension in $\mathbf{Cs}^*$, even up to homotopy: there is no pointed simplicial complex $\mathbf{S}^1$ allowing one to represent $\pi_1(X) = [\mathbf{S}^0, \Omega X]$ as $[\mathbf{S}^1, X]$.)

2.6. **Metric combinatorial homology.** Metric spaces inherit a family of homology theories (*metric combinatorial homology at resolution* ε) via the functors $t_\varepsilon : \mathbf{Mtr_2} \to \mathbf{Cs_2}$

$$H_n^\varepsilon : \mathbf{Mtr_2} \to \mathbf{Ab}, \quad H_n^\varepsilon(X, A) = H_n(t_\varepsilon X, t_\varepsilon A) \quad (0 < \varepsilon < \infty), \tag{1}$$

on the obvious category $\mathbf{Mtr_2}$ of pairs (X, A) of metric spaces ($A \subset X$ with the induced metric).

The axioms of Eilenberg–Steenrod are satisfied *in an adapted form depending on* ε (for excision):

- the *functoriality* and *dimension* axioms hold trivially;
- *exactness* and *naturality* for the homology sequence of a pair (X, A) come from the similar properties (1.5) for the combinatorial pair $(t_\varepsilon X, t_\varepsilon A)$;
- *homotopy invariance* holds for homotopies $\alpha : ([0, 1] \times X, [0, 1] \times A) \to (Y, B)$ in $\mathbf{Mtr_2}$; in fact, choose a finite partition $0 = t_0 < \cdots < t_k = 1$ of the standard interval such that $t_i - t_{i-1} \leqslant \varepsilon$; then α produces a combinatorial homotopy (actually in $\mathbf{Tol_2}$), with *bounded* support $[0, k] \subset \mathbf{Z}$
$$\beta : (\mathbf{Z} \times t_\varepsilon X, \mathbf{Z} \times t_\varepsilon A) \to (t_\varepsilon Y, t_\varepsilon B), \qquad \beta(i, x) = \alpha(t_{(i \vee 0) \wedge k}, x), \tag{2}$$
since $|i - i'| \leqslant 1$ and $d(x, x') \leqslant \varepsilon$ implies $d(\beta(i, x), \beta(i', x')) \leqslant \varepsilon$;
- finally, the *excision* isomorphism $H_n^\varepsilon(X \setminus U, A \setminus U) \to H_n^\varepsilon(X, A)$ holds for metric subspaces $U \subset A \subset X$, *provided that*: if $x \in U$ and $d(x, x') \leqslant \varepsilon$, then $x' \in A$; in fact, under this condition, the combinatorial inclusion map $(t_\varepsilon(X \setminus U), t_\varepsilon(A \setminus U)) = (t_\varepsilon X, t_\varepsilon A)$ is excisive (1.6b): if ξ is linked in X either it is contained in $X \setminus U$, or there is some $x \in \xi \cap U$; but then $\xi \subset A$.

3. Comparison Homomorphisms

We deal now with the combinatorial Hurewicz homomorphism from homotopy to homology (3.1) and the canonical isomorphism from combinatorial homology to the singular homology of a realisation, either the well-known geometric one (3.2), or the "open-spot dilation" (in a particular case, 3.4-5).

3.1. **The combinatorial Hurewicz comparison.** Let X be a pointed combinatorial space. There is a natural Hurewicz homomorphism, for $n \geqslant 1$

$$\begin{aligned} &h_n : \pi_n(X) \to H_n(X), \qquad [a] \mapsto [\Sigma_i a_i], \\ &a_i : \mathbf{2}^n \to X, \qquad a_i(j) = a(i + j) \qquad (i \in \mathbf{Z}^n), \end{aligned} \tag{1}$$

where $a : \mathbf{Z}^n \to X$ is a net *with trivial faces*, all $a_i : \mathbf{2}^n \to X$ are links (degenerate except for finitely many indices i, belonging to the support of a), and the normalised chain $\Sigma_i a_i$ *is a cycle.*

Similarly, for a combinatorial space X,

$$h_0 : \pi_0(X) \to H_0(X), \qquad [a] \mapsto [a], \tag{2}$$

is a mapping of sets, and actually the canonical basis of the free abelian group $H_0(X)$.

It should not be difficult to prove directly the Hurewicz theorem for combinatorial homotopy and homology, adapting the classical proof of the topological case; but we shall deduce it from the latter (in 3.3), via the geometric comparisons in homotopy (I.6.6) and homology (below).

3.2. **The geometric comparison in homology.** Let X be a combinatorial space, $\mathscr{R}X$ its geometric realisation. As proved in [17], there is a natural isomorphism (*geometric comparison*)

$$\Phi_n : H_n(X) \to H_n(\mathscr{R}X), \qquad \Phi_n[\Sigma_i k_i a_i] = [\Sigma_i k_i \hat{a}_i], \tag{1}$$

from combinatorial homology to (singular) homology, which we adapt now to *cubical* chains.

The geometric realisation $\mathscr{R}X$ is the set of all mappings $\lambda : X \to [0,1]$ with linked support $\mathrm{supp}(\lambda)$, such that $\Sigma_x \lambda(x) = 1$. X is embedded in $\mathscr{R}X$, identifying $x \in X$ with its characteristic function. A point of $\mathscr{R}X$ can be viewed as a convex combination $\lambda = \Sigma_i \lambda_i x_i$ of a linked family of X; each (non-empty) linked subset ξ having $p+1$ points spans a *simplex*

$$\Delta(\xi) = \{\lambda \in \mathscr{R}X \mid \mathrm{supp}(\lambda) \subset \xi\}. \tag{2}$$

All $\Delta(\xi)$ are equipped with the euclidean topology (via a bijective correspondence with the standard simplex Δ^p, determined by any linear order of ξ), and $\mathscr{R}X$ with the direct limit topology defined by such subsets: a subset of $\mathscr{R}X$ is open (or closed) if and only if it is so in every $\Delta(\xi)$. (This is generally known in the literature as the *weak* or *coherent* topology, as distinct from the *metric* topology, cf. [17, p. 111]). Each $\Delta(\xi)$ is closed in $\mathscr{R}X$. The *open simplex* $\Delta^\circ(\xi) = \{\lambda \in \mathscr{R}X \mid \mathrm{supp}(\lambda) = \xi\}$ is open *in* $\Delta(\xi)$; $\mathscr{R}X$ is the disjoint union of its open simplices.

The image of a link $a : \mathbf{2}^n \to X$ is a linked subset ξ of X, contained in the convex space $\Delta(\xi) \subset \mathscr{R}X$, and we can consider the *multiaffine* extension $\hat{a} : [0,1]^n \to \mathscr{R}X$ of a (separately affine in each variable). This transformation, plainly consistent with faces and degeneracies, defines a natural homomorphism (1), which is proved in [17] to be iso.

If X is pointed or $n = 0$, this comparison is coherent with the similar isomorphism $\Phi_n : \pi_n(X) \to \pi_n(\mathscr{R}X)$ constructed in I.6.6: we have a commutative diagram

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\Phi_n} & \pi_n(\mathscr{R}X) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_n} \\ H_n(X) & \xrightarrow{\Phi_n} & H_n(\mathscr{R}X) \end{array} \tag{3}$$

with the Hurewicz maps h_n, the combinatorial one at the left (3.1) and the usual, topological one, at the right. This diagram is natural, for maps $f : X \to Y$ in $\mathbf{Cs}^*$ ($\mathbf{Cs}$ for $n = 0$).

3.3. **Corollary.** (a) (Hurewicz) *Let X be a pointed combinatorial space. If X is n-connected ($\pi_k(X)$ trivial for $0 \leqslant k \leqslant n$), then $H_k(X) = 0$ for $0 < k \leqslant n$ and $h_{n+1} : \pi_{n+1}(X) \to H_{n+1}(X)$ is iso (or, for $n = 0$, induces an iso $ab(\pi_1(X)) \to H_1(X)$ from the abelianised group).*

(b) (Whitehead) *If $f : X \to Y$ is a map of connected pointed simplicial complexes and $\pi_k(f)$ is an isomorphism for $1 \leqslant k \leqslant n$, the same holds for $H_k(f)$.*

(c) (Whitehead) *If $f : X \to Y$ is a map of non empty simplicial complexes and for all $x \in X$, $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, f_x)$ is an iso for $0 \leqslant k \leqslant n$, the same holds for $H_k(f)$.*

Proof. (a) Apply the usual (topological) Hurewicz theorem to the commutative diagram 3.2.3.

(b) Again in the diagram 3.2.3 (natural on f), apply a theorem of J.H.C. Whitehead [9, p. 167] saying that, if a map $g : S \to T$ between path-connected pointed spaces induces an iso on the k-homotopy groups, for $1 \leqslant k \leqslant n$, the same holds for singular homology. (There is also a modified version, where one assumes that $\pi_k(f)$ is an iso for $1 \leqslant k < n$ and epi for $k = n$, and concludes the same for $H_k(f)$. Both facts are easily deduced from the exact homotopy and homology sequences of the pair (M_f, X) based on the mapping cylinder of f, linked by Hurewicz homomorphisms.)

(c) Follows from the previous point, via direct sum decomposition over path-components. □

3.4. **The open-spot dilation.** Let X be a metric space and $\varepsilon > 0$. First, we want to compare the tolerance structures $t_\varepsilon X$ and $t_\varepsilon^- X$, where two points are linked iff their distance is $\leqslant \varepsilon$ or $< \varepsilon$, respectively (1.2). As is the case for homotopy groups, the homology groups of the first family determine the ones of the second (but not vice versa, cf. I.7.3)

$$H_n(t_\varepsilon^- X) = \mathrm{colim}_{\eta<\varepsilon}\, H_n(t_\eta X), \tag{1}$$

as a trivial consequence of the finiteness of links: a map $a : \mathbf{2}^n \to t_\varepsilon^- X$ is also a map $\mathbf{2}^n \to t_\eta X$, for $\eta = \mathrm{diam}(a(\mathbf{2}^n)) < \varepsilon$.

Now, let X be a metric subspace of a normed vector space E and $\varepsilon > 0$. Recall (from I.7.4) that the *open-spot realisation* $D_\varepsilon^-(X)$ of X in E (a *dilation operator* considered in mathematical morphology, cf. [8]) is the subspace of E formed by the union of all open d-discs centred at points of X, of radius $\varepsilon/2$ (pointed at the base-point of X, if X is pointed)

$$D_\varepsilon^-(X) = D_\varepsilon^-(X, d) = \{x' \in E \mid d(x, x') < \varepsilon/2 \text{ for some } x \in X\} \supset X. \tag{2}$$

Say that X is t_ε^--*closed* (resp. t_ε-*closed*) in E if $D_\varepsilon^-(X)$ contains the convex envelope of all linked subsets of $t_\varepsilon^- X$ (resp. $t_\varepsilon X$), so that there is a continuous mapping

$$f_\varepsilon^- : \mathscr{R}(t_\varepsilon^- X) \to D_\varepsilon^-(X) \qquad (\text{resp. } f_\varepsilon : \mathscr{R}(t_\varepsilon X) \to D_\varepsilon^-(X)), \tag{3}$$

extending the identity of X and affine on each simplex $\Delta(\xi)$ of the domain. Note that, if X is t_ε-closed, then it is also t_ε^--closed and $f_\varepsilon^- = (\mathscr{R}(t_\varepsilon^- X) \to \mathscr{R}(t_\varepsilon X) \to$

$D_\varepsilon^-(X))$ factors as the map induced by the subobject-inclusion $t_\varepsilon^- X \prec t_\varepsilon X$ followed by f_ε.

3.5. **Theorem (Open-spot comparison isomorphisms in homology).**
(a) *If X is t_ε^--closed in E, there is a canonical isomorphism $\Psi_n^- : H_n(t_\varepsilon^- X) \to H_n(D_\varepsilon^- X)$ between the combinatorial and topological homology groups, which is the composite of the geometric realisation isomorphism Φ_n* (3.2) *with an isomorphism induced by the "affine" map f_ε^-*

$$\Phi_n^- = H_n(f_\varepsilon^-) \cdot \Phi_n = (H_n(t_\varepsilon^- X) \to H_n(\mathscr{R}(t_\varepsilon^- X)) \to H_n(D_\varepsilon^- X)) \quad (n \leqslant 1). \quad (1)$$

(b) *If X is also t_ε-closed, there is a canonical isomorphism $\Psi_n : H_n(t_\varepsilon X) \to H_n(D_\varepsilon^- X)$ consisting of the lower row of the following commutative diagram of isomorphisms*

$$\begin{array}{ccccc} H_n(t_\varepsilon^- X) & \xrightarrow{\Phi_n} & H_n(\mathscr{R}(t_\varepsilon^- X)) & \xrightarrow{H_n(f_\varepsilon^-)} & H_n(D_\varepsilon^- X) \\ \downarrow & & \downarrow & & \| \\ H_n(t_\varepsilon X) & \xrightarrow{\Phi_n} & H_n(\mathscr{R}(t_\varepsilon X)) & \xrightarrow{H_n(f_\varepsilon)} & H_n(D_\varepsilon^- X) \end{array} \quad (1)$$

(*the two vertical arrows being induced by the inclusion $t_\varepsilon^- X \prec t_\varepsilon X$*).

Proof. Let X be t_ε^--closed (resp. t_ε-closed) in E. As proved in theorem I.7.5, the homotopy homomorphism $\pi_n(f_\varepsilon^-, x)$ (resp. $\pi_n(f_\varepsilon, x)$) is an isomorphism, for all $x \in X$ and $n \leqslant 1$. From the Whitehead theorem (the classical, topological one, in form 3.3c) it follows that $H_n(f_\varepsilon^-)$ (resp. $H_n(f_\varepsilon)$) is iso, for $n \leqslant 1$.

Combining this with the geometric comparison Φ_n (3.2), we have the thesis. □

4. Computation of Combinatorial Homology

In this section some computations of homology are given, either directly via the intrinsic Mayer–Vietoris (M-V) sequence, or via the geometric realisation; in some cases, one might similarly use the open-spot dilation (3.4-5). The results in 4.3-4 are used as a support for image analysis.

4.1. **Homological circles.** All the circles C_k ($k \geqslant 3$; 1.3) are homological 1-spheres, since their geometric realisation is the circle. But the homology is also easily computed by the M-V sequence (1.6).

(a) If $k > 3$, C_k is covered by two subspaces U, V which are contractible (being isomorphic to integral intervals) and whose intersection is the discrete object on two points, e.g.

$$U = \{[0], [1], [2]\}, \qquad V = \{[2], \ldots, [k-1], [0]\}, \quad (1)$$

so that, taking into account the homology of a sum and the invariance theorem (2.4), the computation on the M-V sequence proceeds precisely as for the topological circle.

(b) For C_3, take the *subobject* $U = \{[0], [1], [2]\} \prec C_3$ with the tolerance structure $[0]![1]![2]$, and $V = \{[2], [0]\} \subset C_3$, with the induced (chaotic) structure.

Then, U and V are contractible and $U \cap V$ acquires the discrete structure. One concludes as above.

The naturality of the M-V sequence proves also that all maps $p_k : C_{k+1} \to C_k$ induce isomorphism in homology (by the Five Lemma); this suggests that it might be useful to realise a "standard circle" as a pro-object (cf. I.6.5).

4.2. **Homological spheres.** The n-sphere $\Delta S^n \prec \mathbf{e}^{n+1}$ (1.3) has the same homology as the topological n-sphere, since its geometric realisation is homeomorphic to the standard (topological) sphere $\mathbf{S}^n$. Our result can also be proved by induction, as in the topological case, covering ΔS^n ($n \geqslant 1$) with the following subobjects (a face of the standard simplex and the union of all the others)

$$U = \mathbf{e}^n = C\{e_0, \ldots, e_n\} \subset \Delta S^n, \qquad V \prec \Delta S^n \prec \mathbf{e}^{n+1}, \tag{1}$$

where V has the same underlying set $\{e_0, \ldots, e_{n+1}\}$ as ΔS^n, and all its linked sets except U. Then, $U \cap V = \Delta S^{n-1}$.

For the cubical n-sphere $\Box S^n \prec \mathbf{2}^{n+1}$ (whose geometric realisation is a space of dimension $2^n - 1$, having the homotopy type of the n-sphere), the direct proof is similar: cover $\Box S^n$ ($n \geqslant 1$) with the following subobjects (again, a face of the standard cube and the union of all the others)

$$U = \mathbf{2}^n = C\{0,1\}^n \subset \Box S^n, \qquad V \prec \Box S^n \prec \mathbf{2}^{n+1}, \tag{2}$$

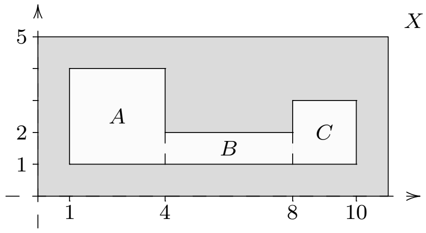
where V has the same underlying set $\{0,1\}^n$ as $\Box S^n$, and all its linked sets except U. Again, $U \cap V = \Box S^{n-1}$.

Finally, the *octahedral* n-sphere $\diamond S^n = \{\pm e_0, \ldots, \pm e_n\} \subset \mathbf{Z}^{n+1}$ has again for geometric realisation the n-sphere. But it suffices to recall that $\diamond S^n = \Sigma^n \mathbf{S}^0$ and apply the suspension isomorphisms (2.5); or also, to apply M-V to the following subspaces U, V, which are immediately contractible to $\pm e_n$

$$U = \diamond S^{n-1} \cup \{-e_n\}, \qquad V = \diamond S^{n-1} \cup \{e_n\}, \qquad U \cap V = \diamond S^{n-1}. \tag{3}$$

4.3. **Metric combinatorial homology and image analysis.** The homology at variable resolution $H_n^\varepsilon(X) = H_n(t_\varepsilon X)$ of a metric subspace $X \subset (\mathbf{R}^n, d_\infty)$ can often be computed directly (as its fundamental group in Part I), using the telescopic retracts introduced in I.3 and the M-V sequence (instead of the van Kampen theorem). This can be of interest within image analysis, as showed by the examples below. Of course, $H_1^\varepsilon(X)$ yields less fine results than the fundamental group (by Hurewicz); but, in higher dimension, homology is generally easier to compute.

Let us consider, as in I.7.1.2, the closed region $X = T \setminus (A \cup B \cup C)$ of the real plane (with the l_∞-metric), endowed with the t_ε-structure (> 0)



$$\begin{aligned} T &= [0,11] \times [0,5] \\ A &=]1,4[\times]1,4[\\ B &= [4,8]\times]1,2[\\ C &=]8,10[\times]1,3[\end{aligned} \tag{1}$$

Then, the same coverings and telescopic homotopies used in I.7.1 for $\pi_1^\varepsilon(X)$ show that $H_0^\varepsilon(X) \cong \mathbf{Z}$ and $H_n^\varepsilon(X) = 0$ for all $n > 1$, while the first homology group gives:

$$H_1^\varepsilon(X) \cong \mathbf{Z} \;\; (0 < \varepsilon < 1;\ 2 \leqslant \varepsilon < 3), \quad \mathbf{Z}^2 \;\; (1 \leqslant \varepsilon < 2), \quad 0 \;\; (3 \leqslant \varepsilon \leqslant \infty), \tag{2}$$

the generators being provided by the 1-chains associated to the loops which generate $\pi_1^\varepsilon(X)$ (I.7.1). These results (including the description of generators) give the same analysis of the metric space X as provided by the fundamental group (I.1.8): at fine resolution ($0 < \varepsilon < 1$), our map presents *one basin* $A \cup B \cup C$; then *two basins* A, C *connected by a bridgeable channel* B ($1 \leqslant \varepsilon < 2$); or *one basin* A with a negligible appendix ($2 \leqslant \varepsilon < 3$); and finally no relevant basin ($\varepsilon \geqslant 3$). Also here, *the finite model* $X \cap (\rho\mathbf{Z} \times \rho\mathbf{Z})$, resulting from a scanner at resolution $\rho = k^{-1}$ (for an *integer* $k \geqslant 2$, in order to simplify the interference with the boundary of X), has the same homology groups for $\varepsilon \geqslant \rho$ (and the same analysis).

Similarly, one proves that the solid metric subspace $X' = T' \setminus (A' \cup B' \cup C') \subset \mathbf{R}^3$, where

$$\begin{aligned} T' &= [0,11] \times [0,5]^2, \qquad & A' &=]1,4[\times]1,4[^2, \\ B' &= [4,8]\times]1,2[^2, \qquad & C' &=]8,10[\times]1,3[^2, \end{aligned} \tag{3}$$

equipped with the t_ε-structure ($\varepsilon > 0$), has $H_0^\varepsilon(X) \cong \mathbf{Z}$, $H_n^\varepsilon(X') = 0$ for all other $n \neq 2$, and

$$H_2^\varepsilon(X') \cong \mathbf{Z} \;\; (0 < \varepsilon < 1;\ 2 \leqslant \varepsilon < 3), \quad \mathbf{Z}^2 \;\; (1 \leqslant \varepsilon < 2), \quad 0 \;\; (3 \leqslant \varepsilon \leqslant \infty). \tag{4}$$

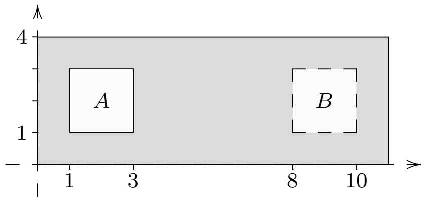
The analysis is analogous to the previous one, 1 dimension up: our object presents *one cavity* $A' \cup B' \cup C'$, at resolution $0 < \varepsilon < 1$; then *two cavities* A', C' *connected by a thin channel* ($1 \leqslant \varepsilon < 2$); or *one cavity* A' with a negligible appendix ($2 \leqslant \varepsilon < 3$); and finally *no relevant cavity* ($\varepsilon \geqslant 3$).

4.4. **Critical values.** Considering the previous examples, one is lead to consider the *variation* of the system of homology groups $H_n^\varepsilon(X) \to H_n^\eta(X)$ ($0 \leqslant \varepsilon \leqslant \eta \leqslant \infty$), for a metric space X, through its *critical values* (cf. Deheuvels [4], Milnor [16]).

Say that ε is a *left regular* (resp. *right regular, regular*) value for this system (i.e., for the combinatorial n-homology of X) if the system itself is constant on a left (resp. right, bilateral) neighbourhood of ε in the extended real interval

$[0, \infty]$. In the contrary, ε is a *left critical* (resp. *right critical, critical*) value; and a *bilateral critical* value if it is both left and right critical.

Thus, in dimension 1, the metric space X considered in 4.3.1 has a right critical value at 0 and left critical values at 1, 2, 3. The metric subspace $Y = T \setminus (A \cup B) \subset \mathbf{R}^2$ represented below has a right critical value at 0 and a *bilateral* critical value at 2



$$T = [0, 11] \times [0, 4]$$
$$A =]1, 3[\times]1, 3[\qquad (1)$$
$$B = [8, 10] \times [1, 3]$$

$$H_1^\varepsilon(Y) \cong \mathbf{Z}^2 \quad (0 < \varepsilon < 2), \qquad \mathbf{Z} \quad (\varepsilon = 2), \qquad 0 \ \ (\varepsilon > 2), \qquad (2)$$

since its homology groups can be computed by the same techniques as above (cf. I.7.2).

4.5. **Proposition.** *For a general metric space X, assume that the real interval $K = [\varepsilon, \eta] \subset [0, \infty]$ does not contain any critical value in dimensions $n-1, n, n+1$ (except possibly a left critical value at ε and a right one at η). Then $H_n^\varepsilon(X) \to H_n^\eta(X)$ is an isomorphism (but not vice versa, cf.* 4.3.2*).*

Proof. K is compact and every $p \in K$ has a K-neighbourhood where the homology system is constant, in the given degrees; by the Lebesgue covering theorem, one can find a finite partition of K, $\varepsilon = p_0 < p_1 < \cdots < p_k = \eta$, whose intervals $[p_{i-1}, p_i]$ are contained in such neighbourhoods. Thus all $H_m(t_{p_{i-1}}X) \to H_m(t_{p_i}X)$ are iso ($m = n-1, n, n+1$), and the relative homologies $H_m(t_{p_i}X, t_{p_{i-1}}X)$ are null, for $m = n, n+1$, by the exact homology sequence of a pair. The exact homology sequence of the triple $(t_{p_{i+1}}X, t_{p_i}X, t_{p_{i-1}}X)$ shows then that also $H_m(t_{p_{i+1}}X, t_{p_{i-1}}X) = 0$, in the same degrees $m = n, n+1$ $(0 < i < k)$. Similarly, by finite induction, $H_m(t_\eta X, t_\varepsilon X) = 0$ for $m = n, n+1$; finally, the thesis follows from the exact homology sequence of the pair $(t_\eta X, t_\varepsilon X)$. □

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Author's address:

Dipartimento di Matematica
Università di Genova
Via Dodecaneso, 35, 16146-Genova
Italy
E-mail: grandis@dima.unige.it