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TWO-DIMENSION-LIKE FUNCTIONS DEFINED ON THE CLASS OF ALL TYCHONOFF SPACES

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Abstract. Two-dimension-like functions are constructed on the class of all Tychonoff spaces. Several of their properties, analogous to those of the classical dimension functions, are established.

1. Introduction. All topological spaces discussed in this paper are assumed to be Tychonoff spaces.

As usual, Ind, ind, and dim denote the classical dimension functions (the large inductive, small inductive, and covering dimensions, respectively).

Let $\mathbb{N}'$ be the union of all integers ≥ -1 and of one element set consisting of a single formal symbol "$+\infty$" provided with an essential order relation.

The set of all natural numbers is denoted by $\mathbb{N}$.

Throughout the paper for any $n \in \mathbb{N}$, the symbol I^n denotes a standard n-cube $I^n \equiv [0;1]^n$, $I^0 \equiv \{0\}$, and $I^{-1} \equiv \varnothing$ where $\varnothing$ stands for the empty set.

The family of all Tychonoff spaces is denoted by T.

The class $\mathcal{K}$ of topological spaces is said to be permissible if $\mathcal{K}$ satisfies the following conditions:

1) for any integer $n \geq -1$ $I^n \in \mathcal{K}$;

2) if $X \in \mathcal{K}$ and $A \subseteq X$, then $A \in \mathcal{K}$;

3) if $X_1, X_2 \in \mathcal{K}$, then $X_1 \times X_2 \in \mathcal{K}$ where $X_1 \times X_2$ is the usual product of spaces.

The function d defined on a permissible class $\mathcal{K}$ of topological spaces with values in $\mathbb{N}'$ is called the generalized dimension-like function (GDF) if a) $d\varnothing = -1$ and b) $dX = dY$ whenever X is homeomorphic to Y.

The GDF d defined on a permissible class $\mathcal{K}$ of topological spaces is said to be of the Tumarkin type if the following conditions $\mathcal{T}_1^{\mathcal{K}} - \mathcal{T}_8^{\mathcal{K}}$ are satisfied:

$\mathcal{T}_1^{\mathcal{K}}$) for any integer $n \geq -1$ $dI^n = n$;

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$\mathcal{T}_2^{\mathcal{K}}$) if $X \in \mathcal{K}$ and A is a locally closed subspace of X (i.e., if $A = F \cup G$, where F is closed and G is open in X), then $dA \leq dX$;

$\mathcal{T}_3^{\mathcal{K}}$) if $X \in \mathcal{K}$ and $X = \bigcup\limits_{i=1}^{\infty} A_i$ where for any $i \in \mathbb{N}$ A_i is a closed subset of the space X, then $dX \leq \sup\limits_{1 \leq i < +\infty} \{dA_i\}$;

$\mathcal{T}_4^{\mathcal{K}}$) for any space $X \in \mathcal{K}$ there exists a Hausdorff compactification bX of the space X such that $dbX \leq dX$;

$\mathcal{T}_5^{\mathcal{K}}$) for every pair of spaces $X_1, X_2 \in \mathcal{K}$ at least one of which is nonempty we have $d(X_1 \times X_2) \leq dX_1 + dX_2$;

$\mathcal{T}_6^{\mathcal{K}}$) if $X \in \mathcal{K}$ and $X = A \cup B$, then $dX \leq dA + dB + 1$;

$\mathcal{T}_7^{\mathcal{K}}$) if $X \in \mathcal{K}$ and there exists a nonnegative integer n such that $dX \leq n$, then the space X can be represented as the union of $n+1$ pairwise disjoint subsets $X_1, X_2, \ldots, X_{n+1}$ with $dX_i \leq 0$ for any $i = 1, 2, \ldots, n+1$;

$\mathcal{T}_8^{\mathcal{K}}$) for any $X \in \mathcal{K}$ and an arbitrary subspace A of the space X there exists a G_δ-set H in X such that $A \subseteq H \subseteq X$ and $dH \leq dA$.

It is well-known fact that on the class of all separable metrizable spaces the classical dimension dim is a GDF of the Tumarkin type. On the other hand, as proved by L. Zambakhidze [1], there exists no GDF of the Tumarkin type on the class T. Moreover, there exists no GDF on T even satisfying the conditions $\mathcal{T}_1^T$, $\mathcal{T}_2^T$, $\mathcal{T}_3^T$, $\mathcal{T}_4^T$, $\mathcal{T}_5^T$ simultaneously [1]. Also, there is no GDF on T satisfying the conditions $\mathcal{T}_1^T$, $\mathcal{T}_2^T$, $\mathcal{T}_8^T$ simultaneously.

We say that a subcollection $\{\mathcal{T}_{i_1}^T, \ldots, \mathcal{T}_{i_k}^T\}$ $(1 \leq i_1 < \cdots < i_k \leq 8$, $k = 1, \ldots, 8)$ of the collection $\{\mathcal{T}_1^T, \ldots, \mathcal{T}_8^T\}$ is realized if there exists a GDF on T which satisfies all conditions $\{\mathcal{T}_{i_1}^T, \ldots, \mathcal{T}_{i_k}^T\}$ simultaneously.

Clearly, if a subcollection $\{\mathcal{T}_{i_1}^T, \ldots, \mathcal{T}_{i_k}^T\}$ $(1 \leq i_1 < \cdots < i_k \leq 8$, $k = 1, \ldots, 8)$ of the collection $\{\mathcal{T}_1^T, \ldots, \mathcal{T}_8^T\}$ is realized, then any subcollection of $\{\mathcal{T}_{i_1}^T, \ldots, \mathcal{T}_{i_k}^T\}$ is realized, too. Also, if $\{\mathcal{T}_{i_1}^T, \ldots, \mathcal{T}_{i_k}^T\}$ is not realized, then no subcollection of the collection $\{\mathcal{T}_1^T, \ldots, \mathcal{T}_8^T\}$ containing the given one is realized.

L. Zambakhidze has shown [1] that the collections $\{\mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T, \mathcal{T}_8^T\}$, $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_7^T\}$, $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_5^T, \mathcal{T}_6^T\}$, $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_6^T, \mathcal{T}_7^T\}$ and $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T\}$ are realized.

In this paper we prove that collections $\{\mathcal{T}_1^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T, \mathcal{T}_8^T\}$ and $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_5^T, \mathcal{T}_7^T\}$ are realized. To this end we construct two GDFs d_1 and d_2 on T such that d_1 satisfies the conditions $\mathcal{T}_1^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T$, $\mathcal{T}_8^T$ and d_2 satisfies the conditions $\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_5^T, \mathcal{T}_7^T$. Moreover, the functions d_1 and d_2 are the extensions of the classical dimension function dim from the class of all separable metrizable spaces over the class T.

2. GDF d_1. Let $X \in T$. It is assumed that $d_1X = \dim X$ if X has a countable base and $d_1X = 0$ otherwise.

Observe that d_1 is a GDF on T and also is the extension of the function dim from the class of all separable metrizable spaces over the class T.

Theorem 1. *The GDF d_1 satisfies the conditions $\mathcal{T}_1^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T, \mathcal{T}_8^T$, that is to say, the subcollection $\{\mathcal{T}_1^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T, \mathcal{T}_7^T, \mathcal{T}_8^T\}$ of the collection $\{\mathcal{T}_1^T, \dots, \mathcal{T}_8^T\}$ is realized.*

Proof. The function d_1 obviously satisfies the conditions $\mathcal{T}_1^T, \mathcal{T}_3^T, \mathcal{T}_4^T, \mathcal{T}_5^T, \mathcal{T}_6^T$, and $\mathcal{T}_7^T$. Therefore it remains for us to prove that d_1 satisfies the condition $\mathcal{T}_8^T$.

Let $X \in T$ and $A \subseteq X$. Assume in the first place that $\omega X \leq \aleph_0$ (here and below ωX denotes the weight of the space X and $\aleph_0$ stands for a countable cardinal number). Then by Tumarkin's theorem [2, Ch. 6, §3, Theorem 14] there exists a G_δ-set H in X such that $A \subseteq H \subseteq X$ and $\dim H \leq \dim A$. Hence, keeping in mind that $\omega A \leq \aleph_0$ and $\omega H \leq \aleph_0$, we have $d_1 H \leq d_1 A$.

Now let $\omega X > \aleph_0$. If $A = \varnothing$, it can be assumed that $H = \varnothing$. If $A \neq \varnothing$, then assume that $H = X$. □

3. GDF d_2. We begin by defining the function H constructed by Hayashi [3].

Definition 1 ([3]). A subset X' of the space $X \in T$ is called quasiclosed in X if there exists a finite family $\{F_1, \dots, F_k\}$ of closed subsets of the space X such that $X' = F_1 \pm F_2 \pm \cdots \pm F_k$, where $+$ and $-$ denote respectively the union and the difference of sets, and whenever $\pm$ is written one should take either $+$ or $-$.

Clearly, every closed subset as well as every open subset of the space X is quasiclosed in X.

The function H is defined on the class T as follows:

Let $X \in T$. $H(X) = -1$ iff $X = \varnothing$; $H(X) = 0$ iff $X \neq \varnothing$ and $X = \bigcup\limits_{i=1}^{\infty} X_i$ where for any $i \in \mathbb{N}$ X_i is quasiclosed in X and $\operatorname{ind} X_i \leq 0$; $H(X) \leq n$ ($n \in \mathbb{N}$) iff $X = X_1 \cup X_2$, where $H(X_1) \leq n-1$ and $H(X_2) \leq 0$; $H(X) = n$ ($n = 0, 1, 2, \dots$) iff $H(X) \leq n$ and $H(X) \not\leq n-1$.

Finally, $H(X) = \infty$ iff the inequality $H(X) \leq n$ does not hold for any $n = -1, 0, 1, 2, \dots$.

Now we shall define the function d_2.

Let $X \in T$. $d_2(X) = -1$ iff $X = \varnothing$; $d_2(X) \leq n$ ($n = 0, 1, 2, \dots$) if $X = \bigcup\limits_{t=1}^{\infty} X_t$ where $H(X_t) \leq 0$ for any $t \in \mathbb{N}$ and $\bigcup\limits_{k=1}^{n+1} X_{t_k} = X$ for any $n+1$ pairwise disjoint natural numbers $t_1, t_2, \dots, t_{n+1}$; $d_2(X) = n$ ($n = 0, 1, 2, \dots$) iff $d_2(X) \leq n$ and $d_2(X) \not\leq n-1$; $d_2(X) = \infty$ if $d_2(X) \not\leq n$ for any $n = -1, 0, 1, 2, \dots$.

Clearly, d_2 is a GDF on the class T.

Lemma 1. *We have $d_2(X) = \dim X$ for any $X \in T$ with a countable base.*

Proof. Let $X \in T$ and $\omega X \le \aleph_0$. Suppose that $d_2(X) \le n$. Then $X = \overset{\infty}{\underset{t=1}{\cup}} X_t$, where $H(X_t) \le 0$ for any $t \ge 1$ and $\overset{n+1}{\underset{i=1}{\cup}} X_{t_i} = X$ for any pairwise disjoint $t_1, t_2, \dots, t_{n+1} \in \mathbb{N}$. Since $\omega(X_t) \le \aleph_0$ for any $t \in \mathbb{N}$, it follows from [3, Theorem 4.3, Corollary 2] that $\dim X_t = \operatorname{ind} X_t = H(X_t) \le 0$. Therefore we have $X = \overset{\infty}{\underset{t=1}{\cup}} X_t$ where $\dim X_t \le 0$ for any $t \in \mathbb{N}$ and $\overset{n+1}{\underset{i=1}{\cup}} X_{t_i} = X$ for any pairwise disjoint $t_1, \dots, t_{n+1} \in \mathbb{N}$. Hence [4, Theorem 1.5.8] $\dim X \le n$.

Conversely, let $\omega X \le \aleph_0$ and $\dim X \le n$, $(n \ge 0)$. Then by Ostrand's theorem [5] $X = \overset{\infty}{\underset{t=1}{\cup}} X_t$, where $\dim X_t \le 0$ for any $t \ge 1$ and $\overset{n+1}{\underset{i=1}{\cup}} X_{t_i} = X$ for any $n+1$ pairwise disjoint natural numbers $t_1, t_2, \dots, t_{n+1}$. Applying again [4, Theorem 4.3, Corollary 2], we obtain $H(X_t) = \dim X_t \le 0$. Hence, by the definition of the function d_2, we have $d_2(X) \le n$. □

Corollary 1. *The GDF d_2 is the extension of the function* dim *from the class of all separable metrizable spaces over the class T.*

Corollary 2. *The equalities $d_2(I^n) = \dim I^n = n$ hold for any integer $n \ge -1$.*

Lemma 2. *We have $d_2(X') \le d_2(X)$ for each $X \in T$ and an arbitrary subspace X' of the space X.*

Proof. Assume that $X \in T$ and X' is an arbitrary subspace of X. Let $d_2(X) \le n$ $(n \ge -1)$. It will be shown that $d_2(X') \le n$ holds too. Indeed, since $d_2(X) \le n$, we have $X = \overset{\infty}{\underset{t=1}{\cup}} X_t$ where $H(X_t) \le 0$ for each $t \ge 1$ and $X = \overset{n+1}{\underset{i=1}{\cup}} X_{t_i}$ for any pairwise disjoint numbers $t_1, \dots, t_{n+1}$. Introduce the notation $X'_t \equiv X_t \cap X'$. Obviously, $X' = \overset{\infty}{\underset{t=1}{\cup}} X'_t$.

Further, since $H(X_t) \le 0$ for any $t \ge 1$, by the definition of the function H we have $X_t = \overset{\infty}{\underset{i=1}{\cup}} X_{ti} \le 0$, where each X_{ti} is quasiclosed in X_t and $\operatorname{ind} X_{ti} \le 0$ $(i = 1, 2, \dots)$. Observe that $X'_t = X_t \cap X' = \big(\overset{\infty}{\underset{i=1}{\cup}} X_{ti}\big) \cap X' = \overset{\infty}{\underset{i=1}{\cup}} (X_{ti} \cap X')$. Since each X_{ti} is quasiclosed in X_t, $X_{ti} \cap (X_t \cap X') = X_{ti} \cap X'$ will be quasiclosed in $X_t \cap X' = X'_t$ [3, Theorem 1.4]. Introduce the notation $X'_{ti} = X_{ti} \cap X'$. Then $X'_t = \overset{\infty}{\underset{i=1}{\cup}} X'_{ti}$, where each X'_{ti} is quasiclosed in X'_t. Moreover, since $X'_{ti} = X_{ti} \cap X' \subseteq X_{ti}$, we have $\operatorname{ind} X'_{ti} \le \operatorname{ind} X_{ti} \le 0$.

Now we shall show that $X'_t = \overset{n+1}{\underset{i=1}{\cup}} X'_{t_i}$ for any pairwise disjoint natural numbers $t_1, \dots, t_{n+1}$. Indeed, $\overset{n+1}{\underset{i=1}{\cup}} X'_{t_i} = \overset{n+1}{\underset{i=1}{\cup}} (X_{t_i} \cap X') = \big(\overset{n+1}{\underset{i=1}{\cup}} X_{t_i}\big) \cap X' = X \cap X' = X'$. The inequality $d_2(X') \le n$ is proved and so is the inequality $d_2(X') \le d_2(X)$. □

Lemma 3. *Let $X \in T$ and $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is quasiclosed in X. Also assume there exists a natural number n such that $d_2(X_i) \leq n$ for any $i \geq 1$. Then $d_2(X) \leq n$.*

Proof. It is obvious that if $n = -1$, the assertion is true.

Let us consider the case $n = 0$. By definition, $d_2(X) = 0$ iff $H(X) = 0$. Applying Theorem 3.2 from [3], we conclude that the assertion of the lemma is true in this case too.

Now consider the case $n \geq 1$. It can be assumed without loss of generality that $X_i \cap X_j = \varnothing$ whenever $i \neq j$. (Indeed, otherwise we have to consider a new covering $\{X_i'\}_{i=1}^{\infty}$ of the space X, where $X_1' = X_1$, $X_k' = X_k \setminus \bigcup_{i=1}^{k-1} X_i'$ for $k > 1$. Then [3, Theorems 1.1 and 1.3] each X_i' is quasiclosed in X and $X_i' \cap X_j' = \varnothing$ whenever $i \neq j$. Since $X_k' \subseteq X_k$ for any $k \geq 1$, by Lemma 2 we have $d_2(X_k') \leq d_2(X_k) \leq n$). By the definition of the function d_2 and since $d_2 X_i \leq n$, we have $X_i = \bigcup_{t=1}^{\infty} X_{it}$, where $H(X_{it}) \leq 0$ for each $t \geq 1$ and $\bigcup_{j=1}^{n+1} X_{it_j} = X_i$ for any pairwise disjoint natural numbers $t_1, \dots, t_{n+1}$.

We introduce the notation $X_{(t)} \equiv \bigcup_{i=1}^{\infty} X_{it}$. It is obvious that $\bigcup_{t=1}^{\infty} X_{(t)} = X$. We shall prove that $H(X_{(t)}) \leq 0$ for any $t \geq 1$.

Since $X_i \cap X_j = \varnothing$ for $i \neq j$, it is obvious that $X_{it} = X_i \cap X_{(t)}$. Hence due to the quasiclosedness of X_i in X this implies [3, Theorem 1.4] that X_{it} is quasiclosed in $X_{(t)}$. On the other hand, since $H(X_{it}) \leq 0$, we have $X_{it} = \bigcup_{k=1}^{\infty} X_{itk}$, where each X_{itk} is quasiclosed in X_{it} (and, accordingly, in X_i and $X_{(t)}$ as well [3, Theorem 1.5]) and for any $i, t, k \geq 1$ we have $\operatorname{ind} X_{itk} \leq 0$. But it is clear that $X_{(t)} = \bigcup_{i,k=1}^{\infty} X_{itk}$ for any $t \geq 1$. Hence, by the definition of the function H, $H(X_{(t)}) \leq 0$ for any $t \geq 1$.

Now let us consider natural numbers $t_1, \dots, t_{n+1}$ such that $t_i \neq t_j$ whenever $i \neq j$ $(i, j = 1, \dots, n+1)$. We have

$$\bigcup_{m=1}^{n+1} X_{(t_m)} = \bigcup_{m=1}^{n+1} \Big(\bigcup_{i=1}^{\infty} X_{it_m} \Big) = \bigcup_{i=1}^{n+1} \Big(\bigcup_{m=1}^{n+1} X_{it_m} \Big) = \bigcup_{i=1}^{\infty} X_i = X.$$

Hence $d_2(X) \leq n$. $\square$

Corollary 3. *let $X \in T$ and $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X. Also assume that there exists a natural number n such that $d_2(X_i) \leq n$ for any $i \geq 1$. Then $d_2(X) \leq n$.*

Lemma 4. *If X_1 is a quasiclosed subset of $X \in T$ and Y_1 is a quasiclosed subset of $Y \in T$, then $X_1 \times Y_1$ is a quasiclosed subset of the space $X \times Y \in T$.*

Proof. By the assumption

$$X_1 = F_0 \pm F_1 \pm \cdots \pm F_{k-1} \pm F_k$$

and

$$Y_1 = \Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{s-1} \pm \Phi_s,$$

where each F_i $(o \leq i \leq k)$ is a closed subset of X and each Φ_j $(0 \leq j \leq s)$ is a closed subset of Y. The sign "+" denotes the usual union of sets, the sign "−" the usual difference of sets, and so whenever $\pm$ is written one should take either $+$ or $-$.

The lemma will be proved by double induction (with respect to k and s).

If $k = s = 0$, then $X_1 = F_0$ and $Y_1 = \Phi_0$, where F_0 is a closed subset of the space X and Φ_0 is a closed subset of the space Y. Therefore $X_1 \times Y_1$ will be a closed subset and thus it will also be a quasiclosed subset of $X \times Y$.

Assume that Lemma 4 has already been proved in two cases: 1) $0 \leq k \leq m-1$ and $0 \leq s \leq n$; 2) $0 \leq k \leq m$ and $0 \leq s \leq n-1$, and prove it for $k = m$ and $s = n$. For this note that the following (easily verifiable) point-set equations hold for any sets A, B, C, D:

(a) $(A \cup B) \times (C \cup D) = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$;
(b) $(A \cup B) \times (C \backslash D) = \{[(A \times C) \cup (B \times C)] \backslash (A \times D)\} \backslash (B \times D)$;
(c) $(A \backslash B) \times (C \cup D) = \{[(A \times C) \cup (A \times D)] \backslash (B \times C)\} \backslash (B \times D)$;
(d) $(A \backslash B) \times (C \backslash D) = [(A \times C) \backslash (A \times D)] \backslash (B \times C)$.

Four cases are possible:

$$(++) \quad \begin{cases} X_1 = F_0 \pm F_1 \pm \cdots \pm F_{m-1} + F_m \\ Y_1 = \Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{n-1} + \Phi_n \end{cases},$$

$$(+-) \quad \begin{cases} X_1 = F_0 \pm F_1 \pm \cdots \pm F_{m-1} + F_m \\ Y_1 = \Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{n-1} - \Phi_n \end{cases},$$

$$(-+) \quad \begin{cases} X_1 = F_0 \pm F_1 \pm \cdots \pm F_{m-1} - F_m \\ Y_1 = \Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{n-1} + \Phi_n \end{cases},$$

$$(--) \quad \begin{cases} X_1 = F_0 \pm F_1 \pm \cdots \pm F_{m-1} - F_m \\ Y_1 = \Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{n-1} - \Phi_n \end{cases}.$$

Let us consider each of these cases separately.

Introduce the notation

$$F_0 \pm F_1 \pm \cdots \pm F_{m-1} \equiv \widetilde{F}_{m-1};$$

$$\Phi_0 \pm \Phi_1 \pm \cdots \pm \Phi_{n-1} \equiv \widetilde{\Phi}_{n-1}.$$

Case (++). Due to (a) we have

$$X_1 \times Y_1 = (\widetilde{F}_{m-1} \cup F_m) \times (\widetilde{\Phi}_{n-1} \cup \Phi_n) = (\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}) \cup \\ \cup (\widetilde{F}_{m-1} \times \widetilde{\Phi}_n) \cup (F_m \times \widetilde{\Phi}_{n-1}) \cup (F_m) \times \Phi_n).$$

By the assumption of induction $\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}$, $\widetilde{F}_{m-1} \times \Phi_n$ and $F_m \times \widetilde{\Phi}_{n-1}$ are quasiclosed subsets of $X \times Y$. Since the sets F_m and Φ_n are closed in X and Y, respectively, $F_m \times \Phi_n$ is closed (and thus is also quasiclosed) in $X \times Y$. This means that the union of these sets will be quasiclosed in $X \times Y$ as well [3, Theorem 1.1].

Case (+−). Applying (b), we have

$$X_1 \times Y_1 = \{[(\widetilde{F}_{m-1} \times \Phi_n) \cup (F_m \times \widetilde{\Phi}_{n-1})] \backslash \\ \backslash (\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1})\} \backslash (F_m \times \Phi_n).$$

By the assumption the sets $\widetilde{F}_{m-1} \times \Phi_n$, $F_m \times \widetilde{\Phi}_{n-1}$ and $\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}$ are quasiclosed in $X \times Y$. The set $F_m \times \Phi_n$ is obviously closed in $X \times Y$. Hence $X_1 \times Y_1$ is quasiclosed in $X \times Y$ [3, Theorems 1.1 and 1.3].

Case (−+). By (c) we have

$$X_1 \times Y_1 = \{[(\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}) \cup (\widetilde{F}_{m-1} \times \Phi_n)] \backslash \\ \backslash (F_m \times \widetilde{\Phi}_{n-1})\} \backslash (F_m \times \Phi_n).$$

By the assumption of induction and Theorem 1.3 from [3] one can prove that $X_1 \times Y_1$ is quasiclosed in $X \times Y$.

Case (−−). From (d) it follows that

$$X_1 \times Y_1 = [(\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}) \backslash (\widetilde{F}_{m-1} \times \Phi_n)] \backslash (F_m \times \widetilde{\Phi}_{n-1}).$$

By the assumption the sets $\widetilde{F}_{m-1} \times \widetilde{\Phi}_{n-1}$, $\widetilde{F}_{m-1} \times \Phi_n$ and $F_m \times \widetilde{\Phi}_{n-1}$ are quasiclosed in $X \times Y$. Hence by Theorem 1.3 from [3] $X_1 \times Y_1$ is quasiclosed in $X \times Y$ as well. □

Proposition 1. *For any pair of spaces* $X, Y \in T$*, if* $\operatorname{ind} X \leq 0$ *and* $\operatorname{ind} Y \leq 0$*, then* $\operatorname{ind} X \times Y \leq 0$*.*

The *proof* is trivial.

Lemma 5. *Let* $X, Y \in T$ *and let either* $X \neq \varnothing$ *or* $Y \neq \varnothing$*. Then* $d_2(X \times Y) \leq d_2(X) + d_2(Y)$*.*

Proof. If either $d_2(X)=0$ and $d_2(Y)=-1$ or $d_2(X)=-1$ and $d_2(Y)=0$, then the inequality $d_2(X\times Y)\le d_2(X)+d_2(Y)$ is obvious.

Assume that $d_2(X)=n\ge 0$ and $d_2(Y)=m\ge 0$. Then $X=\bigcup\limits_{t=1}^{\infty}X_t$, $Y=\bigcup\limits_{l=1}^{\infty}Y_l$, where $H(X_t)\le 0$ and $H(Y_l)\le 0$ for any $t,l\ge 1$, and, moreover, the equalities

$$X=\bigcup_{i=1}^{n+1}X_{t_i},\quad Y=\bigcup_{j=1}^{m+1}Y_{l_j}$$

hold for any sequences $t_1,\dots,t_{n+1}$ and $l_1,\dots,l_{m+1}$ of natural numbers with pairwise disjoint numbers.

Introduce the notation $Z_p\equiv X_p\times Y_p$ for any $p\ge 1$. Since $H(X_p)\le 0$ and $H(Y_p)\le 0$ for each $p\in\mathbb{N}$, we have $X_p=\bigcup\limits_{i=1}^{\infty}X_{pi}$ and $Y_p=\bigcup\limits_{j=1}^{\infty}Y_{pj}$, where each X_{pi} is quasiclosed in X_p and each Y_{pj} is quasiclosed in Y_p, and for any $i,j\in\mathbb{N}$ we have $\operatorname{ind}X_{pi}\le 0$, $\operatorname{ind}Y_{pj}\le 0$.

Lemma 4 implies that $X_{pi}\times Y_{pj}$ is quasiclosed in $X_p\times Y_p=Z_p$ and by Proposition 1 we have $\operatorname{ind}(X_{pi}\times Y_{pj})\le 0$. Moreover, it is obvious that $Z_p=X_p\times Y_p=\bigcup\limits_{i,j=1}^{\infty}(X_{pi}\times Y_{pj})$.

Let us now prove that if we are given $n+m+1$ natural numbers $p_1,p_2,\dots,p_{n+m+1}$ such that $p_i\ne p_j$ for any $i\ne j$ $(1\le i,j\le n+m+1)$, then $\bigcup\limits_{i=1}^{n+m+1}Z_{p_i}=X\times Y$. (This, in particular, implies that $\bigcup\limits_{p=1}^{\infty}Z_p=X\times Y$.)

The inclusion $X\times Y\supseteq Z_{p_1}\cup\dots\cup Z_{p_{n+m+1}}$ is obvious. Let us prove the inverse inclusion. Assume that $(x,y)\in X\times Y$. It remains for us to show that if (x,y) does not belong to some $m+n$ members of the system $\{Z_{p_1},\dots,Z_{p_{m+n+1}}\}$, then (x,y) necessarily belongs to the remaining member of this system.

Consider the case where $(x,y)\in X\times Y$ and $(x,y)\notin Z_{p_1}\cup\dots\cup Z_{p_{m+n}}$. It will be shown that $(x,y)\in Z_{p_{m+n+1}}$. (All other cases are considered analogously.) Let x not belong to exactly k $(0\le k\le m+n)$ members of the system $\{X_{p_i}\}_{i=1}^{m+n}$ and belong to the remaining $m+n-k$ members of this system. Then, since each subsystem of the system $\{X_t\}_{t=1}^{\infty}$ consisting of $n+1$ elements covers the space X, we have $k\le n$.

By the assumption $(x,y)\notin Z_{p_1}\cup\dots\cup Z_{p_{m+n}}$. Now if $x\in X_{p_i}$, we shall necessarily have $y\notin Y_{p_i}$ $(1\le i\le m+n)$. Hence y does not belong to at least $m+n-k$ members of the system $\{Y_{p_i}\}_{i=1}^{m+n}$. Since each subsystem of the system $\{Y_l\}_{l=1}^{\infty}$ consisting of $m+1$ elements covers the space Y, we have $m+n-k\le m$. Therefore $n\le k$.

From the inequalities $k\le n$ and $n\le k$ we obtain the equality $n=k$. Therefore x does not belong to exactly n elements of the system $\{X_{p_i}\}_{i=1}^{m+n}$. Assume that they are sets $X_{p_{i_1}},\dots,X_{p_{i_n}}$ and consider the system $\{X_{p_{i_1}},\dots,X_{p_{i_n}},X_{p_{m+n+1}}\}$. Since the latter system consists of $n+1$ elements, we have $(\bigcup_{j=1}^{n} X_{p_{i_j}}) \cup X_{p_{m+n+1}} = X$ and, consequently, since $x \notin \bigcup_{j=1}^{n} X_{p_{i_j}}$, we have $x \in X_{p_{m+n+1}}$.

Analogously, y does not belong to exactly $m+n-k = m+n-n = m$ elements of the system $\{Y_{p_i}\}_{i=1}^{m+n}$. Assume that they are sets $Y_{p_{j_1}}, \dots, Y_{p_{j_m}}$. (It is obvious that $\{p_{i_1}, \dots, p_{i_n}\} \cup \{p_{j_1}, \dots, p_{j_m}\} = \{p_1, \dots, p_{m+n}\}$ and $\{p_{i_1}, \dots, p_{i_n}\} \cap \{p_{j_1}, \dots, p_{j_m}\} = \varnothing$.) Consider the system $\{Y_{p_{j_1}}, \dots, Y_{p_{j_m}}, Y_{p_{m+n+1}}\}$. Since this system consists of $m+1$ members, we have $(\bigcup_{i=1}^{m} Y_{p_{j_i}}) \cup Y_{p_{m+n+1}} = Y$. But $y \notin \bigcup_{i=1}^{m} Y_{p_{j_i}}$ and thus $y \in Y_{p_{m+n+1}}$. Therefore $(x, y) \in X_{p_{m+n+1}} \times Y_{p_{m+n+1}} \subseteq Z_{p_1} \cup \dots \cup Z_{p_{m+n+1}}$. □

Lemma 6. *Let $X \in T$ and $d_2(X) \le n$ (where $0 \le n < +\infty$). Then there exist $n+1$ subspaces $X_1, \dots, X_{n+1}$ of the space X such that $X = \bigcup_{i=1}^{n+1} X_i$ and $d_2(X_i) \le 0$ holds for any $i = 1, \dots, n+1$.*

Proof. $d_2(X) \le n$ implies $X = \bigcup_{t=1}^{\infty} X_t$ where for each $t \ge 1$ $H(X_t) \le 0$ (which in turn implies $d_2(X_t) \le 0$) and for any pairwise disjoint natural numbers $t_1, \dots, t_{n+1}$ we have $X = \bigcup_{i=1}^{n+1} X_{t_i}$, in particular, $X = \bigcup_{k=1}^{n+1} X_k$ where $d_2(X_k) \le 0$ for any $k = 1, \dots, n+1$. □

Applying Lemmas 2, 5, 6 and Corollaries 2, 3, we arrive at

Theorem 2. *The GDF d_2 satisfies the conditions $\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_5^T, \mathcal{T}_7^T$ simultaneously. In other words, the subsystem $\{\mathcal{T}_1^T, \mathcal{T}_2^T, \mathcal{T}_3^T, \mathcal{T}_5^T, \mathcal{T}_7^T\}$ of the system $\{\mathcal{T}_1^T, \dots, \mathcal{T}_8^T\}$ is realized.*

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