GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 2, 1995, 165-188

ON SOME BOUNDARY VALUE PROBLEMS WITH INTEGRAL CONDITIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. For the functional differential equation $u^{(n)}(t) = f(u)(t)$ we have established the sufficient conditions for solvability and unique solvability of the boundary value problems

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \qquad \int_0^{+\infty} |u^{(m)}(t)|^2 dt < +\infty$$

and

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1),$$
$$\int_0^{+\infty} t^{2j} |u^{(j)}(t)|^2 dt < +\infty \quad (j = 0, \dots, m),$$

where $n \geq 2$, m is the integer part of $\frac{n}{2}$, $c_i \in R$, and f is the continuous operator acting from the space of $(n-1)$-times continuously differentiable functions given on an interval $[0, +\infty[$ into the space of locally Lebesgue integrable functions.

§ 1. FORMULATION OF THE EXISTENCE AND UNIQUENESS THEOREMS

Let $n \geq 2$ and f be a continuous operator acting from the space of $(n-1)$-times continuously differentiable functions given on an interval $R_+ = [0, +\infty[$ into the space of locally Lebesgue integrable functions given on the same interval. Consider the functional differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1.1}$$

by whose solution we shall understand a function $u : R_+ \to R$ which is locally absolutely continuous with its derivatives up to order $n-1$ inclusive

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1991 *Mathematics Subject Classification.* 34K10.
Key words and phrases. Functional differential equation, boundary value problem, integral condition.

1072-947X/95/0300-0165$07.50/0

and satisfies (1.1) almost everywhere on R_+. In this paper we shall be concerned with the problems of the existence and uniqueness of a solution of equation (1.1) satisfying either of the two boundary conditions

$$u^{(i)}(0)=c_i \quad (i=0,\dots,m-1), \quad \int_0^{+\infty} |u^{(m)}(t)|^2 dt<+\infty \tag{1.2}$$

and

$$\begin{gathered} u^{(i)}(0)=c_i \quad (i=0,\dots,m-1), \\ \int_0^{+\infty} t^{2j}|u^{(j)}(t)|^2 dt<+\infty \quad (j=0,\dots,m). \end{gathered} \tag{1.3}$$

For the case $f(u)(t)=g(t,u(t),\dots,u^{(n-1)}(t))$ problems of type (1.1), (1.2) and (1.1), (1.3), as well as their closely related problems of the existence of so-called proper oscillatory and vanishing-at-infinity solutions of the equation

$$u^{(n)}=g\big(t,u(t),\dots,u^{(n-1)}(t)\big),$$

have been studied with a sufficient thoroughness (see [1,2,3] and §§4 and 14 in the monograph [4]). As to the general case, the above-mentioned problems were not previously investigated.

In this paper we establish the sufficient conditions for the solvability and unique solvability of problems (1.1), (1.2) and (1.1), (1.3). In [5] these results are specified for a differential equation with deviating arguments of the form

$$u^{(n)}(t)=g\big(t,u(\tau_0(t)),\dots,u^{(m-1)}(\tau_{m-1}(t))\big),$$

and criteria are found for the existence of a multiparameter family of vanishing-at-infinity proper oscillatory solutions of the above equation.

The following notation will be used throughout the paper.

$C^{n-1}([t_1,t_2])$ and $L([t_1,t_2])$ are respectively the spaces of $(n-1)$-times continuously differentiable and Lebesgue integrable real functions given on the segment $[t_1,t_2]$.

$\widetilde{C}^{n-1}$ is the space of functions $u: R_+\to R$ which are locally absolutely continuous (i.e., absolutely continuous on each finite interval from R_+) together with their derivatives up to order $n-1$ inclusive.

C^{n-1} is a topological space of $(n-1)$-times continuously differentiable real functions given on R_+, where by the convergence of the sequence $(u_k)_{k=1}^{+\infty}$ we understand the uniform convergence of sequences $(u_k^{(i)})_{k=1}^{+\infty}$ $(i=0,\dots,n-1)$ on each finite interval from R_+.

$$C_0^{n-1,m}=\Big\{u\in C^{n-1}: \int_0^{+\infty} |u^{(m)}(t)|^2 dt<+\infty\Big\},$$

$$C^{n-1,m}=\Big\{u\in C^{n-1}:\int_0^{+\infty}t^{2j}|u^{(i)}|^2dt<+\infty\ (i=0,\dots,m)\Big\}.$$

If $u\in C_0^{n-1,m}$, then

$$\|u\|_{0,m}=\Big[\sum_{i=0}^{m-1}|u^{(i)}(0)|^2+\int_0^{+\infty}|u^{(m)}(s)|^2ds\Big]^{1/2};$$

if, however, $u\in C^{n-1,m}$, then

$$\|u\|_m=\Big[\int_0^{+\infty}(1+s)^{2m}|u^{(m)}(s)|^2ds\Big]^{1/2}.$$

L is the space of locally Lebesgue integrable functions $v:R_+\to R$ with the topology of convergence in the mean on each finite interval from R_+.

μ_i^k $(i=0,1,\dots;\ k=2i,2i+1,\dots)$ are the numbers given by the recurrent relations

$$\mu_0^{i+1}=\frac{1}{2},\quad \mu_i^{2i}=1,\quad \mu_{i+1}^k=\mu_{i+1}^{k-1}+\mu_i^{k-2}\quad (k=2i+3,\dots).$$

$$\gamma_n=0\quad\text{for}\quad n\le 3,$$

$$\gamma_n=\sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\quad\text{for}\quad n\ge 4.$$

where m_0 is the integer part of the number $\frac{n}{4}$.

In the sequel it will always be assumed that $f:C^{n-1}\to L$ is a continuous operator.

Theorem 1.1. *Let for any $u\in C_0^{n-1,m}$ the inequalities*

$$(-1)^{n-m-1}u(t)f(u)(t)\ge -a_1(t)\|u\|_{0,m}^2-a_2(t),\tag{1.4}$$

$$|f(u)(t)|\le b(t,|u(t)|,\|u\|_{0,m})\tag{1.5}$$

hold almost everywhere on R_+, where $a_i:R_+\to R_+$ $(i=1,2)$ are measurable functions such that

$$\begin{aligned}&\int_0^{+\infty}(1+t)^{n-2m}a_1(t)dt<\mu_m^n,\\ &\int_0^{+\infty}(1+t)^{n-2m}a_2(t)dt<+\infty,\end{aligned}\tag{1.6}$$

and the function $b:R_+^3\to R_+$ is locally summable with respect to the first argument, nondecreasing with respect to the last two arguments and

$$\lim_{\substack{t\to 0\\ y\to+\infty}}\Big(y^{-2}\int_0^t b(s,x,y)ds\Big)=0\quad\textit{for}\quad x\in R_+.\tag{1.7}$$

Then problem (1.1), (1.2) *has at least one solution.*

Theorem 1.2. *Let for any* u *and* $\overline{u} \in C_0^{n-1,m}$ *the inequality*

$$(-1)^{n-m-1}(u(t)-\overline{u}(t))(f(u)(t)-f(\overline{u})(t)) \geq -a(t)\|u-\overline{u}\|_{0,m}^2 \quad (1.8)$$

hold almost everywhere on R_+, *where* $a : R_+ \to R_+$ *is a measurable function such that*

$$\int_0^{+\infty}(1+t)^{n-2m}a(t)dt < \mu_m^n. \quad (1.9)$$

Then problem (1.1), (1.2) *has at most one solution.*

Theorem 1.3. *Let for any* $u \in C^{n-1,m}$ *the inequalities*

$$(-1)^{n-m-1}u(t)f(u)(t) \geq$$
$$\geq \gamma(1+t)^{-n}|u(t)|^2 - a_1(t)\|u\|_m^2 - a_2(t), \quad (1.10)$$
$$|f(u)(t)| \leq b(t,|u(t)|,\|u\|_m) \quad (1.11)$$

hold almost everywhere on R_+, *where* γ *and* $a_i : R_+ \to R_+$ $(i=1,2)$ *are a positive number and measurable functions such that*

$$\begin{gathered}\int_0^{+\infty}(1+t)^n a_i(t)dt < +\infty \quad (i=1,2),\\ \delta = \frac{n!}{(2m)!}\mu_m^n - \int_0^{+\infty}(1+t)^n a_1(t)dt > 0,\end{gathered} \quad (1.12)$$

$$\gamma + (-1)^m \,\frac{n!}{2} >$$
$$> \frac{m-1}{4}\gamma_n\Big[\frac{\gamma_n}{\delta} + \frac{(m-2)(4m^2-m+3)}{3} + 4\Big]^{m-1}, \quad (1.13)$$

and the function $b : R_+^3 \to R_+$ *is locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition* (1.7). *Then problem* (1.1), (1.3) *has at least one solution.*

Theorem 1.4. *Let for any* u *and* $\overline{u} \in C^{n-1,m}$ *the inequality*

$$(-1)^{n-m-1}(u(t)-\overline{u}(t))(f(u)(t)-f(\overline{u})(t)) \geq$$
$$\geq \gamma(1+t)^{-n}(u(t)-\overline{u}(t))^2 - a(t)\|u-\overline{u}\|_m^2 \quad (1.14)$$

hold almost everywhere on R_+, *where* γ *and* $a : R_+ \to R_+$ *are a positive number and a measurable function such that*

$$\delta = \frac{n!}{(2m)!}\mu_m^n - \int_0^{+\infty}(1+t)^n a(t)dt > 0 \quad (1.15)$$

and inequality (1.13) *is fulfilled. Then problem* (1.1), (1.3) *has at most one solution.*

§ 2. AUXILIARY STATEMENTS

2.1. Operator σ_k. For an arbitrary $i \in \{0, \dots, n-1\}$ we denote by v_i the polynomial of degree not higher than $2n-1$, satisfying the boundary conditions

$$v_i^{(j)}(0) = \delta_{ij}, \quad v_i^{(j)}(1) = 0 \quad (j = 0, \dots, n-1),$$

where δ_{ij} is the Kronecker symbol. Let v^* be the maximal value among maxima of functions $|v_i^{(j)}|$ $(i, j = 0, \dots, n-1)$ on the segment $[0,1]$. For any natural k and function $u \in C^{n-1}([0,k])$ we set

$$\varepsilon_k(u) = (k+2)^{-n}\Big[1 + \sum_{j=0}^{n-1} |u^{(j)}(k)|\Big]^{-2} (v^*)^{-2},$$

$$\sigma_k(u)(t) = \begin{cases} u(t) \quad \text{for} \quad 0 \le t \le k \\ \displaystyle\sum_{i=0}^{n-1} \varepsilon_k^i(u) v_i\Big(\frac{t-k}{\varepsilon_k(u)}\Big) u^{(i)}(k) \quad \text{for} \quad k < t < k + \varepsilon_k(u) \\ 0 \quad \text{for} \quad t \ge k + \varepsilon_k(u) \end{cases}.$$

Lemma 2.1. *For any natural k the operator $\sigma_k : C^{n-1}([0,k]) \to C^{n-1}$ is continuous and for any function $u \in C^{n-1}([0,k])$ satisfying the conditions*

$$u^{(i)}(k) = 0 \quad (i = 0, \dots, m-1) \tag{2.1}$$

we have

$$0 \le \|\sigma_k(u)\|_{0,m}^2 - \sum_{i=0}^{m-1} |u^{(i)}(0)|^2 - \int_0^k |u^{(m)}(s)|^2 ds \le 1, \tag{2.2}$$

$$0 \le \|\sigma_k(u)\|_m^2 - \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds \le 1. \tag{2.3}$$

Proof. The continuity of the operator σ_k is obvious. We shall prove the validity of inequality (2.2). In view of (2.1)

$$0 \le \|\sigma_k(u)\|_{0,m}^2 - \Big(\sum_{i=0}^{m-1} |u^{(i)}(0)|^2\Big) - \int_0^k |u^{(m)}(s)|^2 ds =$$

$$= \int_k^{k+\varepsilon_k(u)} \Big[\sum_{i=m}^{n-1} \varepsilon_k^{i-m}(u) v_i^{(m)}\Big(\frac{s-k}{\varepsilon_k(u)}\Big) u^{(i)}(k)\Big]^2 ds \le$$

$$\leq \varepsilon_k(u)(v^*)^2\Big[\sum_{i=m}^{n-1}|u^{(i)}(k)|\Big]^2 < 1.$$

Inequality (2.3) is proved likewise. □

2.2. Lemmas on A Priori Estimates. In this subsection we shall derive a priori estimates of the function $u \in \widetilde{C}^{n-1}$ which for some natural k satisfy either of the two systems of differential inequalities

$$|u^{(n)}(t)| \leq b_0(t, |u(t)|, \|\sigma_k(u)\|_{0,m}), \quad (-1)^{n-m-1}u^{(n)}(t)u(t) \geq \\ \geq -a_1(t)\|\sigma_k(u)\|^2_{0,m} - a_2(t) \quad \text{for} \quad 0 \leq t \leq k \tag{2.4}$$

and

$$|u^{(n)}(t)| \leq b_0(t, |u(t)|, \|\sigma_k(u)\|_{m}), \quad (-1)^{n-m-1}u^{(n)}(t)u(t) \geq \\ \geq \gamma(1+t)^{-n}|u(t)|^2 - a_1(t)\|\sigma_k(u)\|^2_{m} - a_2(t) \quad \text{for} \quad 0 \leq t \leq k \tag{2.5}$$

and the boundary conditions

$$\begin{gathered} |u^{(i)}(0)| \leq \rho_0 \quad (i = 0, \dots, m-1), \\ u^{(j)}(k) = 0 \quad (j = 0, \dots, n-m-1), \end{gathered} \tag{2.6}$$

where $\rho_0 > 0$ and m is the integer part of the number $\frac{n}{2}$.

Along with the above-mentioned numbers μ_i^k we shall introduce the numbers μ_{ij}^k $(i = 0, 1, \dots;\ j = i, i+1, \dots;\ k = i+j+1, i+j+2, \dots)$ using the following recurrent relations:

$$\mu_{00}^k = \frac{1}{2}, \quad \mu_{0j}^k = 1 \ \ (k = 1, 2, \dots;\ \ j = 1, \dots, k-1),$$

$$\mu_{ii}^{2i+1} = \frac{1}{2}, \quad \mu_{i\,k-i-1}^{k} = 1 \quad (i = 1, 2, \dots;\ k = 2i+2, 2i+3, \dots),$$

$$\mu_{ij}^k = \mu_{ij}^{k-1} + \mu_{i-1\,j-1}^{k-2}$$

$$(i = 1, 2, \dots;\ j = i, i+1, \dots;\ k = i+j+2, i+j+3, \dots).$$

The following three lemmas are proved in [4] (see Lemmas 4.1-4.3 and 5.1).

Lemma 2.2. *If the functions u and $w : [0, t_0] \to R$ are absolutely continuous together with their derivatives up to order $n-1$ inclusive, then*

$$\int_0^{t_0} w(t)u(t)u^{(n)}(t)dt =$$

$$= \sum_{i=0}^{n-m-1}\ \sum_{j=i}^{n-1-i} (-1)^{n-1-j}\mu_{ij}^n\big(w^{(n-1-i-j)}(t_0)u^{(i)}(t_0)u^{(j)}(t_0) - \\ - w^{(n-1-i-j)}(0)u^{(i)}(0)u^{(j)}(0)\big) +$$

$$+\sum_{i=0}^{m}(-1)^{n-i}\mu_i^n\int_0^{t_0}w^{(n-2i)}(t)|u^{(i)}(t)|^2dt.$$

Lemma 2.3. *Let $m\geq 2$, $t_0>0$, $r_0\geq 0$ and the function $u:[0,t_0]\to R$ be m-times continuously differentiable and*

$$(1+t_0)^{2i}u^{(i)}(t_0)u^{(i-1)}(t_0)-i(1+t_0)^{2i-1}|u^{(i-1)}(t_0)|^2-$$
$$-u^{(i)}(0)u^{(i-1)}(0)+i|u^{(i-1)}(0)|^2\leq r_0\quad(i=1,\dots,m-1).$$

Then for any $\eta>\frac{1}{3}(m-2)(4m^2-m+3)$ the estimates

$$\int_0^{t_0}(1+t)^{2i}|u^{(i)}(t)|^2dt\leq 2\eta^{m-2}r_0+\alpha_i(\eta)\int_0^{t_0}|u(t)|^2dt+$$
$$+\beta_i(\eta)\int_0^{t_0}(1+t)^{2m}|u^{(m)}(t)|^2dt\quad(i=1,\dots,m-1),$$

hold, where

$$\begin{gathered}\alpha_1(\eta)=(m-1)\big(1+\frac{\eta}{4}\big),\\ \alpha_i(\eta)=(m-i)\big(1+\frac{\eta}{4}\big)\prod_{j=1}^{i-1}\Big(\eta-\frac{(j-1)(4j^2+7j+6)}{3}\Big)\\ (i=2,\dots,m-1),\\ \beta_i(\eta)=\prod_{j=i}^{m-1}\Big(\eta-\frac{(j-1)(4j^2+7j+6)}{3}\Big)^{-1}\quad(i=1,\dots,m-1).\end{gathered}\tag{2.7}$$

Lemma 2.4. *Let $\delta>0$ and the function $u:[0,\delta]\to R$ be i-times continuously differentiable. Then there exists a point $t^*\in[0,\delta]$ such that*

$$|u^{(i)}(t^*)|\leq(i+1)!(2i+1)^{i+\frac{1}{2}}\delta^{-\frac{1}{2}-i}\Big(\int_0^{\delta}|u(t)|^2dt\Big)^{\frac{1}{2}}.$$

Lemma 2.5. *Let $r_1>0$ and the function $b_0:[0,1]\times R_+^2\to R_+$ be summable with respect to the first argument, nondecreasing with respect to the last two arguments and*

$$\lim_{\substack{t\to 0\\ y\to+\infty}}\Big(y^{-2}\int_0^t b_0(s,x,y)ds\Big)=0\quad\text{for}\quad x\in R_+.\tag{2.8}$$

Then there exists a positive number r_2 such that any function $u:[0,1]\to R$, absolutely continuous together with its derivatives up to order $n-1$ inclusive and satisfying the inequalities

$$\int_0^1 |u^{(n)}(s)u(s)|ds + \int_0^1 |u^{(m)}(s)|^2 ds \le r_1\Big(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)|\Big), \tag{2.9}$$

$$|u^{(n)}(t)| \le b_0\Big(t, |u(t)|, r_1^{\frac{1}{2}} \Big(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)|\Big)^{\frac{1}{2}}\Big) \quad \textit{for} \quad 0 \le t \le 1, \tag{2.10}$$

admits the estimate

$$|u^{(i)}(0)| \le r_2 \quad (i = m, \dots, n-1). \tag{2.11}$$

Proof. By virtue of condition (2.8) there exist numbers $\delta \in]0, 1]$ and $y_0 \in R_+$ such that

$$\int_0^\delta b_0(s, 8nr_1, y)ds \le \frac{1}{8nr_1} y^2 \quad \text{for} \quad y \ge y_0,$$

and therefore

$$\int_0^\delta b_0(s, 8nr_1, y)ds \le \le \int_0^\delta b_0(s, 8nr_1, y_0)ds + \frac{1}{8nr_1} y^2 \quad \text{for} \quad y \ge 0. \tag{2.12}$$

According to Lemma 2.4 there exist point $t_i \in [0, \delta]$ $(i = m, \dots, n-1)$ such that

$$|u^{(i)}(t_i)| \le (i - m + 1)!(2i - 2m + 1)^{i-m+\frac{1}{2}} \delta^{-\frac{1}{2}-i+m} \Big(\int_0^\delta |u^{(m)}(s)|^2 ds\Big)^{\frac{1}{2}} \quad (i = m, \dots, n-1).$$

Therefore from the equalities

$$u^{(i)}(t) = u^{(i)}(t_i) + \int_{t_i}^t u^{(i+1)}(s)ds \quad (i = m, \dots, n-1)$$

we find

$$|u^{(i)}(t)| \le \sum_{j=i}^{n-1} \delta^{j-i} |u^{(j)}(t_j)| + \delta^{n-1-i} \int_0^\delta |u^{(n)}(s)|ds \le \le n^n \delta^{-n} \Big(\int_0^1 |u^{(m)}(s)|^2 ds\Big)^{\frac{1}{2}} + \int_0^\delta |u^{(n)}(s)|ds \quad \text{for} \quad 0 \le t \le \delta \quad (i = m, \dots, n-1).$$

Hence by (2.9) it follows that

$$\sum_{i=m}^{n-1}|u^{(i)}(0)| \leq n^{1+n}\delta^{-n}\Big(\int_0^1 |u^{(m)}(s)|^2 ds\Big)^{\frac{1}{2}} + n\int_0^\delta |u^{(n)}(s)|ds \leq$$
$$\leq n^{2+2n}\delta^{-2n}r_1 + \frac{1}{4r_1}\int_0^1 |u^{(m)}(s)|^2 ds + n\int_0^\delta |u^{(n)}(s)|ds \leq$$
$$\leq n^{2+2n}\delta^{-2n}r_1 + \frac{1}{4}\Big(1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big) + n\int_0^\delta |u^{(n)}(s)|ds. \tag{2.13}$$

Let I_1 and I_2 be respectively the set of points of the interval $[0,\delta]$ at which the inequalities $|u(t)| \leq 8nr_1$ and $|u(t)| > 8nr_1$ hold. Then on account of inequalities (2.9), (2.10), and (2.12) we have

$$\int_0^\delta |u^{(n)}(s)|ds = \int_{I_1} |u^{(n)}(s)|ds + \int_{I_2} |u^{(n)}(s)|ds \leq$$
$$\leq \int_{I_1} b_0\bigg(s, 8nr_1, r_1^{\frac{1}{2}}\Big(1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big)^{\frac{1}{2}}\bigg)ds + \frac{1}{8nr_1}\int_{I_2}|u^{(n)}(s)u(s)|ds \leq$$
$$\leq \int_0^\delta b_0(s,8nr_1,y_0)ds + \frac{1}{4n}\Big(1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big).$$

Taking the latter inequality into account, from (2.13) we obtain

$$\sum_{i=m}^{n-1}|u^{(i)}(0)| \leq n^{2+2n}\delta^{-2n}r_1 + n\int_0^\delta b_0(s,8nr_1,y_0)ds + \frac{1}{2}\Big(1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big).$$

This immediately implies estimates (2.11), where

$$r_2 = 2n^{2+2n}\delta^{-2n}r_1 + 2n\int_0^\delta b_0(s,8nr_1,y_0)ds + 1$$

is the positive number independent of u. $\square$

Lemma 2.6. *Let $\rho_0 > 0$, the functions $a_i : R_+ \to R_+$ $(i = 1,2)$ be measurable and satisfy conditions* (1.6)*, and the function $b_0 : R_+^3 \to R_+$ be locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition* (2.8)*. Then there exists a positive number r such that any function $u \in \widetilde{C}^{n-1}$ satisfying for some natural k conditions* (2.4) *and* (2.6) *admits the estimates*

$$\int_0^k |u^{(m)}(s)|^2 ds + \int_0^k (1+s)^{n-2m}|u^{(n)}(s)u(s)|ds \leq r, \tag{2.14}$$
$$|u^{(i)}(t)| \leq r(1+t)^{m-i-\frac{1}{2}} \quad \text{for} \quad 0 \leq t \leq k \quad (i = 0,\dots,m-1), \tag{2.15}$$

$$|u^{(i)}(t)| \le r(1+t)^{n-i-1} + \int_0^t (t-s)^{n-i-1} b_0\big(s, r(1+s)^{m-\frac{1}{2}}, r\big)ds$$
$$\text{for} \quad 0 \le t \le k \quad (i = m, \dots, n-1). \tag{2.16}$$

Proof. By virtue of (1.6)

$$\varepsilon = \mu_m^n - \int_0^{+\infty} (1+s)^{n-2m} a_1(s)ds > 0, \tag{2.17}$$

$$\rho_1 = \Big(1 + m\rho_0^2\Big)\mu_m^n + \int_0^{+\infty} (1+s)^{n-2m} a_2(s)ds < +\infty. \tag{2.18}$$

For

$$r_1 = (\varepsilon^{-1} + \varepsilon^{-1}\mu_m^n + 2)\rho_1 + (\varepsilon^{-1} + \varepsilon^{-1}\mu_m^n + 1)m\rho_0 + 1 + m^2\rho_0^2$$

we shall choose $r_2 > 0$ such that the conclusion of Lemma 2.5 is valid and put $r = r_1(1 + r_2 + mr_2) + m\rho_0$.

Let $u \in \widetilde{C}^{n-1}$ be an arbitrary function satisfying, for some natural k, inequalities (2.4) and (2.6). Then

$$(-1)^{n-m}(1+t)^{n-2m}u^{(n)}(t)u(t) + w_k(t) =$$
$$= (1+t)^{n-2m}a_1(t)\|\sigma_k(u)\|_{0,m}^2 + (1+t)^{n-2m}a_2(t),$$

where

$$w_k(t) = (1+t)^{n-2m}|(-1)^{n-m}u^{(n)}(t)u(t) -$$
$$-a_1(t)\|\sigma_k(u)\|_{0,m}^2 - a_2(t)|. \tag{2.19}$$

Hence due to Lemma 2.2 we find

$$\mu_m^n \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s)ds = l(u) +$$
$$+\|\sigma_k(u)\|_{0,m}^2 \int_0^k (1+s)^{n-2m}a_1(s)ds + \int_0^k (1+s)^{n-2m}a_2(s)ds, \tag{2.20}$$

where

$$l(u) = \sum_{i=0}^{m-1} (-1)^{m-i} u^{(i)}(0)u^{(n-1-i)}(0) \quad \text{for} \quad n = 2m$$

and

$$l(u) = -\frac{1}{2}|u^{(m)}(0)|^2 + \sum_{i=0}^{m-1} (-1)^{m-i}\Big[(i+1)u^{(n-2-i)}(0) -$$
$$-u^{(n-1-i)}(0)\Big]u^{(i)}(0) \quad \text{for} \quad n = 2m+1.$$

By conditions (2.2) and (2.6)

$$\|\sigma_k(u)\|_{0,m}^2 \le 1 + m\rho_0^2 + \int_0^k |u^{(m)}(s)|^2 ds \tag{2.21}$$

and

$$l(u) \le m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|.$$

Taking into account these inequalities and conditions (1.6), (2.17), and (2.18), from (2.19) and (2.20) we obtain

$$\begin{gathered}\int_0^k w_k(s)ds \ge \int_0^k (1+s)^{n-2m}|u^{(n)}(s)u(s)|ds - \\ -\|\sigma_k(u)\|_{0,m}^2 \int_0^k (1+s)^{n-2m} a_1(s)ds - \int_0^k (1+s)^{n-2m} a_2(s)ds \ge \\ \ge \int_0^k (1+s)^{n-2m}|u^{(n)}(s)u(s)|ds - \mu_m^n \int_0^k |u^{(m)}(s)|^2 ds - \rho_1, \end{gathered} \tag{2.22}$$

$$\begin{gathered}\mu_m^n \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s)ds \le (\mu_m^n - \varepsilon) \int_0^k |u^{(m)}(s)|^2 ds + \rho_1 + \\ + m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|\end{gathered}$$

and

$$\varepsilon \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s)ds \le \rho_1 + m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|. \tag{2.23}$$

Since w_k is nonnegative, from (2.22) and (2.23) we obtain

$$\int_0^k |u^{(m)}(s)|^2 ds \le \varepsilon^{-1}\rho_1 + \varepsilon^{-1} m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|, \tag{2.24}$$

$$\begin{gathered}\int_0^k (1+s)^{n-2m}|u^{(n)}(s)u(s)|ds \le (\varepsilon^{-1}\mu_m^n + 2)\rho_1 + \\ +(1+\varepsilon^{-1}\mu_m^n) m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|,\end{gathered}$$

$$\begin{gathered}\int_0^k |u^{(m)}(s)|^2 ds + \int_0^k (1+s)^{n-2m}|u^{(n)}(s)u(s)|ds \le \\ \le r_1\Big(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)|\Big).\end{gathered} \tag{2.25}$$

On the other hand, using (2.21) and (2.24), from (2.4) we have

$$|u^{(n)}(t)|\le b_0\Big(t,|u(t)|,r_1^{\frac{1}{2}}\Big(1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big)^{\frac{1}{2}}\Big) \text{ for } 0\le t\le k. \quad (2.26)$$

Therefore inequalities (2.9) and (2.10) are fulfilled. Thus estimates (2.11) hold by virtue of the choice of r_2.

With (2.11) taken into account, inequalities (2.25) and (2.26) imply estimates (2.14) and (2.16). On the other hand, using (2.6) and (2.11), from (2.25) we find

$$|u^{(i)}(t)|=\Big|\sum_{j=i}^{m-1}\frac{t^{j-i}}{(j-i)!}u^{(j)}(0)+\frac{1}{(m-1-i)!}\int_0^t(t-s)^{m-1-i}u^{(m)}(s)ds\Big|\le$$
$$\le m\rho_0(1+t)^{m-1-i}+\Big[\int_0^t(t-s)^{2m-2-2i}ds\Big]^{\frac{1}{2}}\Big[\int_0^t|u^{(m)}(s)|^2ds\Big]^{\frac{1}{2}}\le$$
$$\le r(1+t)^{m-i-\frac{1}{2}} \quad \text{for} \quad 0\le t\le k \quad (i=0,\dots,m-1).$$

Therefore estimates (2.15) hold as well. □

Lemma 2.7. *Let γ and $a_i: R_+\to R_+$ $(i=1,2)$ be a positive constant and measurable functions, satisfying conditions* (1.12) *and* (1.13)*, $\rho_0>0$, and $b_0: R_+^3\to R_+$ be locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition* (2.8)*. Then there is a positive number r such that any function $u\in\widetilde{C}^{n-1}$, satisfying for some natural k conditions* (2.5) *and* (2.6)*, admits estimates*

$$\sum_{i=0}^{m}\int_0^k(1+s)^{2i}|u^{(i)}(s)|^2ds+\int_0^k(1+s)^n|u^{(n)}(s)u(s)|ds\le r, \quad (2.27)$$
$$|u^{(i)}(t)|\le r(1+t)^{-i-\frac{1}{2}} \quad \textit{for} \quad 0\le t\le k \quad (i=0,\dots,m-1), \quad (2.28)$$
$$|u^{(i)}(t)|\le r(1+t)^{n-i-1}+\int_0^t(t-s)^{n-i-1}b_0\big(s,r(1+s)^{-\frac{1}{2}},r\big)ds$$
$$\textit{for} \quad 0\le t\le k \quad (i=m,\dots,n-1). \quad (2.29)$$

Proof. According to (1.13) there exists $\varepsilon\in]0,\delta[$ such that

$$\gamma+(-1)^m\,\frac{n!}{2}>\frac{m-1}{4}\gamma_n(\eta+4)^{m-1}+\varepsilon, \quad (2.30)$$

where

$$\eta=\frac{\gamma_n}{\delta-\varepsilon}+\frac{(m-2)(4m^2-m+3)}{3}.$$

Let

$$\rho_1 = \sum_{i=1}^{2} \int_0^{+\infty} (1+s)^n a_i(s)ds, \quad \rho_2 = (1+\rho_0^2) \sum_{i=0}^{m-1} \sum_{j=i}^{n-1-i} \frac{n!}{(1+i+j)!} \mu_{ij}^n,$$

$$\rho_3 = 2m\rho_0^2 \eta^{m-2} \gamma_n, \quad r_1 = \Big[\varepsilon^{-1}\big(1+\gamma+\frac{n!}{(2m)!}\mu_m^n\big) + 2\Big](\rho_1+\rho_2+\rho_3)+1.$$

Choose $r_2 > 0$ such that the conclusion of Lemma 2.5 is valid and put

$$r = 2(m-1)m\rho_0^2\eta^{m-2} + r_1\big[(m-1)^2(\eta+4)^{m-1} + m\big](1+nr_2).$$

Let $u \in \widetilde{C}^{n-1}$ be an arbitrary function satisfying for some natural k conditions (2.5) and (2.6). Then

$$(-1)^{n-m}(1+s)^n u^{(n)}(s)u(s) + \gamma|u(s)|^2 + w_k(s) =$$
$$(1+s)^n a_1(s)\|\sigma_k(u)\|_m^2 + (1+s)^n a_2(s) \quad \text{for} \quad 0 \le s \le k,$$

where

$$w_k(s) = \big|(-1)^{n-m}(1+s)^n u^{(n)}(s)u(s) + \gamma|u(s)|^2 -$$
$$-(1+s)^n a_1(s)\|\sigma_k(u)\|_m^2 - (1+s)^n a_2(s)\big|.$$

Hence by virtue of Lemma 2.2 we find

$$\sum_{i=0}^{m}(-1)^{m-i}\frac{n!}{(2i)!}\mu_i^n \int_0^k (1+s)^{2i}|u^{(i)}(s)|^2 ds + \gamma \int_0^k |u(s)|^2 ds +$$
$$+\int_0^k w_k(s)ds = l(u) + \|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s)ds + \int_0^k (1+s)^n a_2(s)ds,$$

where

$$l(u) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} (-1)^{m-1-j} \frac{n!}{(i+j+1)!} \mu_{ij}^n u^{(i)}(0)u^{(j)}(0).$$

If we set $c_m(u) = 0$ for $m = 1$,

$$c_m(u) = \sum_{i=1}^{m-1}(-1)^{m-i-1}\frac{n!}{(2i)!}\mu_i^n \int_0^k (1+s)^{2i}|u^{(i)}(s)|^2 ds \quad \text{for} \quad m > 1,$$

then the latter equality can be rewritten as

$$\frac{n!}{(2m)!}\mu_m^n \int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \Big[\gamma + (-1)^m \ \frac{n!}{2}\Big] \int_0^k |u(s)|^2 ds +$$
$$+\int_0^k w_k(s)ds = c_m(u) + l(u) + \|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s)ds +$$

$$+\int_0^k (1+s)^n a_2(s)ds. \tag{2.31}$$

By conditions (1.12), (2.3), and (2.6)

$$\|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s)ds + \int_0^k (1+s)^n a_2(s)ds \leq$$
$$\leq \Big(\frac{n!}{(2m)!}\mu_m^n - \delta\Big)\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \rho_1, \tag{2.32}$$
$$l(u) \leq \sum_{i=0}^{m-1}\sum_{j=i}^{n-1-i}(-1)^{m-1-j}\frac{n!}{(1+i+j)!}\,\mu_{ij}^n u^{(i)}(0)u^{(j)}(0) \leq$$
$$\leq \sum_{i=0}^{m-1}\Big[\rho_0^2\sum_{j=i}^{m-1}\frac{n!}{(1+i+j)!}\,\mu_{ij}^n + \rho_0\sum_{j=m}^{n-1-i}\frac{n!}{(1+i+j)!}\mu_{ij}^n|u^{(j)}(0)|\Big] \leq$$
$$\leq \rho_2\Big[1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big]. \tag{2.33}$$

If $m>1$, then on account of Lemma 2.3 we have

$$c_m(u) \leq \sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\int_0^k (1+s)^{2m-2-4j}|u^{(m-1-2j)}(s)|^2 ds \leq$$
$$\leq \rho_3 + \sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\alpha_{m-1-2j}(\eta)\int_0^k |u(s)|^2 ds +$$
$$+\sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\beta_{m-1-2j}(\eta)\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds.$$

On the other hand, by virtue of the inequality

$$\eta > 1 + \frac{(m-2)(4m^2-m+3)}{3}$$

(2.7) implies clearly that

$$\alpha_i(\eta) \leq \frac{m-1}{4}(\eta+4)^{m-1},$$
$$\beta_i(\eta) \leq \Big[\eta - \frac{(m-2)(4m^2-m+3)}{3}\Big]^{-1} = \frac{\delta-\varepsilon}{\gamma_n} \tag{2.34}$$
$$(i=1,\dots,m-1).$$

Therefore

$$c_m(u) \le \rho_3 + \frac{m-1}{4}(\eta+4)^{m-1}\gamma_n \int_0^k |u(s)|^2 ds + (\delta-\varepsilon)\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds.$$

If along with this inequality we take into consideration inequalities (2.30), (2.32), and (2.33), then from (2.31) we have

$$\varepsilon\Big[\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds\Big] + \int_0^k w_k(s)ds \le$$
$$\le (\rho_1+\rho_2+\rho_3)\Big[1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big].$$

Therefore

$$\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds \le$$
$$\le \varepsilon^{-1}(\rho_1+\rho_2+\rho_3)\Big[1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big], \tag{2.35}$$

$$\int_0^k (1+s)^n|u^{(n)}(s)u(s)|ds \le \int_0^k w_k(s)ds + \gamma\int_0^k |u(s)|^2 ds +$$
$$+\Big[1+\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds\Big]\int_0^k (1+s)^n a_1(s)ds + \int_0^k (1+s)^n a_2(s)ds \le$$
$$\le \rho_1 + \Big[\varepsilon^{-1}\big(\gamma+\frac{n!}{(2m)!}\mu_m^n\big)+1\Big](\rho_1+\rho_2+\rho_3)\Big[1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big],$$

$$\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds +$$
$$+\int_0^k (1+s)^n|u^{(n)}(s)u(s)|ds \le r_1\Big[1+\sum_{i=m}^{n-1}|u^{(i)}(0)|\Big]. \tag{2.36}$$

On the other hand, using (2.3) and (2.35), from (2.5) we obtain inequality (2.26). Therefore inequalities (2.9) and (2.10) are fulfilled. Thus estimates (2.11) hold by virtue of the choice of r_2.

By inequalities (2.11), (2.34), (2.36) and Lemma 2.3 we have

$$\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds +$$
$$+\int_0^k (1+s)^n|u^{(n)}(s)u(s)|ds \le r_1(1+nr_2),$$

$$\sum_{i=1}^{m-1}\int_0^k (1+s)^{2i}|u^{(i)}(s)|^2ds \leq 2m(m-1)\rho_0^2\eta^{m-2} +$$
$$+\sum_{i=1}^{m-1}\alpha_i(\eta)\int_0^k |u(s)|^2ds + \sum_{i=1}^{m-1}\beta_i(\eta)\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2ds \leq$$
$$\leq 2m(m-1)\rho_0^2\eta^{m-2} + (m-1)\Big[(m-1)(\eta+4)^{m-1}+1\Big]\times$$
$$\times\Big[\int_0^k (1+s)^{2m}|u^{(m)}(s)|^2ds + \int_0^k |u(s)|^2ds\Big] \leq$$
$$\leq 2m(m-1)\rho_0^2\eta^{m-2} + r_1\big[(m-1)^2(\eta+4)^{m-1}+m-1\big](1+nr_2).$$

Therefore estimate (2.27) is valid.

In view of (2.6)

$$|u^{(i)}(t)| = \Big|\int_t^k u^{(i+1)}(s)ds\Big| \leq$$
$$\leq \Big|\int_t^k (1+s)^{-2i-2}ds\Big|^{\frac{1}{2}}\Big[\int_t^k (1+s)^{2i+2}|u^{(i+1)}(s)|^2ds\Big]^{\frac{1}{2}} \leq$$
$$\leq (1+t)^{-i-\frac{1}{2}}\Big[\int_t^k (1+s)^{2i+2}|u^{(i+1)}(s)|^2ds\Big]^{\frac{1}{2}}$$
$$\text{for}\quad 0\leq t\leq k \quad (i=0,\dots,m-1).$$

Hence, with (2.27) taken into account, we obtain estimates (2.28). As to estimates (2.29), they follow from inequalities (2.10), (2.11), and (2.28). □

2.3. Some Properties of Functions from the Classes $C_0^{n-1,m}$ and $C^{n-1,m}$.

Lemma 2.8. *If $u\in C_0^{n-1,m}$, then*

$$\lim_{t\to+\infty} t^{i+\frac{1}{2}-m}u^{(i)}(t) = 0 \quad (i=0,\dots,m-1) \tag{2.37}$$

and for any constants c_{ij} $(i=0,\dots,n-m-1; j=i,\dots,n-1-i)$ the function

$$w(t) = \sum_{i=0}^{n-m-1}\sum_{j=i}^{n-1-i} c_{ij}t^{i+j+1-2m}u^{(i)}(t)u^{(j)}(t)$$

satisfies the condition

$$\liminf_{t\to+\infty}|w(t)| = 0. \tag{2.38}$$

Lemma 2.9. *If $u \in C^{n-1,m}$, then*

$$\lim_{t\to+\infty} t^{i+\frac{1}{2}} u^{(i)}(t) = 0 \quad (i = 0, \dots, m-1) \tag{2.39}$$

and for any constants c_{ij} $(i = 0, \dots, n-m-1; j = i, \dots, n-1-i)$ the function

$$w(t) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} c_{ij} t^{i+j+1} u^{(i)}(t) u^{(j)}(t)$$

satisfies condition (2.38).

Lemma 2.10. *If $m \geq 2$, $\quad r_0 \geq 0$, $u \in C^{n-1,m}$,*

$$i|u^{(i-1)}(0)|^2 - u^{(i)}(0)u^{(i-1)}(0) \leq r_0 \quad (i = 1, \dots, m-1),$$

then for any $\eta > \frac{1}{3}(m-2)(4m^2 - m + 3)$ we have the estimates

$$\int_0^{+\infty} (1+t)^{2i} |u^{(i)}(t)|^2 dt \leq 2\eta^{m-2} r_0 + \alpha_i(\eta) \int_0^{+\infty} |u(t)|^2 dt +$$
$$+ \beta_i(\eta) \int_0^{+\infty} (1+t)^{2m} |u^{(m)}(t)|^2 dt \quad (i = 1, \dots, m-1).$$

These lemmas follow immediately from Lemmas 4.3-4.5 in the monograph [4].

2.4. Lemma on the Solvability of an Auxiliary Two-Point Boundary Value Problem. Let $t_0 \in]0, +\infty[$, $c_i \in R$ $(i = 0, \dots, m-1)$, $\overline{c}_j \in R$ $(j = 0, \dots, n-m-1)$, $p : [0, t_0] \to R$ be a summable function and $q : C^{n-1}([0, t_0]) \to L([0, t_0])$ be a continuous operator. Consider the boundary value problem

$$u^{(n)}(t) = p(t)u(t) + q(u)(t), \tag{2.40}$$

$$\begin{aligned} u^{(i)}(0) &= c_i \quad (i = 0, \dots, m-1), \\ u^{(j)}(t_0) &= \overline{c}_j \quad (j = 0, \dots, n-m-1). \end{aligned} \tag{2.41}$$

Lemma 2.11. *Let $(-1)^{n-m-1} p(t) \geq 0$ for $0 \leq t \leq t_0$, and let there exist a summable function $q^* : [0, t_0] \to R_+$ such that the inequality*

$$|q(u)(t)| \leq q^*(t) \quad \textit{for} \quad 0 \leq t \leq t_0 \tag{2.42}$$

holds for any function $u \in C^{n-1}([0, t_0])$ satisfying the boundary conditions (2.41). *Then problem* (2.40), (2.41) *is solvable.*

Proof. We shall show in the first place that the homogeneous problem

$$u^{(n)}(t) = p(t)u(t), \tag{2.40$_0$}$$

$$\begin{aligned} &u^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \\ &u^{(j)}(t_0) = 0 \quad (j = 0, \dots, n-m-1) \end{aligned} \tag{2.41$_0$}$$

has only a trivial solution.

Let u be an arbitrary solution of problem (2.40$_0$), (2.41$_0$). Then, since the function $(-1)^{n-m}p$ is nonpositive, we have

$$(-1)^{n-m}u^{(n)}(t)u(t) + |p(t)||u(t)|^2 = 0.$$

On integrating both sides of this equality from 0 to t_0, by virtue of Lemma 2.2 and conditions (2.41$_0$) we obtain

$$\mu_m^n \int_0^{t_0} |u^{(m)}(t)|^2 dt + \int_0^{t_0} t^{n-2m}|p(t)||u(t)|^2 dt + l_0(u) = 0,$$

where $l_0(u) = 0$ for $n = 2m$ and $l_0(u) = \frac{1}{2}|u^{(m)}(0)|^2$ for $n = 2m+1$. Hence it is clear that $u(t) \equiv 0$.

Since problem (2.40$_0$), (2.41$_0$) has only a trivial solution, problem (2.40), (2.41) is equivalent to the integral equation

$$u(t) = u_0(t) + \int_0^{t_0} G(t,s)q(u)(s)ds, \tag{2.43}$$

where u_0 is the solution of the homogeneous equation (2.40$_0$) under the boundary conditions (2.41) and G is the Green function of problem (2.40$_0$), (2.41$_0$).

By Shauder's principle [6] the continuity of the operator $q : C^{n-1}([0,t_0]) \to L([0,t_0])$ and condition (2.42) guarantees the existence of at least one solution of the integral equation (2.43). $\square$

§ 3. PROOFS OF THE EXISTENCE AND UNIQUENESS THEOREMS

Proof of Theorem 1.1. Let

$$\rho_0 = \max\{|c_i| \ i = 0, \dots, m-1\}, \quad b_0(t,x,y) = b(t,x,y),$$

r be the number from Lemma 2.6, $r_0 = 2m^2\rho_0^2 + 2r + 1$, and

$$\chi(s) = \begin{cases} 1 \quad \text{for} \quad 0 \le s \le r_0 \\ 2 - \dfrac{s}{r_0} \quad \text{for} \quad r_0 < s < 2r_0 \\ 0 \quad \text{for} \quad s \ge 2r_0 \end{cases}. \tag{3.1}$$

For any natural k put $q_k(u)(t) = \chi(\|\sigma_k(u)\|_{0,m}) f(\sigma_k(u))(t)$ and consider the boundary value problem

$$u^{(n)}(t) = q_k(u)(t), \tag{3.2}$$

$$\begin{aligned} &u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \\ &u^{(j)}(k) = 0 \quad (j = 0, \dots, n-m-1). \end{aligned} \tag{3.3}$$

By Lemma 2.1 the continuity of the operator $f : C^{n-1} \to L$ implies the continuity of the operator $q_k : C([0,k]) \to L([0,k])$. Let $u \in C^{n-1}([0,k])$ be an arbitrary function satisfying the boundary conditions (3.3). Then in view of (1.5)

$$|q_k(u)(t)| \le \chi(\|\sigma_k(u)\|_{0,m}) b(t, |u(t)|, \|\sigma_k(u)\|_{0,m}) \quad \text{for} \quad 0 \le t \le k.$$

On the other hand,

$$\begin{aligned} |u(t)| = \Big| \sum_{i=0}^{m-1} \frac{t^i}{i!} u^{(i)}(0) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} u^{(m)}(s) ds \Big| \le \\ \le m\rho_0 (1+t)^{m-1} + t^{m-\frac{1}{2}} \Big[\int_0^t |u^{(m)}(s)|^2 ds \Big]^{\frac{1}{2}} \le \\ \le (1+t)^{m-\frac{1}{2}} \big(m\rho_0 + \|\sigma_k(u)\|_{0,m} \big) \quad \text{for} \quad 0 \le t \le k. \end{aligned}$$

Therefore

$$|q_k(u)(t)| \le q^*(t) \quad \text{for} \quad 0 \le t \le k,$$

where $q^*(t) = b(t, m\rho_0 + 2r_0)(1+t)^{m-\frac{1}{2}}, 2r_0)$ and $q^* \in L([0,k])$. Since all the conditions of Lemma 2.11 are fulfilled for problem (3.2), (3.3), it is solvable. Let u_k be some solution of this problem. By inequalities (1.4) and (1.5)

$$\begin{aligned} (-1)^{n-m-1} u_k^{(n)}(t) u_k(t) = \\ = (-1)^{n-m-1} \chi(\|\sigma_k(u_k)\|_{0,m}) \sigma_k(u_k)(t) f(\sigma_k(u_k))(t) \ge \\ \ge -a_1(t) \|\sigma_k(u_k)\|_{0,m}^2 - a_2(t) \quad \text{for} \quad 0 \le t \le k, \end{aligned}$$

$$|u_k^{(n)}(t)| \le b_0(t, |u_k(t)|, \|\sigma_k(u_k)\|_{0,m}) \quad \text{for} \quad 0 \le t \le k. \tag{3.4}$$

Therefore by virtue of Lemmas 2.1 and 2.6 we have the estimates

$$\begin{aligned} |u_k^{(i)}(t)| \le r(1+t)^{m-i-\frac{1}{2}} \quad (i = 0, \dots, m-1), \\ |u_k^{(i)}(t)| \le r(1+t)^{n-1-i} + \int_0^t (t-s)^{n-1-i} b_0\big(s, r(1+s)^{m-\frac{1}{2}}, r\big) ds \\ (i = m, \dots, n-1) \quad \text{for} \quad 0 \le t \le k, \end{aligned} \tag{3.5}$$

$$\int_0^k |u_k^{(m)}(s)|^2 ds \le r, \quad \|\sigma_k(u_k)\|_{0,m} \le r_0. \tag{3.6}$$

Let us extend u_k to the entire R_+ using the equality

$$u_k(t) = \sigma_k(u_k)(t) \quad \text{for} \quad t \geq k. \tag{3.7}$$

Then due to (3.1) and (3.6) we have

$$u_k^{(n)}(t) = f(u_k)(t) \quad \text{for} \quad 0 \leq t \leq k. \tag{3.8}$$

From (3.4)–(3.6) it is clear that the sequences $\big(u_k^{(i)}\big)_{k=1}^{+\infty}$ $(i = 0, \dots, n-1)$ are uniformly bounded and equicontinuous on each finite segment of R_+. Therefore by the Arzella-Ascoli lemma there exists a subsequence $\big(u_{k_j}\big)_{j=1}^{+\infty}$ of $(u_k)_{k=1}^{+\infty}$ such that $\big(u_{k_j}^{(i)}\big)_{j=1}^{+\infty}$ $(i = 0, \dots, n-1)$ uniformly converges on each finite segment of R_+.

By the continuity of the operator $f : C^{n-1} \to L$ and equality (3.8) it is clear that the function $u(t) = \lim_{j \to +\infty} u_{k_j}(t)$ for $t \in R_+$ is a solution of equation (1.1). On the other hand, from (3.3) and (3.6) it follows that u satisfies conditions (1.2). $\square$

Proof of Theorem 1.2. Let u and $\overline{u}$ be two arbitrary solutions of problem (1.1), (1.2). Putting $u_0(t) = u(t) - \overline{u}(t)$, we obtain

$$u_0^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \qquad \int_0^{+\infty} |u_0^{(m)}(s)|^2 ds < +\infty.$$

On the other hand, by condition (1.8) we have

$$(-1)^{n-m}(1+t)^{n-2m} u_0(t) u_0^{(n)}(t) \leq (1+t)^{n-2m} a(t) \|u_0\|_{0,m}^2 \quad \text{for} \quad t \in R_+.$$

After integrating this inequality from 0 to t, by Lemma 2.2 we obtain

$$\mu_m^n \int_0^t |u_0^{(m)}(s)|^2 ds \leq w(t) + \|u_0\|_{0,m}^2 \int_0^t (1+s)^{n-2m} a(s) ds,$$

where

$$w(t) = (n-2m) \sum_{i=0}^{n-m-1} (-1)^{n-m-i} (i+1) u_0^{(i)}(t) u_0^{(n-2-i)}(t) -$$
$$-(1+t)^{n-2m} \sum_{i=0}^{n-m-1} (-1)^{n-m-i} u_0^{(i)}(t) u_0^{(n-1-i)}(t).$$

However, since by Lemma 2.8 the function w satisfies condition (2.38), from the latter inequality we find

$$\mu_m^n \|u_0\|_{0,m}^2 = \mu_m^n \int_0^{+\infty} |u_0^{(m)}(s)|^2 ds \leq \|u_0\|_{0,m}^2 \int_0^{+\infty} (1+s)^{n-2m} a(s) ds.$$

Hence on account of (1.9) we obtain $\|u_0\|_{0,m} = 0$, i.e., $u(t) \equiv \overline{u}(t)$. $\square$

Proof of Theorem 1.3. Let

$$\rho_0 = \max\{|c_i| : \ i = 0, \dots, m-1\}, \ \ b_0(t,x,y) = b(t,x,y) + \gamma(1+t)^{-n}x,$$

r be the number from Lemma 2.7, $r_0 = r + 1$, and χ be the function given by equality (3.1). For any natural k we put

$$q_k(u)(t) = \chi(\|\sigma_k(u)\|_m)\Big[f(\sigma_k(u))(t) - (-1)^{n-m-1}\gamma(1+t)^{-n}u(t)\Big]$$

and consider the equation

$$u^{(n)}(t) = (-1)^{n-m-1}\gamma(1+t)^{-n}u(t) + q_k(u)(t) \tag{3.9}$$

with the boundary conditions (3.3).

According to Lemma 2.1 the operator $q_k : C^{n-1}([0,k]) \to L([0,k])$ is continuous. On the other hand, by (1.11), for any $u \in C^{n-1}([0,k])$ we have

$$|q_k(u)(t)| \le \chi(\|\sigma_k(u)\|_m) b_0(t, |u(t)|, \|\sigma_k(u)\|_m) \quad \text{for} \quad 0 \le t \le k.$$

But

$$|u(t)| = |\sigma_k(u)(t)| = \frac{1}{(m-1)!}\Big|\int_t^{+\infty}(t-s)^{m-1}[\sigma_k(u)(s)]^{(m)}ds\Big| \le$$
$$\le \frac{1}{(m-1)!}\Big|\int_t^{+\infty}(t-s)^{2m-2}(1+s)^{-2m}ds\Big|^{\frac{1}{2}}\|\sigma_k(u)\|_m \le$$
$$\le (1+t)^{-\frac{1}{2}}\|\sigma_k(u)\|_m \quad \text{for} \quad 0 \le t \le k.$$

Therefore

$$|q_k(u)(t)| \le q^*(t) \quad \text{for} \quad 0 \le t \le k,$$

where $q^*(t) = b_0(t, 2r_0(1+t)^{-\frac{1}{2}}, 2r_0)$ and $q^* \in L([0,k])$. Thus all the conditions of Lemma 2.11 are fulfilled for problem (3.9), (3.3). Therefore it has at least one solution. Let u_k be some solution of this problem. Then due to inequalities (1.10) and (1.11) we shall have

$$(-1)^{n-m-1}u_k(t)u_k^{(n)}(t) = \gamma(1+t)^{-n}\big[1 - \chi(\|\sigma_k(u_k)\|_m)\big]|u_k(t)|^2 +$$
$$+(-1)^{n-m-1}\chi(\|\sigma_k(u_k)\|_m)\sigma_k(u_k)(t)f(\sigma_k(u_k)(t)) \ge$$
$$\ge \gamma(1+t)^{-n}|u_k(t)|^2 - a_1(t)\|\sigma_k(u_k)\|_m^2 - a_2(t) \quad \text{for} \quad 0 \le t \le k,$$
$$|u_k^{(n)}(t)| \le \gamma(1+t)^{-n}|u_k(t)| + b(t, |u_k(t)|, \|\sigma_k(u_k)\|_m) =$$
$$= b_0(t, |u_k(t)|, \|\sigma_k(u_k)\|_m) \quad \text{for} \quad 0 \le t \le k. \tag{3.10}$$

Thus by Lemmas 2.1 and 2.7 we have the estimates

$$|u_k^{(i)}(t)| \le r(1+t)^{-i-\frac{1}{2}} \quad (i = 0, \dots, m-1),$$

$$|u_k^{(i)}(t)| \le r(1+t)^{n-1-i} + \int_0^t (t-s)^{n-i-1} b_0\big(s, r(1+s)^{-\frac{1}{2}}, r\big) ds$$
$$(i = m, \dots, n-1) \quad \text{for} \quad 0 \le t \le k, \tag{3.11}$$

$$\sum_{i=0}^m \int_0^k (1+s)^{2i} |u_k^{(i)}(s)|^2 ds \le r, \quad \|\sigma_k(u_k)\|_m \le r_0. \tag{3.12}$$

Let us extend u_k to the entire R_+ using equality (3.7). Then identity (3.8) will be fulfilled by (3.1) and (3.12).

According to estimates (3.10)–(3.12) the sequences $\big(u_k^{(i)}\big)_{k=1}^{+\infty}$ $(i=0,\dots,n-1)$ are uniformly bounded and equicontinuous on each finite segment of R_+. Therefore there exists a subsequence $\big(u_{k_j}\big)_{j=1}^{+\infty}$ of $\big(u_k\big)_{k=1}^{+\infty}$ such that $\big(u_{k_j}^{(i)}\big)_{j=1}^{+\infty}$ $(i = 0, \dots, n-1)$ uniformly converges on each finite segment of R_+.

Since the operator $f : C^{n-1} \to L$ is continuous, from conditions (3.3), (3.8), and (3.12) it follows that the function

$$u(t) = \lim_{j \to +\infty} u_{k_j}(t) \quad \text{for} \quad t \in R_+$$

is a solution of problem (1.1), (1.3). □

Proof of Theorem 1.4. Let u and $\overline{u}$ be two arbitrary solutions of problem (1.1), (1.3). If we put $u_0(t) = u(t) - \overline{u}(t)$, then

$$u_0^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} t^{2j} |u_0^{(j)}(t)|^2 dt < +\infty \quad (j = 0, \dots, m).$$

On the other hand by condition (1.14) we have

$$(-1)^{n-m}(1+t)^n u_0(t) u_0^{(n)}(t) + \gamma |u_0(t)|^2 \le (1+t)^n a(t) \|u_0\|_m^2 \quad \text{for} \quad t \in R_+.$$

After integrating this inequality from 0 to t, by Lemma 2.2 we obtain

$$\sum_{i=0}^m (-1)^{m-i} \frac{n!}{(2i)!} \mu_i^n \int_0^t (1+s)^{2i} |u_0^{(i)}(s)|^2 ds +$$
$$+ \gamma \int_0^t |u_0(s)|^2 ds \le w(t) + \|u_0\|_m^2 \int_0^t (1+s)^n a(s) ds \quad \text{for} \quad t \in R_+,$$

where

$$w(t) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} (-1)^{m-j} \frac{n!}{(1+i+j)!} \mu_{ij}^n (1+t)^{1+i+j} u^{(i)}(t) u^{(j)}(t).$$

But by Lemma 2.9 the function w satisfies condition (2.38). Taking this fact and condition (1.15) into account, from the latter inequality we find

$$\delta \|u_0\|_m^2 + \Big[\gamma + (-1)^m \, \frac{n!}{2}\Big] \int_0^{+\infty} |u_0(s)|^2 ds \le c_m(u_0), \tag{3.13}$$

where

$$c_m(u_0)=0 \quad \text{for} \quad m=1,$$

$$c_m(u_0)=\sum_{i=1}^{m-1}(-1)^{m-i-1}\frac{n!}{(2i)!}\mu_i^n\int_0^{+\infty}(1+s)^{2i}|u_0^{(i)}(s)|^2ds \ \text{ for } \ m>1.$$

In view of (1.13), for a sufficiently small $\varepsilon\in\,]0,\delta[$ inequality (2.30) is fulfilled, where

$$\eta=\frac{\gamma_n}{\delta-\varepsilon}+\frac{(m-2)(4m^2-m+3)}{3}.$$

If $m>1$, then by virtue of Lemma 2.10 we have

$$c_m(u_0)\le$$

$$\le\sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\int_0^{+\infty}(1+s)^{2m-2-4j}|u_0^{(m-1-2j)}(s)|^2ds\le$$

$$\le\sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\alpha_{m-1-2j}(\eta)\int_0^{+\infty}|u_0(s)|^2ds+$$

$$+\sum_{j=0}^{m_0-1}\frac{n!}{(2m-2-4j)!}\mu_{m-1-2j}^n\beta_{m-1-2j}(\eta)\|u_0\|_m^2\le$$

$$\le\frac{m-1}{4}(\eta+4)^{m-1}\gamma_n\int_0^{+\infty}|u_0(s)|^2ds+(\delta-\varepsilon)\|u_0\|_m^2.$$

If along with this estimate we take into account inequality (2.30), then from (3.13) we obtain

$$\|u_0\|_m^2+\int_0^{+\infty}|u_0(s)|^2ds\le 0.$$

Hence it is clear that $u_0(t)\equiv 0$, i.e., $u(t)\equiv\overline{u}(t)$. $\square$

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(Received 08.12.1993)

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