Georgian Mathematical Journal
Volume 10 (2003), Number 2, 289–310

A UNIFIED CHARACTERIZATION OF q-OPTIMAL AND MINIMAL ENTROPY MARTINGALE MEASURES BY SEMIMARTINGALE BACKWARD EQUATIONS

M. MANIA AND R. TEVZADZE

Abstract. We give a unified characterization of q-optimal martingale measures for $q \in [0, \infty)$ in an incomplete market model, where the dynamics of asset prices are described by a continuous semimartingale. According to this characterization the variance-optimal, the minimal entropy and the minimal martingale measures appear as the special cases $q = 2$, $q = 1$ and $q = 0$ respectively. Under assumption that the Reverse Hölder condition is satisfied, the continuity (in L^1 and in entropy) of densities of q-optimal martingale measures with respect to q is proved.

2000 Mathematics Subject Classification: 60H30, 91B28, 90C39.
Key words and phrases: Backward semimartingale equation, q-optimal martingale measure, minimal entropy martingale measure, contingent claim pricing.

1. Introduction and the Main Results

An important tool of Mathematical Finance is to replace the basic probability measure by an equivalent martingale measure, sometimes also called a pricing measure. It is well known that prices of contingent claims can usually be computed as expectations under a suitable martingale measure. The choice of the pricing measure may depend on the attitude towards risk of investors or on the criterion relative to which the quality of the hedging strategies is measured. In this paper we study the q-optimal martingale measures using the Semimartingale Backward Equations (SBE for short) introduced by Chitashvili [3].

The q-optimal martingale measure is a measure with the minimal L^q-norm among all signed martingale measures. The q-optimal martingale measures (for $q > 1$) were introduced by Grandits and Krawczyk [16] in relation to the closedness in L^p of a space of stochastic integrals. On the one hand, the q-optimal martingale measure is a generalization of the variance-optimal martingale measure introduced by Schweizer [37], which corresponds to the case $q = 2$. Schweizer [38] showed that if the quadratic criterion is used to measure the hedging error, then the price of a contingent claim (the mean-variance hedging price) is the mathematical expectation of this claim with respect to the variance-optimal martingale measure. The variance optimal martingale measure also plays a crucial role in determining the mean-variance hedging strategy (see, e.g., [33], [6], [14], [20]).

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

On the other hand, it was shown by Grandits [18] (in the finite discrete time case) and by Grandits and Rheinländer [17] (for the continuous process X) that if the reverse Hölder condition is satisfied, then the densities of q-optimal martingale measures converge as $q \downarrow 1$ in L^1 (and in entropy) to the density of the minimal entropy martingale measure. The problem of finding the minimal entropy martingale measure is dual to the problem of maximizing the expected exponential utility from terminal wealth (see [7], [34]). Note also that the q-optimal martingale measures for $q < 1$ (in particular, for $q = 1/2$ it defines the Hellinger distance martingale measure) are also closely related to the utility maximization problem (see [15], [35]).

The aim of this paper is to give a unified characterization of q-optimal martingale measures for $q \in [0, \infty)$ in an incomplete semimartingale market model. We express the densities of q-optimal martingale measures in terms of a solution of the corresponding semimartingale backward equation, where the index q appears as a parameter. According to this characterization the variance-optimal, the minimal entropy and the minimal martingale measures appear as the special cases $q = 2$, $q = 1$ and $q = 0$, respectively. Besides, the above mentioned convergence result of Grandits and Rheinländer [17] naturally follows from the continuity properties of solutions of SBEs with respect to q. We show that the same convergence is valid if $q \to 1$, and if $q \downarrow 0$, then the densities of the q-optimal martingale measures converge to the density of the minimal martingale measure. Moreover, we prove that the rate of convergence in entropy distance is $|q-1|$, although we require an additional condition of continuity of the filtration, not imposed in [17].

To formulate the main statements of this paper, let us give some basic definitions and assumptions.

Let $X = (X_t, t \in [0,T])$ be an R^d-valued semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, F = (F_t, t \in [0,T]), P)$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and T is a finite time horizon. The process X may be interpreted to model the dynamics of the discounted prices of some traded assets.

Denote by $\mathcal{M}^e$ the set of equivalent martingale measures of X, i.e., set of measures Q equivalent to P and such that X is a local martingale under Q. Let $Z_t(Q)$ be the density process of Q relative to the basic measure P. For any $Q \in \mathcal{M}^e$, denote by M^Q a P-local martingale such that $Z^Q = \mathcal{E}(M^Q) = (\mathcal{E}_t(M^Q), t \in [0,T])$, where $\mathcal{E}(M)$ is the Doleans-Dade exponential of M.

Let

$$\mathcal{M}_q^e = \{Q \in \mathcal{M}^e : EZ_T^q(Q) < \infty\},$$
$$\mathcal{M}_1^e = \{Q \in \mathcal{M}^e : EZ_T(Q) \ln Z_T(Q) < \infty\}.$$

Assume that

A) X is a continuous semimartingale,

B) $\mathcal{M}_q^e \neq \varnothing$ if $q \geq 1$, and $\mathcal{M}^e \neq \varnothing$ if $q < 1$.

Then it is well known that X satisfies the structure condition (SC), i.e., X admits the decomposition

$$X_t = X_0 + \int_0^t d\langle M\rangle_s \lambda_s + M_t \quad \text{a.s. for all} \quad t \in [0,T], \tag{1.1}$$

where M is a continuous local martingale and λ is a predictable R^d-valued process. If the local martingale $\widehat{Z}_t = \mathcal{E}_t(-\lambda \cdot M)$, $t \in [0,T])$ is a strictly positive martingale, then $d\widehat{P}/dP = \widehat{Z}_T$ defines an equivalent probability measure called the minimal martingale measure for X (see [12]). In general (since X is continuous), any element of $\mathcal{M}^e$ is given by the density $Z(Q)$ which is expressed as an exponential martingale of the form $\mathcal{E}(-\lambda \cdot M + N)$, where N is a local martingale strongly orthogonal to M. Here we use the notation $\lambda \cdot M$ for the stochastic integral with respect to M.

Let us consider the following optimization problems:

$$\min_{Q \in \mathcal{M}_q^e} E\mathcal{E}_T^q(M^Q), \qquad q > 1, \tag{1.2}$$

$$\max_{Q \in \mathcal{M}^e} E\mathcal{E}_T^q(M^Q), \qquad 0 < q < 1, \tag{1.3}$$

$$\min_{Q \in \mathcal{M}_q^e} E^Q \ln \mathcal{E}_T(M^Q), \quad q = 1. \tag{1.4}$$

Provided that conditions A) and B) are satisfied, these optimization problems admit a unique solution in the class of equivalent martingale measures (see [5], [16], [27] for $q = 2$, $q > 1$, $q < 1$, respectively, and [31], [13] for the case $q = 1$). Therefore we may define the q-optimal martingale measures for $q > 1$ and $q < 1$ and the minimal entropy martingale measure as solutions of optimization problems (1.2), (1.3) and (1.4), respectively.

The main statement of this paper (Theorem 3.1), for simplicity formulated here (and proved in section 3) in the one-dimensional case, gives a necessary and sufficient condition for a martingale measure to be q-optimal.

We show that the martingale measure $Q^*(q)$ is q-optimal if and only if $dQ^*(q) = \mathcal{E}_T(M^{Q^*(q)})dP$, where

$$M^{Q^*(q)} = -\lambda \cdot M + \frac{1}{Y_-(q)} \cdot L(q),$$

and the triple $(Y(q), \psi(q), L(q))$, where $Y(q)$ is a strictly positive special semimartingale, $\psi(q)$ is a predictable M-integrable process and $L(q)$ is a local martingale orthogonal to M, is a unique solution of the semimartingale backward equation

$$Y_t = Y_0 + \frac{q}{2}\int_0^t \frac{(\lambda_s Y_{s-} + \psi_s)^2}{Y_{s-}} d\langle M\rangle_s + \int_0^t \psi_s dM_s + L_t, \quad Y_T = 1 \tag{1.5}$$

in a certain class (see Definition 3.1) of semimartingales.

In Section 4 we study the dependence of solutions of SBE (1.5) on the parameter q and additionally assume that:

C) the filtration F is continuous,

B^*) there exists a martingale measure Q that satisfies the Reverse Hölder inequality for some $q_0 > 1$.

In Theorem 4.2 we prove that for any $0 \le q \le q_0$, $0 \le q' \le q_0$

$$\|M^{Q^*(q)} - M^{Q^*(q')}\|_{BMO_2} \le const\ |q - q'|^{\frac{1}{2}}. \tag{1.6}$$

According to Theorem 3.2 of Kazamaki [22], the mapping $\varphi : M \to \mathcal{E}(M) - 1$ of BMO_2 into H^1 is continuous. Therefore, in particular,(1.6) implies that

$$\|\mathcal{E}(M^{Q^*(q)}) - \mathcal{E}(M^{Q(E)})\|_{H^1} \to 0 \quad \text{as} \quad q \to 1, \tag{1.7}$$

$$\|\mathcal{E}(M^{Q^*(q)}) - \mathcal{E}(-\lambda \cdot M)\|_{H^1} \to 0 \quad \text{as} \quad q \downarrow 0, \tag{1.8}$$

where $Q(E) = Q^*(1)$ is the minimal entropy martingale measure and $\mathcal{E}(-\lambda \cdot M)$ is the density process of the minimal martingale measure. The convergence (1.7)) was proved by Grandits and Rheinländer [17] in the case $q \downarrow 1$ and then by Santacroce [36] for the case $q \uparrow 1$.

Moreover, it follows from Theorem 4.3 that

$$I(Q^*(q), Q^*(q')) \le const\ |q - q'|, \tag{1.9}$$

where $I(Q, R)$ is the relative entropy, or the Kullback–Leibler distance, of the probability measure Q with respect to the measure R and is defined as

$$I(Q, R) = E^R \frac{dQ}{dR} \ \ln \frac{dQ}{dR}.$$

In particular, (1.9) implies the convergence of q-optimal martingale measures (as $q \downarrow 1$) in entropy to the minimal entropy martingale measure, which was proved by Grandits and Rheinländer [17] without assumption C).

Backward stochastic differential equations (BSDE) have been introduced by Bismut [1] for the linear case as equations for the adjoint process in the stochastic maximum principle. A nonlinear BSDE (with Bellman generator) was first considered by Chitashvili [3]. He derived the semimartingale BSDE (or SBE), which can be considered as a stochastic version of the Bellman equation for a stochastic control problem, and proved the existence and uniqueness of a solution (see also [4]). The theory of BSDEs driven by the Brownian motion was developed by Pardoux and Peng [32] for more general generators. They obtained the well-posedness results for generators satisfying the uniform Lipschitz condition. The results of Pardoux and Peng were generalized by Kobylansky [23] for generators with quadratic growth.

BSDEs appear in numerous problems of Mathematical Finance (see, e.g., [11]). In several works BSDEs and the dynamic programming approach were also used to determine different martingale measures. By Laurent and Pham [24] the dynamic programming approach was used to determine the variance-optimal martingale measure in the case of Brownian filtration. Rouge and El Karoui [11] derived a BSDE related to the minimal entropy martingale measure for diffusion models and used the above-mentioned result of Kobylansky to show the existence of a solution. The dynamic programming method was also applied in [26], [27], [28] to determine the q-optimal and minimal entropy martingale measures in the semimartingale setting. In [27] the density of the q-optimal martingale measure was expressed in terms of an SBE derived for the value process

$$\widetilde{V}_t(q) = \operatorname*{ess\,inf}_{Q \in \mathcal{M}_q^e} E(\mathcal{E}_{tT}^q(M^Q) \mid F_t), \quad q > 1,$$

corresponding to the problem (1.2). As shown in Proposition 2.3, the solution $Y(q)$ of (1.5) is related to $\widetilde{V}(q)$ by the equality

$$Y_t(q) = (\widetilde{V}_t(q))^{\frac{1}{1-q}}.$$

As compared with the SBE derived in [27] for $\widetilde{V}(q)$, equation (1.5) has the following advantages:
1) equation (1.5), unlike the equation for $\widetilde{V}(q)$, in the case $q = 1$ determines the minimal entropy martingale measure and
2) equation (1.5) is of the same form with and without the assumption of the continuity of the filtration, whereas the equation for $\widetilde{V}(q)$ becomes much more complicated (there appear additional jump terms in the generator, see, e.g., [29]) when the filtration is not continuous.

It was shown in [30] that the value function for the mean-variance hedging problem is a quadratic trinomial and a system of SBEs for its coefficients was derived. It was proved that the first coefficient of this trinomial coincides with $\widetilde{V}(2)^{-1}$ and satisfies equation (1.5) for $q = 2$, which is simpler than the equation for $\widetilde{V}(2)$. This fact was more explicitly pointed out by Bobrovnytska and Schweizer [2], who gave a description of the variance optimal martingale measure using equation (1.5) for $q = 2$.

After finishing this paper (during the reviewing process), we received a copy of the paper by Hobson [19] who also studied q-optimal martingale measures using BSDEs driven by the brownian motion, called in [19] the fundamental representation equation. In a general diffusion market model, assuming that a solution of representation equation exists, this solution is used to characterize the q-optimal martingale measure, with the minimal entropy martingale measure arising when $q = 1$. Note that one can derive the fundamental representation equation from equation (1.5) using the Itô formula for $lnY(q)$ and the boundary condition. Therefore Theorem 3.1 implies the existence of a solution of the representation equation of [19] for the models considered in that paper. It should be mentioned that in [19] the representation equation was explicitly solved in the case of Markovian stochastic volatility models with correlation.

For all unexplained notations concerning the martingale theory used below we refer the reader to [21], [8] and [25]. About BMO-martingales and the reverse Hölder conditions see [9] and [22].

2. Basic Optimization Problems and Auxiliary Results

In this section we introduce the basic optimization problems and study some properties of the corresponding value processes.

Instead of condition B) we shall sometimes use a stronger condition:

B*) there exists a martingale measure Q that satisfies the reverse Hölder $R_q(P)$ inequality if $q>1$, and the reverse Hölder $R_{L\ln L}(P)$ inequality if $q=1$, i.e., there is a constant C such that

$$E(\mathcal{E}^q_{\tau,T}(M^Q)\mid F_\tau)\le C \quad \text{if} \quad q>1,$$
$$E(\mathcal{E}_{\tau,T}(M^Q)\ln\mathcal{E}_{\tau,T}(M^Q)\mid F_\tau)\le C \quad \text{if} \quad q=1$$

for any stopping time τ.

Here and in the sequel we use the notation

$$\mathcal{E}_{\tau,T}(N)=\frac{\mathcal{E}_T(N)}{\mathcal{E}_\tau(N)}=\mathcal{E}_T(N-N_{\cdot\wedge\tau})$$

for a semimartingale N and

$$\langle N\rangle_{\tau,T}=\langle N\rangle_T-\langle N\rangle_\tau$$

for a local martingale N for which the predictable characteristic $\langle N\rangle$ exists.

Remark 2.1. Condition B*) implies that $M^Q\in BMO_2$ for $M^Q=-\lambda\cdot M+N$, where N is a local martingale orthogonal to M (see [22] for $q>1$ and [17] for the case $q=1$). Since $\langle\lambda\cdot M\rangle_{\tau,t}<\langle-\lambda\cdot M+N\rangle_{\tau,t}$ for any $\tau<t\le T$, we have that $\lambda\cdot M$ also belongs to BMO and the minimal martingale measure exists.

We recall that a uniformly integrable martingale $M=(M_t,t\in[0,T])$ belongs to the class BMO_2 if and only if M is of bounded jumps and for a constant C

$$E^{1/2}(\langle M\rangle_{\tau,T}\mid F_\tau)\le C, \quad P\text{-a.s.}$$

for every stopping time τ. The smallest constant with this property (or $+\infty$ if it does not exist) is called the BMO_2 norm of M and is denoted by $\|M\|_{BMO_2}$.

Let H^1 be the space of martingales N with $\|N\|_{H^1}=\sup_{t\le T}|N_t|<\infty$. Note that H^1 is the dual space of BMO_2 (see [8]).

Denote by Π_p the class of predictable X-integrable processes such that

$$E\mathcal{E}^p_T(\pi\cdot X)<\infty \quad \text{for} \quad p<\infty, \tag{2.1}$$
$$Ee^{\langle\pi\cdot X\rangle_T}<\infty \quad \text{for} \quad p=\pm\infty. \tag{2.2}$$

Remark 2.2. For all $\pi\in\Pi_p$ the strategy $\widetilde{\pi}=\pi\mathcal{E}(\pi\cdot X)$ belongs to the class $\mathcal{H}_2$ of [35] since $1+\widetilde{\pi}\cdot X=\mathcal{E}(\pi\cdot X)$ is a Q-supermartingale for all $Q\in\mathcal{M}^e$, as a positive Q-local martingale.

We consider the following optimization problems:

$$\begin{aligned}
&\min_{\pi\in\Pi_p}E\mathcal{E}^p_T(\pi\cdot X), && p>1,\\
&\min_{\pi\in\Pi_p}Ee^{\langle\pi\cdot X\rangle_T}, && p=\pm\infty,\\
&\max_{\pi\in\Pi_p}E\mathcal{E}^p_T(\pi\cdot X), && 0<p<1,\\
&\min_{\pi\in\Pi_p}E\mathcal{E}^p_T(\pi\cdot X), && p<0.
\end{aligned}\tag{2.3}$$

and

$$
\begin{aligned}
&\min_{Q\in\mathcal{M}_q^e} E\mathcal{E}_T^q(M^Q), && q>1,\\
&\min_{Q\in\mathcal{M}_q^e} E^Q \ln \mathcal{E}_T(M^Q), && q=1,\\
&\min_{Q\in\mathcal{M}^e} E\mathcal{E}_T^q(M^Q), && q<0,\\
&\max_{Q\in\mathcal{M}^e} E\mathcal{E}_T^q(M^Q), && 0\le q<1.
\end{aligned}
\tag{2.4}
$$

Throughout the paper we assume that $q=\frac{p}{p-1}$.

It is well known that if condition B) is satisfied, then each of these optimization problems admits a unique solution.

Let Q^* be a q-optimal martingale measure and let π^* be the optimal strategy for problem (2.3). It follows from [15],[35] (see also [16], [13], [17]) that

$$\mathcal{E}_T^{p-1}(\pi^*\cdot X)=c\mathcal{E}_T(M^{Q^*}) \quad \text{for} \quad p<\infty, \tag{2.5}$$

$$e^{(\pi^*\cdot X)_T}=c\mathcal{E}_T(M^{Q^*}) \quad \text{for} \quad p=\pm\infty. \tag{2.6}$$

Moreover, $\mathcal{E}(\pi^*\cdot X)(\pi^*\cdot X$ if $p=\pm\infty)$ is a martingale with respect to Q^*.

Thus the q-optimal martingale measure M^{Q^*} and the optimal strategy π^* of the corresponding dual problem are related by the equality

$$\mathcal{E}_T^p(\pi^*\cdot X)=c^q\mathcal{E}_T^q(M^{Q^*}).$$

Now let us define the value processes

$$
V_t(p)=\begin{cases}
\operatorname*{ess\,inf}_{\pi\in\Pi_p} E(\mathcal{E}_{tT}^p(\pi\cdot X)|F_t), & p>1,\\
\operatorname*{ess\,inf}_{\pi\in\Pi_p} E(\exp(\int_t^T \pi_s dX_s)|F_t), & p=\pm\infty\\
\operatorname*{ess\,inf}_{\pi\in\Pi_p} E(\mathcal{E}_{tT}^p(\pi\cdot X)|F_t), & p<0,\\
\operatorname*{ess\,sup}_{\pi\in\Pi_p} E(\mathcal{E}_{tT}^p(\pi\cdot X)|F_t), & 0<p<1,
\end{cases}
\tag{2.7}
$$

and

$$
\widetilde{V}_t(q)=\begin{cases}
\operatorname*{ess\,inf}_{Q\in\mathcal{M}_q^e} E(\mathcal{E}_{tT}^q(M^Q)|F_t), & q>1,\\
\operatorname*{ess\,inf}_{Q\in\mathcal{M}_q^e} E(\mathcal{E}_{tT}(M^Q)\ln\mathcal{E}_{tT}(M^Q)|F_t), & q=1\\
\operatorname*{ess\,sup}_{Q\in\mathcal{M}^e} E(\mathcal{E}_{tT}^q(M^Q)|F_t), & 0\le q<1,\\
\operatorname*{ess\,inf}_{Q\in\mathcal{M}_q^e} E(\mathcal{E}_{tT}^q(M^Q)|F_t), & q<0,
\end{cases}
\tag{2.8}
$$

corresponding to problems (2.3) and (2.4), respectively.

The optimality principle for the problem (2.3) can be proved in a standard manner (see, e.g., [10], [24]) and takes the following form.

Proposition 2.1. *There exists an RCLL semimartingale, denoted as before by $V_t(p)$, such that for each $t \in [0,T]$*

$$V_t(p) = \begin{cases} \operatorname*{ess\,inf}_{\pi\in\Pi_p} E(\mathcal{E}^p_{tT}(\pi\cdot X)|F_t) \ \ a.s. & \text{for} \quad p>1 \ \text{ or } \ p<0, \\ \operatorname*{ess\,inf}_{\pi\in\Pi_\infty} E(e^{(\pi\cdot X)_T-(\pi\cdot X)_t}|F_t) \ \ a.s. & \text{for} \quad p=\pm\infty. \end{cases}$$

$V_t(p)$ is the largest RCLL process equal to 1 at time T such that the process $V_t(p)\mathcal{E}^p_t(\pi\cdot X)$ (the process $V_t(p)e^{(\pi\cdot X)_t}$ if $p=\pm\infty$) is a submartingale for every $\pi\in\Pi_p$.

Moreover, π^* is optimal if and only if $V_t(p)\mathcal{E}^p_t(\pi^*\cdot X)$ ($V_t(p)e^{(\pi\cdot X)_t}$ for $p=\pm\infty$) is a martingale.

Proposition 2.2. *For all $t\in[0,T]$ the value process $V_t(p)$ is an increasing function of p on $[-\infty,1)$ and $[1,\infty]$ separately. Moreover, if $p>1$ and $p'<0$, then $V_t(p)\le V_t(p')$ a.s. for all $t\in[0,T]$.*

Proof. Let p,p' be such that $p>p'>1$ or $0>p>p'$. It is sufficient to consider the case $p'-p+1>0$. Applying successively the optimality of $\pi^*(p)$, representation (2.5), the Bayes formula, the Hölder inequality and the fact that $\mathcal{E}(\pi^*(p)\cdot X)$ is a martingale with respect to $Q^*(p)$, we get

$$\begin{aligned} V_t(p') &= E[\mathcal{E}^{p'}_{tT}(\pi^*(p')\cdot X)/F_t] \le E[\mathcal{E}^{p'}_{tT}(\pi^*(p)\cdot X)/F_t] \\ &= E[\mathcal{E}^{p-1}_{tT}(\pi^*(p)\cdot X)\mathcal{E}^{p'-p+1}_{tT}(\pi^*(p)\cdot X)/F_t] \\ &= \frac{c\mathcal{E}_t(M^{Q^*})}{\mathcal{E}^{p-1}_t(\pi^*(p)\cdot X)} E^{Q^*}[\mathcal{E}^{p'-p+1}_{tT}(\pi^*(p)\cdot X)/F_t] \\ &\le \frac{c\mathcal{E}_t(M^{Q^*})}{\mathcal{E}^{p-1}_t(\pi^*(p)\cdot X)} = \frac{E(\mathcal{E}^{p-1}_T(\pi^*(p)\cdot X)/F_t)}{\mathcal{E}^{p-1}_t(\pi^*(p)\cdot X)} = V_t(p). \end{aligned}$$

If $p=\infty$, then

$$V_t(p') \le E[\mathcal{E}^{p'}_{tT}(\pi\cdot X)/F_t] \le E[e^{p'(\pi\cdot X)_T-p'(\pi\cdot X)_t}/F_t]$$

for all $\pi\in\Pi_{p'}$ and hence $V(p')\le V(\infty)$. Similarly, $V(-\infty)\le V(p)$ for $p<0$.

Now suppose that $0<p'<p<1$. By the Hölder inequality and by the optimality of $\pi^*(p)$ we have

$$V_t(p') \le E[\mathcal{E}^p_{tT}(\pi^*(p')\cdot X)/F_t]^{\frac{p'}{p}} \le E[\mathcal{E}^p_{tT}(\pi^*(p)\cdot X)/F_t]^{\frac{p'}{p}} = V_t(p)^{\frac{p'}{p}}.$$

Since $V_t(p)\ge 1$, we get $V_t(p')\le V_t(p)$ a.s.

If $p>0>p'$, then $V_t(p)\ge 1\ge V_t(p')$.

Assume now that $p>1$ and $p'<0$ and denote $r=\max\{p,-p'\}$. Thus $p'\ge -r>0$, and taking into account that V is increasing on $[-\infty,1)$ and $(1,\infty]$, respectively, we have

$$\begin{aligned} V_t(p') \ge V_t(-r) &= E[\mathcal{E}^{-r}_{tT}(\pi^*(-r)\cdot X)/F_t] \\ &= E[\mathcal{E}^r_{tT}(-\pi^*(-r)\cdot X)e^{r<\pi^*(-r)\cdot X>_{tT}}/F_t] \\ &\ge E[\mathcal{E}^r_{tT}(-\pi^*(-r)\cdot X)/F_t] \ge E[\mathcal{E}^r_{tT}(\pi^*(r)\cdot X)/F_t] \\ &= V_t(r) \ge V_t(p). \end{aligned}$$

□

Corollary 2.1. $Y(q) = V(\frac{q}{q-1})$ *is a decreasing function on* $(-\infty, \infty)$.

Proposition 2.3. *The value processes defined by* (2.7) *and* (2.8) *are related by*

$$V(p) = \widetilde{V}(q)^{1-p} \quad for \quad q \neq 1$$

and

$$V(\infty) = e^{-\widetilde{V}(1)} \quad for \quad q = 1.$$

Moreover,

$$V(p)\mathcal{E}^{p-1}(\pi^* \cdot X) = c\mathcal{E}(M^{Q^*}) \quad for \quad q \neq 1,$$
$$V(\infty)e^{(\pi^* \cdot X)} = c\mathcal{E}(M^{Q^*}) \quad for \quad q = 1.$$

Proof. Let us first consider the case $q \neq 1$. Using the optimality of $Q^* = Q^*(q)$, the Bayes rule and representation (2.5), we have

$$\widetilde{V}_t(q) = \frac{1}{\mathcal{E}_t^q(M^{Q^*})} E[\mathcal{E}_T^q(M^{Q^*})/\mathcal{F}_t] = \frac{1}{\mathcal{E}_t^{q-1}(M^{Q^*})} E^{Q^*}[\mathcal{E}_T^{q-1}(M^{Q^*})/\mathcal{F}_t]$$
$$= \bar{c}\, \frac{E(\mathcal{E}_T(\pi^* \cdot X)/F_t)}{\mathcal{E}_t^{q-1}(M^{Q^*})} = \bar{c} \frac{\mathcal{E}_t(\pi^* \cdot X)}{\mathcal{E}_t^{q-1}(M^{Q^*})}.$$

Therefore, taking into account $c = \bar{c}^{1-p}$ and $p = \frac{q}{q-1}$, we obtain

$$\widetilde{V}_t(q)^{p-1} = c^{-1}\, \frac{\mathcal{E}_t^{p-1}(\pi^* \cdot X)}{\mathcal{E}_t(M^{Q^*})}$$

and

$$\widetilde{V}_t(q)^{1-p} = c\, \frac{\mathcal{E}_t(M^{Q^*})}{\mathcal{E}_t^{p-1}(\pi^* \cdot X)} = cc^{-1}\, \frac{E[\mathcal{E}_T^{p-1}(\pi^* \cdot X)/F_t]}{\mathcal{E}_t^{p-1}(\pi^* \cdot X)} = E[\mathcal{E}_{tT}^{p-1}(\pi^* \cdot X)/F_t].$$

On the other hand, using similar arguments we obtain

$$V_t(p) = E[\mathcal{E}_{tT}^p(\pi^* \cdot X)/F_t] = \frac{E[\mathcal{E}_T^{p-1}(\pi^* \cdot X)\mathcal{E}_T(\pi^* \cdot X)/F_t]}{\mathcal{E}_t^p(\pi^* \cdot X)}$$
$$= c\, \frac{E[\mathcal{E}_T(M^{Q^*})\mathcal{E}_T(\pi^* \cdot X)/F_t]}{\mathcal{E}_t^p(\pi^* \cdot X)} = c \frac{E^{Q^*}[\mathcal{E}_T(\pi^* \cdot X)/F_t]\mathcal{E}_t(M^{Q^*})}{\mathcal{E}_t^p(\pi^* \cdot X)}$$
$$= c\, \frac{\mathcal{E}_t(M^{Q^*})}{\mathcal{E}_t^{p-1}(\pi^* \cdot X)} = \frac{E[\mathcal{E}_T^{p-1}(\pi^* \cdot X)/F_t]}{\mathcal{E}_t^{p-1}(\pi^* \cdot X)} = E[\mathcal{E}_{tT}^{p-1}(\pi^* \cdot X)/F_t]. \tag{2.9}$$

Therefore $\widetilde{V}_t(q)^{1-p} = E[\mathcal{E}_{t,T}^p(\pi^* \cdot X)/F_t] = V_t(p)$. Besides, we have $V_t(p) = c\frac{\mathcal{E}_t(M^{Q^*})}{\mathcal{E}_t^{p-1}(\pi^* \cdot X)}$.

Now let us consider the case $q = 1$. Since

$$\tilde{V}_t(1) = E^{Q^*}(\ln \mathcal{E}_{tT}(M^{Q^*})/\mathcal{F}_t) = E^{Q^*}(\ln \mathcal{E}_T(M^{Q^*})/\mathcal{F}_t) - \ln \mathcal{E}_t(M^{Q^*}),$$

from representation (2.6) we have

$$\tilde{V}_t(1) = c + \int_0^t \pi_s^* dX_s - \ln E(e^{c + \int_0^T \pi_s^* dX_s}/\mathcal{F}_t).$$

Therefore

$$\exp(-V_t(\infty)) = e^{-\int_0^t \pi_s^* dX_s} E\Big(e^{\int_0^T \pi_s^* dX_s}/\mathcal{F}_t\Big) = E(e^{\int_t^T \pi_s^* dX_s}/\mathcal{F}_t) = V(\infty).$$

Moreover, (2.6) also implies

$$V_t(\infty) = E[e^{(\pi^* \cdot X)_T}/\mathcal{F}_t]e^{-(\pi^* \cdot X)_t} = c\mathcal{E}_t(M^{Q^*})e^{-(\pi^* \cdot X)_t}. \qquad \square$$

Remark 2.3. Equality (2.9) implies that the optimal strategy π^* satisfies

$$E[\mathcal{E}_{tT}^p(\pi^* \cdot X)/F_t] = E[\mathcal{E}_{tT}^{p-1}(\pi^* \cdot X)/F_t].$$

Note that this fact in discrete time and in the case $q = 2$ was observed by Schweizer [38].

Lemma 2.1. *Let there exists a martingale measure that satisfies the reverse Hölder $R_{q_0}(P)$ inequality for some $q_0 > 1$ and let $Y(q) = V(\frac{q}{q-1})$. Then there is a constant $c > 0$ such that*

$$\inf_{0 \le q \le q_0} Y_t(q) \ge c \quad \textit{for all} \quad t \in [0,T] \;\; \textit{a.s.} \tag{2.10}$$

Proof. The $R_{q_0}(P)$ inequality implies that

$$1 \le \tilde{V}_t(q_0) \le C,$$

where C is a constant from the $R_{q_0}(P)$ condition.

By Proposition 2.3 we have

$$Y(q) = V\left(\frac{q}{q-1}\right) = \tilde{V}(q)^{\frac{1}{1-q}}.$$

Therefore

$$Y_t(q_0) \ge C^{\frac{1}{1-q_0}}. \tag{2.11}$$

Since by Corollary of Proposition 2.2 $Y_t(q) \ge Y_t(q_0)$ for any $q \le q_0$, inequality (2.10) follows from (2.11). $\square$

3. A Semimartingale Backward Equation for the Value Process

In this section we derive a SBE for $V(p)$ and write the expression for the optimal strategy π^*. Then, using relations (2.5) and (2.6) we construct the corresponding optimal martingale measures.

We say that the process B strongly dominates the process A and write $A \prec B$ if the difference $B - A \in \mathcal{A}_{\mathrm{loc}}^+$, i.e., is a locally integrable increasing process. Let $(A^Q, Q \in \mathcal{Q})$ be a family of processes of finite variations, zero at time zero. Denote by $\operatorname*{ess\,inf}_{Q \in \mathcal{Q}}(A^Q)$ the largest process of finite variation, zero at time zero, which is strongly dominated by the process $(A_t^Q, t \in [0,T])$ for every $Q \in \mathcal{Q}$, i.e., this is an "ess inf" of the family $(A^Q, Q \in \mathcal{Q})$ relative to the partial order $\prec$.

Now we give the definition of the class of processes for which the uniqueness of the solution of the considered SBE will be proved.

Definition 1. We say that Y belongs to the class D_p if $Y\mathcal{E}^p(\pi \cdot X) \in D$ for every $\pi \in \Pi_p$.

Recall that the process X belongs to the class D if the family of random variables $X_\tau I_{(\tau\le T)}$ for all stopping times τ is uniformly integrable.

Remark 3.1. The value process $V(p)$ for $p>1$ or $p<0$ belongs to the class D_p since $V(p)\mathcal{E}^p(\pi\cdot X)$ is a positive submartingale for any $\pi\in\Pi_p$.

Remark 3.2. Suppose that there exists a martingale measure that satisfies the reverse Hölder inequality $R_q(P)$. Then by Theorem 4.1 of Grandits and Krawchouk [16] $E\sup_{t\le T}\mathcal{E}^p_t(\pi\cdot X)\le CE\mathcal{E}^p_T(\pi\cdot X)$ and the process $\mathcal{E}^p_t(\pi\cdot X)$ belongs to the class D for every $\pi\in\Pi_p$. Therefore, any bounded positive process Y belongs to the class D_p if the $R_q(P)$ condition is satisfied.

Since X is continuous, the process $\mathcal{E}^p_t(\pi\cdot X)$ is locally bounded for any $p\in R$ and Proposition 2.1 implies that the process $V(p)$ is a special semimartingale with respect to the measure P with the canonical decomposition

$$V_t(p)=\overline{m}_t(p)+A_t(p),\quad \overline{m}\,(p)\in M_{loc},\quad A(p)\in\mathcal{A}_{loc}. \tag{3.1}$$

Let

$$\overline{m}_t(p)=\int_0^t\varphi_s(p)dM_s+m_t(p),\quad \langle m(p),M\rangle=0, \tag{3.2}$$

be the Galtchouk–Kunita–Watanabe decomposition of $\overline{m}\,(p)$ with respect to M.

Theorem 3.1. *Let conditions* A) *and* B) *be satisfied. Let* $q\in[0,\infty)$ *and* $p=\frac{q}{q-1}$. *Then*

a) *the value process* $V(p)$ *is a solution of the semimartingale backward equation*

$$\begin{aligned}Y_t(q)=Y_0(q)&+\frac{q}{2}\int_0^t Y_{s-}(q)\Big(\lambda_s+\frac{\psi_s(q)}{Y_{s-}(q)}\Big)'d\langle M\rangle_s\Big(\lambda_s+\frac{\psi_s(q)}{Y_{s-}(q)}\Big)\\&+\int_0^t\psi_s(q)dM_s+L_t(q),\quad t<T,\end{aligned}\tag{3.3}$$

with the boundary condition

$$Y_T(q)=1. \tag{3.4}$$

This solution is unique in the class of positive semimartingales from D_p.

Moreover, the martingale measure Q^* *is* q*-optimal if and only if*

$$M^{Q^*}=-\lambda\cdot M+\frac{1}{Y_-(q)}\cdot L(q) \tag{3.5}$$

and the strategy π^* *is optimal if and only if*

$$\pi^*=\begin{cases}(1-q)(\lambda+\frac{\psi(q)}{Y_-(q)}) & \textit{for}\ \ q\neq1\\ -\lambda-\frac{\psi(q)}{Y_-(q)} & \textit{for}\ \ q=1\end{cases}$$

for $Y(q)\in D_p$.

b) *If, in addition, condition* B^*) *is satisfied, then the value process* $V(p)$ *is the unique solution of the semimartingale backward equation* (3.3), (3.4) *in the class of semimartingales* Y *satisfying the two-sided inequality*

$$c\le Y_t(q)\le C\quad \textit{for all}\quad t\in[0,T]\quad \textit{a.s.} \tag{3.6}$$

for some positive constants $c < C$.

Proof. For simplicity, we consider the case $d = 1$. In the multidimensional case the proof is similar.

Existence. Let us show that $Y(q) = V(p)$ satisfies (3.3), (3.4). Suppose that $1 < p < \infty$ or $-\infty < p < 0$ (i.e., we first consider the case $q \neq 1$). By the Itô formula we have

$$\mathcal{E}_t^p(\pi \cdot X) = 1 + p\int_0^t \mathcal{E}_s^p(\pi \cdot X)\Big[\pi_s dX_s + \frac{p-1}{2}\pi_s^2 d\langle M\rangle_s\Big].$$

Using (3.1), (3.2) and the Itô formula for the product, we obtain

$$V_t(p)\mathcal{E}_t^p(\pi \cdot X) = V_0(p) + \int_0^t \mathcal{E}_s^p(\pi \cdot X)\Big[dA_s(p) + pV_{s-}(p)\pi_s\lambda_s d\langle M\rangle_s + p\varphi_s(p)\pi_s d\langle M\rangle_s + \frac{p(p-1)}{2}\,\pi_s^2 V_{s-}(p)d\langle M\rangle_s\Big] + \text{martingale}.$$

By the optimality principle (since the optimal strategy for the problem (2.3) exists) we get

$$A_t(p) = -\operatorname*{ess\,inf}_{\pi\in\Pi_p}\int_0^t \Big[p(V_{s-}(p)\lambda_s + \varphi_s(p))\pi_s + \frac{p(p-1)}{2}\,\pi_s^2 V_{s-}(p)\Big]d\langle M\rangle_s$$

and

$$A_t(p) = -\int_0^t \Big[p(V_{s-}(p)\lambda_s + \varphi_s(p))\pi_s^* + \frac{p(p-1)}{2}\,\pi_s^{*2} V_{s-}(p)\Big]d\langle M\rangle_s \tag{3.7}$$

if and only if π^* is optimal.

It is evident that there exists a sequence of stopping times τ_n, with $\tau_n \uparrow T$, such that

$$V_{\tau_n\wedge t-}(p) \ge \frac{1}{n}, \quad \int_0^{\tau_n\wedge t}\lambda_s^2 d\langle M\rangle_s \le n, \quad \int_0^{\tau_n\wedge t}\varphi_s^2 d\langle M\rangle_s \le n.$$

Then the strategy $\pi^n = (1-q)(\lambda + \frac{\varphi(p)}{V_-(p)})1_{[0,\tau_n]}$ belongs to the class Π_p for every $n \ge 1$ and

$$\operatorname*{ess\,inf}_{\pi\in\Pi_p}\Big|\pi_s + (q-1)\Big(\lambda_s + \frac{\varphi_s(p)}{V_{s-}(p)}\Big)\Big|^2 V_{s-}(p)$$
$$\le 2(p-1)^2\frac{|\lambda_s V_{s-}(p) + \varphi_s(p))|^2}{V_{s-}(p)}\,1_{(\tau_n\le s)} \to 0, \quad \text{as} \quad n\to\infty.$$

Therefore

$$A_t(p) = \frac{q}{2}\int_0^t \frac{|\lambda_s V_{s-}(p) + \varphi_s(p))|^2}{V_{s-}(p)}\,d\langle M\rangle_s \tag{3.8}$$

and (3.1)–(3.2) imply that $V_t(p)$ satisfies (3.3), (3.4). Moreover, comparing (3.7) and (3.8), we have

$$\frac{p}{2(p-1)V(p)}\big((p-1)\pi^* V(p) + \lambda V(p) + \varphi(p)\big)^2 = 0 \quad \mu^{\langle M\rangle}\text{-a.e.}$$

and hence $\pi^* = (1-q)(\lambda + \frac{\varphi(p)}{V_-(p)})$.

For $p = \pm\infty$, $V(p)e^{(\pi\cdot X)}$ admits a decomposition

$$V_t(p)e^{(\pi\cdot X)_t} = V_0(p) + \int_0^t V_{s-}(p)e^{(\pi\cdot X)_s}\left(\lambda_s\pi_s + \pi_s\frac{\varphi_s}{V_{s-}} + \frac{1}{2}\,\pi_s^2\right)d\langle M\rangle_s$$
$$+ \int_0^t e^{(\pi\cdot X)_s}dA_s(p) + \text{martingale}.$$

Thus, in a similar manner one can obtain

$$A_t(p) = -\operatorname*{ess\,inf}_{\pi\in\Pi_p}\int_0^t \left[\big(V_{s-}(p)\lambda_s + \varphi_s(p)\big)\pi_s + \frac{1}{2}\,\pi_s^2 V_{s-}(p)\right]d\langle M\rangle_s$$
$$= \frac{1}{2}\int_0^t V_{s-}\left(\lambda_s + \frac{\varphi_s}{V_{s-}}\right)^2 d\langle M\rangle_s$$

and $\pi^* = -\lambda - \frac{\varphi}{V_-}$.

Uniqueness. Suppose that $Y = Y(q)$ is a strictly positive solution of (3.3),(3.4) and $Y \in D_p$. Since Y satisfies equation (3.3) and $Y\mathcal{E}^p(\pi\cdot X) \in D$ for all $\pi \in \Pi_p$, by the Itô formula we obtain

$$Y_t\mathcal{E}_t^p(\pi\cdot X) = Y_0 + \frac{p(p-1)}{2}\int_0^t Y_{s-}\left|\pi_s + \frac{1}{p-1}\left(\lambda_s + \frac{\psi_s}{Y_{s-}}\right)\right|^2 d\langle M\rangle_s$$
$$+\text{local martingale}. \tag{3.9}$$

Hence $Y_t\mathcal{E}_t^p(\pi\cdot X)$ is a P-submartingale and $Y_t \le E[\mathcal{E}_{tT}^p(\pi\cdot X)/\mathcal{F}_t]$ for every $\pi \in \Pi_p$. Therefore

$$Y_t \le V_t(p). \tag{3.10}$$

On the other hand, (3.9) implies that $Y\mathcal{E}^p(\pi^0\cdot X)$ is a positive P-local martingale for $\pi^0 = (1-q)(\lambda + \frac{\psi}{Y_-})$. Thus it is a supermartingale and

$$Y_t \ge E[\mathcal{E}_{t,T}^p(\pi^0\cdot X)/\mathcal{F}_t].$$

Taking $t = 0$ in the latter inequality, from (3.10) we obtain

$$E\mathcal{E}_T^p(\pi^0\cdot X) \le Y_0 \le V_0(p) < \infty$$

and, hence, $\pi^0 \in \Pi_p$. Therefore $Y_t \ge V_t(p)$ and from (3.10) we obtain $Y(q) = V(p)$. By the uniqueness of the Doob–Meyer decomposition

$$L(q) = m(p) \quad \text{and} \quad \psi(q) = \varphi(p).$$

For $p = \pm\infty$ the proof of the uniqueness is similar.

Let us show now that the q-optimal martingale measure admits representation (3.5).

From Proposition 2.3 (for $1 < p < \infty$ or $-\infty < p < 0$) we have $V_t(p)\mathcal{E}_t^{p-1}(\pi^*\cdot X) = c\mathcal{E}_t(M^{Q^*})$ and after equalizing the orthogonal martingale parts, we obtain $\mathcal{E}_{t-}^{p-1}(\pi^*\cdot X)dm_t(p) = c\mathcal{E}_{t-}(M^{Q^*})dN_t^*$, where $M^{Q^*} = -\lambda\cdot M + N^*$. Thus $N_t^* = \int_0^t \frac{1}{V_{s-}(p)}\,dm_s(p)$ and $M^{Q^*} = -\lambda\cdot M + \frac{1}{V_-(p)}\cdot m(p)$.

Hence, the processes M^{Q^*} and $-\lambda\cdot M+\frac{1}{V_-(p)}\cdot m(p)$ are indistinguishable and

$$\mathcal{E}_T(M^{Q^*})=\mathcal{E}_T\Big(-\lambda\cdot M+\frac{1}{V_-(p)}\cdot m(p)\Big)\in\mathcal{M}_q^e. \tag{3.11}$$

For $p=\pm\infty$, similarly to the case $p<\infty$, we again get $e^{(\pi^*\cdot X)_t}dm_t = c\mathcal{E}_t(M^{Q^*})dN_t^*$ and also

$$N_t^*=c\int_0^t\mathcal{E}_s(M^{Q^*})e^{-(\pi^*\cdot X)_s}dm_s=\int_0^t\frac{1}{V_{s-}(\infty)}\,dm_s(\infty).$$

Conversely, let a measure $\widetilde{Q}$ be of the form (3.5), where $Y(q)$ is a solution of (3.3), (3.4) from the class D_p. Then by the uniqueness of a solution we have $Y(q)=V(p)$, $L(q)=m(p)$ and hence

$$\mathcal{E}\Big(-\lambda\cdot M+\frac{1}{Y_-(q)}\cdot L(q)\Big)=\mathcal{E}\Big(-\lambda\cdot M+\frac{1}{V_-(p)}\cdot m(p)\Big)$$

Therefore, by (3.11) $\widetilde{Q}$ is q-optimal.

b) If condition A^*) is satisfied, then $1\le\tilde{V}(q)\le C$. Thus $C^{\frac{1}{1-p}}\le V(p)\le 1$. On the other hand, by Remark 3.2 all bounded positive semimartingales belong to the class D_p and hence $V(p)$ is a unique solution in this class. □

Remark 3.3. If $q=0$, then $p=0$ and the class $D_p=D_0$ coincides with the class D. Since for $q=0$ any solution Y of (3.3) is a local martingale and any local martingale from the class D is a martingale, we have that any solution Y of (3.3), (3.4) from D equals to 1 for all $t\in[0,T]$ (as a martingale with $Y_T=1$). Therefore $L(0)=0$ and by (3.5) $M^{Q^*(0)}=-\lambda\cdot M$.

Remark 3.4. The existence and uniqueness of a solution of (3.3)–(3.4) in case $q=2$ follows from Theorem 3.2 of [29]. For the case $q=2$ by Bobrovnytska and Schweizer [2] proved the well-posedness of (3.3), (3.4) and representation 3.5 for the variance-optimal martingale measure, under general filtration. In [2] the uniqueness of a solution was proved for a class of processes $(Y(2),\psi(2),L(2))$ such that

$$\frac{1}{Y(2)}\mathcal{E}^2(M^Q)\in D\quad\text{for all}\quad Q\in\mathcal{M}_2^e\quad\text{and} \tag{3.12}$$

$$\mathcal{E}\Big(-\lambda\cdot M+\frac{1}{Y_-(2)}\cdot\tilde{L}(2)\Big)\in\mathcal{M}^2(P). \tag{3.13}$$

The same class was used in [26] to show the uniqueness of a solution of a SBE derived for $\widetilde{V}(2)=\frac{1}{V(2)}$ under an additional condition of the continuity of the filtration. Although the class D_2 (from Definition 3.1) as well, as the class of processes satisfying (3.12)–(3.13), include the class of processes satisfying the two-sided inequality (3.6), they are not comparable. Therefore the union of these classes (which needs a better description) should enlarge the class of uniqueness.

4. Connection of the Minimal Entropy and Q-Optimal Martingale Measures

In this section we study the dependence of solutions of the SBE (3.3) on the parameter q assuming that the filtration F is continuous. We show that if condition B^*) for some $q_0 > 1$ is satisfied, then the solution process $Y(q)$, the BMO_2-norm's square of the martingale part of $Y(q)$ and the entropy distance satisfy the uniform Lipschitz condition with respect to q, which implies the convergence of the densities of q-optimal martingale measures to the densities of the minimal entropy and minimal martingale measures as $q \to 1$ and $q \downarrow 0$, respectively.

Hereafter we suppose that condition C) is satisfied, i.e, we assume that any local martingale is continuous.

Let us first prove some auxiliary statements.

Lemma 4.1. *For all $q \in [0, \infty)$ the martingale part $\widehat{L}(q) = \psi(q) \cdot M + L(q)$ of any solution $Y(q) \in D$ of the equation* (3.3)*,* (3.4) *belongs to the class BMO_2 and*

$$\|\widehat{L}(q)\|_{BMO_2} \le 1, \quad \text{for all} \quad q \in [0, \infty). \tag{4.1}$$

Proof. If $Y(q) \in D$ then $(Y_t(q), t \in [0, T])$ is a submartingale and

$$Y_t(q) \le E(Y_T(q)/F_t) = 1. \tag{4.2}$$

Using the Itô formula for $Y_T^2(q) - Y_\tau^2(q)$ and the boundary condition $Y_T(q) = 1$, we have

$$\langle\widehat{L}(q)\rangle_T - \langle\widehat{L}(q)\rangle_\tau + 2\int_\tau^T Y_s(q)\psi_s(q)dM_s + 2\int_\tau^T Y_s(q)dL_s(q)$$
$$+q\int_\tau^T (\lambda_s Y_s(q) + \psi_s(q))^2 d\langle M\rangle_s = 1 - Y_\tau^2(q) \tag{4.3}$$

for any stopping time τ. Without loss of generality we may assume that $L(q)$ and $\psi \cdot M$ are square integrable martingales, otherwise one can use localization arguments. Therefore, if we take conditional expectations in (4.3) we obtain

$$E(\langle\widehat{L}(q)\rangle_T - \langle\widehat{L}(q)\rangle_\tau | F_\tau) + qE\bigg(\int_\tau^T (\lambda_s Y_s(q) + \psi_s(q))^2 d\langle M\rangle_s / F_\tau\bigg) \le 1 \tag{4.4}$$

for any stopping time τ, which implies that $\widehat{L}$ belongs to the space BMO_2 and the estimate (4.1) also holds. □

Lemma 4.2. *Let there exist a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$ and for any $0 \le q \le q_0$ let $Y(q)$ be a bounded positive solution of* (3.3)*,* (3.4)*. Then for any $0 \le q \le q_0, 0 \le q' \le q_0$*

$$|Y_t(q) - Y_t(q')| \le const\, |q - q'|. \tag{4.5}$$

Proof. By Theorem 3.1b we have $Y(q) = V(\frac{q}{q-1})$ and Proposition 3.3 with $R_q(P)$ condition imply that

$$Y_t(q) \ge C^{\frac{1}{1-q}} \quad \text{for all} \quad t \in [0, T] \quad \text{a.s.} \tag{4.6}$$

Moreover, it follows from Lemma 2.1 that

$$\inf_{0\le q\le q_0} Y_t(q) \ge c \quad \text{for all} \quad t\in[0,T] \quad \text{a.s.} \tag{4.7}$$

for a constant $c>0$.

If the triple $(Y(q),\psi(q),L(q))$ is a solution of (3.3), then using the Itô formula for $\ln Y_t(q)$ we have $(\bar Y(q),\bar\psi(q),\bar L(q))$, where $\bar Y(q)=\ln Y(q)$, $\bar\psi(q)=\frac{\psi(q)}{Y(q)}$, $\bar L(q)=\frac{1}{Y}\cdot L(q)$, satisfies the SBE

$$\bar Y_t(q)=\bar Y_0(q)+\int_0^t\left(\frac{q}{2}\lambda_s^2+q\lambda_s\bar\psi_s(q)+\frac{q-1}{2}\bar\psi_s^2(q)\right)d\langle M\rangle_s \\ -\frac12\langle\bar L(q)\rangle_t+\int_0^t\bar\psi_s(q)dM_s+\bar L_t(q),\quad t<T, \tag{4.8}$$

with the boundary condition

$$\bar Y_T=0. \tag{4.9}$$

Taking the difference of two equations (4.8) for q and q' we have

$$\bar Y_t(q)-\bar Y_t(q')=\bar Y_0(q)-\bar Y_0(q')+\frac{q-q'}{2}\int_0^t(\lambda_s+\bar\psi_s(q))^2d\langle M\rangle_s \\ +\int_0^t(\bar\psi_s(q)-\bar\psi_s(q'))\left(\frac{q'-1}{2}\,\bar\psi_s(q)+\frac{q'-1}{2}\,\bar\psi_s(q')+q'\lambda_s\right)d\langle M\rangle_s \\ -\frac12\langle\bar L(q)\rangle_t+\frac12\langle\bar L(q')\rangle_t+\int_0^t(\bar\psi_s(q)-\bar\psi_s(q'))dM_s+\bar L_t(q)-\bar L_t(q'). \tag{4.10}$$

For any $q,q'\in[0,q_0]$ let us define the measure $Q=Q(q,q')$ by $dQ=\mathcal{E}_T(N(q,q'))dP$, where

$$N=N(q,q')=\left(\frac{q'-1}{2}\,\bar\psi(q)+\frac{q'-1}{2}\,\bar\psi(q')+q'\lambda\right)\cdot M+\frac12\left(\bar L(q)+\bar L(q')\right).$$

Lemma 4.1 and inequality (4.6) imply that $N=N(q,q')$ is a BMO_2-martingale hence for any $q,q'\in[0,q_0]$, $Q=Q(q,q')$ is a probability measure equivalent to P.

Denote by $\bar N$ the martingale part of $\bar Y(q)-\bar Y(q')$, i.e.,

$$\bar N=\bar N(q,q')=(\bar\psi_s(q)-\bar\psi_s(q'))\cdot M+\bar L(q)-\bar L(q').$$

Therefore by the Girsanov theorem,

$$\bar Y(q)-\bar Y(q')-\frac{q-q'}{2}\,(\lambda-\bar\psi(q))^2\cdot\langle M\rangle=\bar N-\langle\bar N,N\rangle$$

is a local martingale with respect to the measure Q. Moreover, since $\bar N$ also belongs to the class BMO_2 (again by (4.6) and Lemma 4.1), according to Proposition 11 of [9]

$$\bar N-\langle\bar N,N\rangle\in BMO_2(Q)$$

for any fixed q and q' from $[0,q_0]$.

Thus, using the martingale property and the boundary condition $\bar{Y}_T(q) = 0$, we obtain

$$\bar{Y}(q) - \bar{Y}(q') = -\frac{q-q'}{2} E^Q\bigg(\int_t^T (\lambda_s - \bar{\psi}_s(q))^2 d\langle M\rangle_s / F_t\bigg),$$

or

$$\frac{Y(q)}{Y(q')} = e^{\frac{q'-q}{2} E^Q\left(\int_t^T (\lambda_s - \bar{\psi}_s(q))^2 d<M>_s / F_t\right)}. \tag{4.11}$$

Note that, this implies that $q \geq q' \Rightarrow Y_t(q) \leq Y_t(q')$ for any $q, q' \in [0, q_0]$, which gives the proof of Proposition 2.2 and its Corollary as a consequence of a special comparison assertion for SBEs.

Let us show that

$$E^{Q(q,q')}\bigg(\int_t^T (\lambda_s - \bar{\psi}_s(q))^2 d\langle M\rangle_s / F_t\bigg) \leq const, \tag{4.12}$$

where the constant does not depend on q and q'.

Since $\lambda \cdot M \in BMO_2$ (see Remark 2.1), inequality (4.7) and Lemma 4.1 imply that the BMO_2 norms of $N(q, q')$ and $(\lambda - \bar{\psi}(q)) \cdot M$ are uniformly bounded for $q, q' \leq q_0$

$$\sup_{0\leq q,q'\leq q_0} \|N(q,q')\|_{BMO_2} < \infty, \tag{4.13}$$

$$\sup_{0\leq q,q'\leq q_0} \|(\lambda - \bar{\psi}(q)) \cdot M\|_{BMO_2} < \infty. \tag{4.14}$$

According to Theorem 3.1 of Kazamaki [22], (4.13) implies that

$$\sup_{0\leq q,q'\leq q_0} E(\mathcal{E}^{\alpha}_{t,T}(N(q,q'))/F_t) < const \tag{4.15}$$

for some $\alpha > 1$. Taking n so that $\frac{n}{n-1} \leq \alpha$ and using the conditional Hölder inequality, we have

$$E^{Q(q,q')}\bigg(\int_t^T (\lambda_s - \bar{\psi}_s(q))^2 d\langle M\rangle_s / F_t\bigg)$$
$$\leq E^{\frac{n-1}{n}}(\mathcal{E}^{\frac{n}{n-1}}_{t,T}(N(q,q'))/F_t) E^{\frac{1}{n}}(\langle(\lambda - \bar{\psi}(q)) \cdot M\rangle^n_{t,T}/F_t). \tag{4.16}$$

On the other hand, it follows from the energy inequality (see [22, p. 28]) that

$$E(\langle(\lambda - \bar{\psi}(q)) \cdot M\rangle^n_{t,T}/F_t) \leq n!\|(\lambda - \bar{\psi}(q)) \cdot M\|^{2n}_{BMO_2}.$$

Therefore from (4.14), (4.15) and (4.16) we conclude that the estimate (4.12) holds.

Thus it follows from (4.11) and (4.12) that

$$\begin{aligned} 1 \leq \frac{Y(q)}{Y(q')} \leq e^{const\ (q'-q)} \quad \text{if} \quad q' \geq q, \\ 1 \geq \frac{Y(q)}{Y(q')} \geq e^{const\ (q'-q)} \quad \text{if} \quad q' \leq q. \end{aligned} \tag{4.17}$$

Since $Y_t(q)$ is uniformly bounded from below by a positive constant and decreasing with respect to q, we obtain from (4.17)

$$|Y_t(q) - Y_t(q')| \leq const\ |q - q'|.$$

□

Lemma 4.3. *Let there exist a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$ and for any $0 \le q \le q_0$ let $Y(q)$ be a bounded positive solution of* (3.3), (3.4). *Then for any $0 \le q \le q_0, 1 < q' \le q_0$*

$$\|L(q) - L(q')\|_{BMO_2} \le const\ |q - q'|^{\frac{1}{2}}, \tag{4.18}$$

where $L(q)$ is the martingale part of $Y(q)$.

Proof. Applying the Itô formula for $(Y_T(q) - Y_T(q'))^2 - (Y_t(q) - Y_t(q'))^2$ and the boundary condition $Y_T(q) = 1$, we have

$$\begin{aligned}\langle L(q) - L(q')\rangle_{tT} + 2\int_t^T (Y_s(q) - Y_s(q'))d(A_s(q) - A_s(q'))\\ + 2\int_t^T (Y_s(q) - Y_s(q'))(\varphi_s(q) - \varphi_s(q'))dM_s\\ + 2\int_t^T (Y_s(q) - Y_s(q'))d(L_s(q) - L_s(q')) \le 0.\end{aligned}$$

Since by Corollary of Proposition 2.2 $Y(q) \ge Y(q')$ if $q \le q'$, by taking conditional expectations we obtain (e.g., if $q \ge q'$)

$$\begin{aligned}E(\langle L(q) - L(q')\rangle_{tT}/F_\tau) \le 2E\bigg(\int_t^T (Y_s(q') - Y_s(q))dA_s(q)/F_t\bigg)\\ \le const\ (q - q')E(A_T(q) - A_t(q)/F_t),\end{aligned} \tag{4.19}$$

where the last inequality follows from Lemma 4.2. On the other hand, Lemma 2.1, Lemma 4.1 and the boundedness of $Y(q)$ imply that

$$E(A_T(q) - A_t(q)/F_t) = E\bigg(\int_t^T \frac{(\lambda_s Y_s(q) + \psi_s(q))^2}{Y_s(q)} d\langle M\rangle_s/F_t\bigg) \le const,$$

which (together with (4.19)) results in the estimate (4.18). □

Theorem 4.1. *Let there exist a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$. Then for any $0 \le q \le q_0$, $0 < q' \le q_0$*

$$\|M^{Q^*(q)} - M^{Q^*(q')}\|_{BMO_2} \le const\ |q - q'|^{\frac{1}{2}}. \tag{4.20}$$

Proof. From Theorem 3.1 we have

$$M^{Q^*(q)} - M^{Q^*(q')} = \frac{1}{Y_-(q)} \cdot L(q) - \frac{1}{Y_-(q')} \cdot L(q')$$

and

$$
\begin{aligned}
&E\Big(\Big\langle \frac{1}{Y_-(q)}\cdot L(q) - \frac{1}{Y_-(q')}\cdot L(q')\Big\rangle_{\tau,T}/F_\tau\Big)\\
&\quad\le 2E\Big(\Big\langle\Big(\frac{1}{Y_-(q)} - \frac{1}{Y_-(q')}\Big)\cdot L(q)\Big\rangle_{\tau,T}/F_\tau\Big)\\
&\qquad + 2E\Big(\Big\langle \frac{1}{Y_-(q)}\cdot (L(q)-L(q'))\Big\rangle_{\tau,T}/F_\tau\Big)\\
&\quad= 2E\Big(\int_\tau^T \frac{(Y_{s-}(q)-Y_{s-}(q'))^2}{Y_{s-}^2(q)Y_{s-}^2(q')}\,d\langle L(q)\rangle_s/F_\tau\Big)\\
&\qquad + 2E\Big(\int_\tau^T \frac{1}{Y_{s-}^2(q')}\,d\langle L(q)-L(q')\rangle_s/F_\tau\Big).
\end{aligned}
\tag{4.21}
$$

Since the process $Y_-(q)$ is uniformly bounded from bellow by a strictly positive constant, it follows from Lemma 4.2 that the right-hand side of (4.21) is estimated by

$$
const\ ((q-q')^2\|L(q)\|^2_{BMO_2} + \|L(q)-L(q')\|^2_{BMO_2}).
$$

Thus Lemma 4.1 and Lemma 4.3 results in (4.20). □

Corollary 4.1. *Let there exists a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$. Then*

$$
\begin{aligned}
&\|\mathcal{E}(M^{Q^*(q)}) - \mathcal{E}(M^{Q(E)})\|_{H^1} \to 0 \quad \text{as} \quad q\to 1,\\
&\|\mathcal{E}(M^{Q^*(q)}) - \mathcal{E}(-\lambda\cdot M)\|_{H^1} \to 0 \quad \text{as} \quad q\downarrow 0,
\end{aligned}
\tag{4.22}
$$

where $Q(E)=Q^*(1)$ is the minimal entropy martingale measure.

The proof follows from Theorem 4.2 and from Theorem 3.2 of Kazamaki [22].

Theorem 4.2. *Let there exists a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$. Then for any $0\le q\le q_0$, $0<q'\le q_0$*

$$
I(Q^*(q),Q^*(q')) \le const\ |q-q'|. \tag{4.23}
$$

Proof. By the definition of the entropy distance we have

$$
\begin{aligned}
I\big(Q^*(q),Q^*(q')\big) &= E\mathcal{E}(M^{Q^*(q)})\ \ln\frac{\mathcal{E}(M^{Q^*(q)})}{\mathcal{E}(M^{Q^*(q')})}\\
&= E^{Q^*(q)}\Big[M_T^{Q^*(q)} - \frac{1}{2}\langle M^{Q^*(q)}\rangle_T - M_T^{Q^*(q')} + \frac12\langle M^{Q^*(q')}\langle_T\Big].
\end{aligned}
$$

Since $M^{Q^*(q)}\in BMO_2$, by Proposition 11 of [9] the processes

$$
M^{Q^*(q)} - \langle M^{Q^*(q)}\rangle \quad\text{and}\quad M^{Q^*(q')} - \langle M^{Q^*(q)}, M^{Q^*(q')}\rangle
$$

are BMO_2-martingales with respect to the measure $Q^*(q)$. Therefore

$$
I\big(Q^*(q),Q^*(q')\big) = \frac12 E^{Q^*(q)}\langle M^{Q^*(q)} - M^{Q^*(q')}\rangle_T. \tag{4.24}
$$

Since $\lambda \cdot M \in BMO_2$ (see Remark 2.1), inequality (4.7) and Lemma 4.1 imply that the BMO_2 norms of $M^{Q^*(q)}$ are uniformly bounded for $q, q' \le q_0$. Therefore Theorem 3.1 of Kazamaki [22] implies that

$$\sup_{0 \le q, q' \le q_0} E(\mathcal{E}_T^{\alpha}(M^{Q^*(q)})) < \infty \tag{4.25}$$

for some $\alpha > 1$. Taking n so that $\frac{n}{n-1} \le \alpha$ and using (4.25), the Hölder inequality and the energy inequality (see [22, p. 28]) we have that

$$\begin{aligned} &E^{Q^*(q)} \langle M^{Q^*(q)} - M^{Q^*(q')} \rangle_T \\ &\le E^{\frac{n-1}{n}} \mathcal{E}_T^{\frac{n}{n-1}}(M^{Q^*(q)}) E^{\frac{1}{n}} \langle M^{Q^*(q)} - M^{Q^*(q')} \rangle_T^n \\ &\le const \, \|M^{Q^*(q)} - M^{Q^*(q')}\|_{BMO_2}^2 \le const \, |q - q'|, \end{aligned} \tag{4.26}$$

where the last inequality follows from Theorem 4.2.

Therefore from (4.24) and (4.26) it follows that the estimate (4.23) holds. □

Corollary 4.2. *Let there exists a martingale measure that satisfies the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$. Then*

$$\lim_{q \downarrow 1} Q^*(q) = Q(E) \quad \textit{in entropy.} \tag{4.27}$$

Remark 4.1. Corollary 4.2 was proved in [17] without assumption of the continuity of the filtration and under assumption that the Reverse Hölder $R_{LlnL}(P)$ inequality is satisfied. As proved in the same paper, the Reverse Hölder $R_{LlnL}(P)$ inequality implies that the Reverse Hölder $R_q(P)$ inequality for some $q_0 > 1$ is satisfied.

Acknowledgments

We would like to thank N. Lazrieva for useful remarks.

The research was supported by INTAS Grant No. 99–00559 and by Grant of Georgian Academy of Sciences No. 10.1.

References

1. J. M. Bismut, Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44**(1973), 384–404.
2. O. Bobrovnytska and M. Schweizer, Mean-variance hedging and stochastic control: Beyend the Brownian setting. *Preprint,* 2002.
3. R. Chitashvili, Martingale ideology in the theory of controlled stochastic processes. *Probability theory and mathematical statistics* (*Tbilisi,* 1982), 73–92, *Lecture Notes in Math.,* 1021, *Springer, Berlin,* 1983.
4. R.Chitashvili and M. Mania, Optimal locally absolutely continuous change of measure. Finite set of decisions, I, II. *Stochastics* **21**(1987), No. 2, 131–185; **21**(1987), No. 3, 187–229.
5. F. Delbaen and W. Schachermayer, Variance-optimal martingale measure for continuous processes. *Bernoulli* **2**(1996), No. 1, 81-105.

6. F. Delbaen, P. Monat, W. Schachermayer, W. Schweizer, and C. Stricker, Weighted norm inequalities and hedging in incomplete markets. *Finance Stoch.* **1**(1997), 781–227.

7. F. Delbaen, P. Grandits, Th. Rheinlände, D. Samperi, W. Schweizer, and Ch. Stricker, Exponential hedging and entropic penalties. *Math. Finance* **12**(2002), No. 2, 99–123.

8. C. Dellacherie and P. A. Meyer, Probabilités et potentiel. Chapitres V à VIII. Th´rie des martingales. Revised edition. *Actualités Scientifiques et Industrielles,* 1385. *Hermann, Paris,* 1980.

9. K. Doleans-Dade and P. A. Meyer, Inégalités de normes avec poinds, Séminaire de Probabilités XIII. *Lecture Notes in Math.,* 721, 204–215. *Springer, Berlin,* 1979.

10. R. J. Elliott, Stochastic calculus and applications. *Springer, New York,* 1982.

11. N. El Karoui, S. Peng, and M. C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* **7**(1997), No. 1, 1–71.

12. H. Föllmer and M. Schweizer, Hedging of contingent claims under incomplete information. *Applied stochastic analysis* (*London,* 1989), 389–414, *Stochastics Monogr.,* 5, *Gordon and Breach, New York,* 1991.

13. M. Frittelli, The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance* **10**(2000), No. 1, 39–52.

14. C. Gourieroux, J. P. Laurent, and H. Pham, Mean-variance hedging and numeraire. *Math. Finance* **8**(1998), No. 3, 179–200.

15. T. Goll and L. Rüschendorf, Minimax and minimal distance martingale measures and their relationship to portfolio pptimization. *Finance Stoch.* **5**(2001), 557–581.

16. P. Grandits and L. Krawczyk, Closeness of some spaces of stochastic integrals. *Séminaire de Probabilités,* XXXII, 73–85, *Lecture Notes in Math.,* 1686, *Springer, Berlin,* 1998.

17. P. Grandits and T. Rheinländer , On the minimal entropy martingale measures. *Ann. Probab.* **30**(2002), No. 3, 1003–1038.

18. P. Grandits, The p-optimal martingale measure and its asymptotic relation with the minimal entropy martingale measure. *Bernoulli* **5**(1999), No. 2, 225–247.

19. D. Hobson, Stochastic volatility models, correlation and the q-optimal measure. *Preprint,* 2002.

20. D. Heath, E. Platen, and M. Schweizer, A comparision of two quadratic approaches to hedging in incomplete markets. *Math. Finance* **11**(2001), No. 4, 385–413.

21. J. Jacod, Calcul stochastique et problémes de martingales. *Lecture Notes in Mathematics,* 714. *Springer, Berlin,* 1979.

22. N. Kazamaki, Continuous exponential martingales and BMO. *Lecture Notes in Mathematics,* 1579. *Springer-Verlag, Berlin,* 1994.

23. M. Kobylanski, Backward stochastic differential equation and partial differential equations with quadratic growth. *Ann. Probab.* **28**(2000), No. 2, 558–602.

24. J. P. Laurent and H. Pham, Dynamic programming and mean-variance hedging. *Finance Stoch.* **3**(1999), 83–110.

25. R. Sh. Liptser and A. N. Shiryayev, Martingale theory. (Russian) *Nauka, Moscow,* 1986.

26. M. Mania and R. Tevzadze, A Semimartingale Bellman equation and the variance-optimal martingale measure. *Georgian Math. J.* **7**(2000), No. 4, 765–792.

27. M. Mania, M. Santacroce, and R. Tevzadze, A Semimartingale backward equation related to the p-optimal martingale measure and the minimal price of contingent claims. *Stoch. Processes and Related Topics, Stochastics Monogr.*, 12, 169–202. *Taylor & Francis*, 2002.
28. M. Mania, M. Santacroce, and R. Tevzadze, A Semimartingale BSDE related to the minimal entropy martingale measure. *Finance Stoch.* **7**(2003), No. 3, 385–402.
29. M. Mania and R. Tevzadze, Backward stochastic PDE and hedging in incomplete markets. *Proc. A. Razmadze Math. Inst.* **130**(2002), 39–72.
30. M. Mania and R. Tevzadze, A Semimartingale Bellman equation and the variance-optimal martingale measure under general information flow. *Tech. Report, Friedrich-Schiller-Universität Jena,* Math/Inf/03, 2002. Submitted to *SIAM Journal on Control and Opt.*, 24 pages.
31. Y. Miyahara, Canonical Martingale measures of incomplete assets markets. *Probability theory and mathematical statistics* (*Tokyo,* 1995), 343–352, *World Sci. Publishing, River Edge, NJ,* 1996.
32. E. Pardoux and S. G. Peng, Adapted solution of a backward stochastic differential equation. *Systems Control Lett.* **14**(1990), No. 1, 55–61.
33. H. Pham, T. Rheinländer, and M. Schweizer, Mean-variance hedging for continuous processes: new proofs and examples. *Finance Stoch.* **2**(1998), No. 2, 173–198.
34. R. Rouge and N. El Karoui, Pricing via utility maximization and entropy. *Math. Finance* **10**(2000), No. 2, 259–276.
35. W. Schachermayer, A super-martingale property of the optimal portfolio process. *Preprint,* 2002.
36. M. Santacroce, On the convergence of the p-optimal martingale measures to the minimal entropy martingale measure. *Technical Report of the Universitá del Piemonte Orientale "A. Avogadro",* 2002.
37. M. Schweizer, Mean-variance hedging for general claims. *Ann. Appl. Probab.* **2**(1992), 171–179.
38. M. Schweizer, Approximation pricing and the variance optimal martingale measure. *Ann. Probab.* **24**(1996), 206–236.

(Received 15.12.2002)

Authors' addresses:

M. Mania
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0143
Georgia
E-mail: mania@rmi.acnet.ge

R. Tevzadze
Institute of Cybernetics
Georgian Academy of Sciences
1, S. Euli St., Tbilisi 0186
Georgia