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WEIGHTED REVERSE WEAK TYPE INEQUALITY FOR GENERAL MAXIMAL FUNCTIONS

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ABSTRACT. Necessary and sufficient conditions are found to be imposed on a pair of weights, for which a weak type two-weighted reverse inequality holds, in the case of general maximal functions defined in homogeneous type spaces.

§ 1. Definition and Formulation of the Basic Results

By a homogeneous type space (X, ρ, μ) we mean a topological space X with measure μ and a quasimetric, i.e., a function $\rho : X \times X \to R^1_+$ satisfying the conditions

(1) $\rho(x, y) = 0 \iff x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \eta(\rho(x, z) + \rho(z, y))$, where $\eta > 0$ does not depend on $x, y, z \in X$.

Furthermore, it is assumed that

(4) all balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$ are μ-measurable and the measure μ satisfies the doubling condition

$$0 < \mu B(x, 2r) \leq d_2 \mu B(x, r) < \infty, \quad x \in X, \quad 0 < r < \infty;$$

(5) for any open set $U \subset X$ and point $x \in U$ there exists a ball $B(x, r)$ with the condition $B(x, r) \subset U$;

(6) continuous functions with compact support are dense in $L^1(X, d\mu)$.

In addition to this, it is required that the space X have no atoms, i.e., $\mu\{x\} = 0$ for any point x from X.

Let f be a locally summable function, $x \in X$ and $t \geq 0$. We introduce the following maximal function:

$$Mf(x, t) = \sup \frac{1}{\mu B} \int_B |f| \, d\mu,$$

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where the lowest upper bound is taken over all balls B containing the point x and having a radius greater than $t/2$.

If $X=\mathbb{R}^n$, μ is the Lebesgue measure, ρ is the Euclidean metric, and $t=0$, then $Mf(x,0)$ transforms to the classical Hardy–Littlewood maximal function and for $n=1$ and $t\geq 0$ it transforms to the maximal function considered by Carleson when estimating the Poisson integrals.

By a weight function w we shall mean a locally summable nonnegative function $w: X\to R^1_+$ and by a measure β a measure in $X\times[0,\infty)$ defined in the product of σ-algebras generated by balls in X and by intervals in $[0,\infty)$.

The merit of this paper is in finding the criterion for the existence of a weak type reverse two-weighted inequality for the maximal functions $Mf(x,t)$. We thereby generalize the results obtained by K. Anderson and W.-S. Young [1] and B. Muckenhoupt [2] for the classical Hardy–Littlewood maximal function and improve the result obtained in [3].

It should also be noted that the criterion for straight two-weighted inequalities of the weak type was obtained by F. Ruiz and J. Torrea [4].

In what follows $\widehat{B}$ will denote a cylinder $B\times[0,2\,\mathrm{rad}\,B)$, N the absolute constant $N=\eta(1+2\eta)$, NB the ball $NB=NB(x,r)=B(x,Nr)$, and d_N a minimal constant for which $\mu(NB)\leq d_{N}\mu B$; $c,\ c_1,\ c_2,\dots$ are positive constants.

This paper gives the proofs of the following theorems.

Theorem 1. *Let B_0 be some ball in X. The following conditions are equivalent:*

(1) *for any function $f\in L^1(X,w\,d\mu)$, $\operatorname{supp} f\subset B_0$, and any λ, $\lambda\geq\lambda_0=\frac{d_N}{\mu B_0}\int_{B_0}|f|\,d\mu$,*

$$\beta\{(x,t)\in\widehat{B_0}:\ Mf(x,t)>\lambda\}\geq\frac{c_1}{\lambda}\int\limits_{\{x\in B_0:|f(x)|>\lambda\}}|f|w\,d\mu; \tag{1}$$

(2) *for any ball B such that $B\cap B_0\neq\varnothing$ and $B\subset NB_0$*

$$\frac{\beta(\widehat{NB}\cap\widehat{B_0})}{\mu B}\geq c_2\operatorname*{ess\,sup}_{x\in B\cap B_0}w(x). \tag{2}$$

Theorem 2. *Let $\mu X=\infty$. The following conditions are equivalent:*
(1) *for any function $f\in L^1(X,w\,d\mu)$ and any $\lambda>0$*

$$\beta\{(x,t)\in X\times[0,\infty):\ Mf(x,t)>\lambda\}\geq\frac{c_3}{\lambda}\int\limits_{\{x\in X:|f(x)|>\lambda\}}|f|w\,d\mu; \tag{3}$$

(2) *for any ball* B

$$\frac{\beta(\widehat{NB})}{\mu B} \geq c_4 \operatorname*{ess\,sup}_{x\in B} w(x). \tag{4}$$

Theorem 3. *Let* B_0*,* w*, and* β *satisfy condition* (2)*. Then if*

$$\int\limits_{\widehat{B}_0} Mf(x,t)d\beta < \infty$$

for the function f*, we have*

$$\int\limits_{B_0} |f|(1+\log^+|f|)w\, d\mu < \infty.$$

Theorem 4. *Let* w *and* β *satisfy condition* (4)*. Then if*

$$\int\limits_{\{(x,t):Mf(x,t)\geq 1\}} Mf(x,t)d\beta < \infty$$

for the function $f \in L^1(X, w\,d\mu)$*, we have*

$$\int\limits_{X} |f|\log^+|f| w\, d\mu < \infty.$$

.

Corollary. *For nontrivial* w *and* β *the pair of inequalities*

$$\frac{c_5}{\lambda} \int\limits_{\{x\in X:|f(x)|>\lambda\}} |f|w\,d\mu \leq \beta\{(x,t)\in X\times[0,\infty): Mf(x,t)>\lambda\} \leq$$

$$\leq \frac{c_6}{\lambda} \int\limits_{\{x\in X:|f(x)|>\frac{\lambda}{2}\}} |f|w\,d\mu$$

hold for all $f \in L^1(x, w\,d\mu)$ *if and only if*

$$\beta\widehat{B} \sim \mu B, \quad 0 < c_7 \leq w(x) \leq c_8 < \infty$$

for any ball B *and any point* $x \in X$*.*

§ 2. The Covering Lemma

In the first place note that the following statement holds in quasimetric spaces: from any covering of a set $E \subset X$ we can find at most a countable subcovering. Further we have (see [5])

Lemma 1. *Let E be a bounded set from X and a ball $B_x = B(x, r_x)$ (with center at x) be given for any point $x \in E$. Then from the covering $\{B_x\}_{x\in E}$ we can find at most a countable subfamily of nonintersecting balls $(B_k)_{k\geq 1}$ such that*

$$\bigcup_{k\geq 1} NB_k \supset E.$$

The essence of the requirement that $\mu\{x\} = 0$, $x \in X$, mentioned in §1 becomes clear after formulating

Lemma 2. *A homogeneous type space has no atoms if and only if for any $\delta > 0$ an arbitrary set E with positive measure has a subset $E_\delta \subset E$ with the condition $0 < \mu E_\delta < \delta$.*

Proof. Let $\mu\{x_0\} > 0$. Then the set $E = \{x_0\}$ does not contain a subset of a positive measure smaller than μE. One aspect of the proof of the lemma becomes thereby obvious.

Let, conversely, $\mu\{x\} = 0$ for all $x \in X$ and E be an arbitrary set of positive measure. The continuity of measure implies that for each $x \in E$ there exists a ball B_x with center at x such that $\mu B_x < \delta$. According to the remark made at the beginning of this section, from the system of balls $\{B_x\}_{x\in E}$ we can find a countable subfamily $(B_k)_{k\geq 1}$ covering B_0. Hence we have

$$\mu E = \mu\Big(\bigcup_{k\geq 1} (B_k \cap E)\Big) \leq \sum_{k\geq 1} \mu(B_k \cap E).$$

Therefore there exists $k_0 \geq 1$ such that $\mu(B_{k_0} \cap E) > 0$. So, assuming $E_\delta = B_{k_0} \cap E$, we obtain $E_\delta \subset E$ and

$$0 < \mu E_\delta \leq \mu B_{k_0} < \delta. \qquad \square$$

Lemma 3. *Let $\Omega \subset X \times [0, \infty)$ be a set such that if $(x, t) \in \Omega$, then $(x, \tau) \in \Omega$ for all τ, $0 \leq \tau < t$. Let the projection Ω_X of the set Ω on X be a bounded set and $\Omega_0 \subset \Omega_X$ be a set of all x from Ω_X for which $\widehat{B}(x, r) \subset \Omega$ with some radius $r > 0$. Then there exists a sequence of balls $(B_i)_{i\geq 1}$ such that*

(1) $\frac{1}{N} B_i \cap \frac{1}{N} B_j = \varnothing$, $i \neq j$;

(2) $\Omega_0 = \bigcup_i B_i = \bigcup_i NB_i$;

(3) $\bigcup_i \widehat{NB}_i \subset \Omega$;

(4) $\widehat{3\eta NB}_i \cap (X \times [0, \infty) \backslash \Omega) \neq \varnothing$, $i = 1, 2, \ldots$;

(5) $\sum_i \chi_{\widehat{NB_i}}(x,t) \le \theta \chi_\Omega(x,t)$,
where $\theta \ge 1$ does not depend on $x \in X$ and $t \ge 0$.

Proof. Let $F = X \times [0,\infty)\backslash\Omega$. We introduce the value

$$\operatorname{dist}(x,F) \stackrel{\text{def}}{=} \sup\{r : \widehat{B}(x,r) \subset \Omega\}, \quad x \in \Omega_X.$$

It is clear that

$$0 < \operatorname{dist}(x,F) < \infty$$

for any point $x \in \Omega_0$.

Let us take

$$r_x = \frac{\operatorname{dist}(x,F)}{2\eta N^2}$$

for any $x \in \Omega_0$. The system of balls $\{B(x,r_x)\}_{x\in\Omega_0}$ covers Ω_0. By Lemma 1 there exists a sequence $(B(x_i,r_{x_i}))_{i\ge1}$ of nonintersecting balls such that

$$\Omega_0 \subset \bigcup_{i\ge1} B(x_i, N r_{x_i}).$$

Setting $r_i = N r_{x_i}$, $B_i = B(x_i,r_i)$, we shall have

$$\Omega_0 \subset \bigcup_{i\ge1} B_i \quad \text{and} \quad \frac{1}{N}B_i \cap \frac{1}{N}B_j = \varnothing \quad \text{for} \quad i \ne j.$$

Statement (1) is thereby proved.

To prove statement (3) note that

$$N r_i = N^2 r_{x_i} = \frac{\operatorname{dist}(x_i,F)}{2\eta} < \operatorname{dist}(x_i,F).$$

Therefore, by definition of the value "dist," we shall have

$$\widehat{NB_i} \subset \Omega$$

for each $i \ge 1$.

Further, for the cylinder $\widehat{3\eta NB_i}$ we obtain

$$\operatorname{rad}(3\eta NB_i) = 3\eta N^2 r_{x_i} = \frac{3}{2}\operatorname{dist}(x_i,F) > \operatorname{dist}(x_i,F).$$

Therefore statement (4) is true.

Now we shall prove statement (2). Since $\Omega_0 \subset \bigcup_{i\ge1} B_i$, it is sufficient for us to prove that $NB_i \subset \Omega_0$ for all $i = 1,2,\dots$.

Let us fix NB_i and show that $\operatorname{dist}(x,F) > 0$ for any point $x \in NB_i$.

Assume the opposite: $\operatorname{dist}(x,F) = 0$. Then $\widehat{B}(x,\alpha) \cap F \ne \varnothing$ for any $\alpha > 0$. Therefore there is $(y,t) \in \widehat{B}(x,\alpha) \cap F$. We shall consider two cases:

(a) $t \geq 2\operatorname{dist}(x_i, F)$; then

$$Nr_i = \frac{\operatorname{dist}(x_i, F)}{2\eta} \leq \frac{t}{4\eta} < \frac{\alpha}{2\eta} < \alpha.$$

(b) $t < 2\operatorname{dist}(x_i, F)$; then $y \notin B(x_i, \operatorname{dist}(x_i, F))$, since otherwise $(y, t) \in \widehat{B}(x_i, \operatorname{dist}(x_i, F)) \subset \Omega$.

Thus we have

$$2\eta Nr_i = \operatorname{dist}(x_i, F) \leq \rho(x_i, y) \leq \eta(\rho(x_i, x) + \rho(x, y)) < \eta(Nr_i + \alpha).$$

Therefore $Nr_i < \alpha$.

So in both cases we find that if $x \in NB_i$, then $\operatorname{rad} NB_i < \alpha$ for any $\alpha > 0$, i.e., $\operatorname{rad} NB_i = 0$, which leads to the contradiction.

We have thereby proved that $\operatorname{dist}(x, F) > 0$ for any $x \in NB_i$ and therefore $x \in \Omega_0$.

Finally, we shall prove the validity of statement (5).

Let $x \in NB_i$. As shown above, $\operatorname{dist}(x, F) > 0$. Consider the cylinder $\widehat{B}(x, 2\operatorname{dist}(x, F))$. By the definition of the value "dist" we have $\widehat{B}(x, 2\operatorname{dist}(x, F)) \cap F \neq \varnothing$ and therefore there exists

$$(y, t) \in \widehat{B}(x, 2\operatorname{dist}(x, F)) \cap F.$$

We shall consider two cases:

(a) $t \geq 2\operatorname{dist}(x_i, F)$; then

$$Nr_i = \frac{\operatorname{dist}(x_i, F)}{2\eta} \leq \frac{t}{4\eta} < \frac{\operatorname{dist}(x, F)}{\eta} < 2\operatorname{dist}(x, F);$$

(b) $t < 2\operatorname{dist}(x_i, F)$; then $y \notin B(x_i, \operatorname{dist}(x_i, F))$, since otherwise $(y, t) \in \widehat{B}(x_i, \operatorname{dist}(x_i, F)) \subset \Omega$.

Thus we have

$$2\eta Nr_i = \operatorname{dist}(x_i, F) \leq \rho(x_i, y) \leq \eta(\rho(x_i, x) + \rho(x, y)) < \\ < \eta(Nr_i + 2\operatorname{dist}(x, F)).$$

Therefore $Nr_i < 2\operatorname{dist}(x, F)$.

So in both cases we find that if $x \in NB_i$, then

$$Nr_i < 2\operatorname{dist}(x, F).$$

Fix an arbitrary point x. Let $NB_i \ni x$ and $y \in NB_i$. Then

$$\rho(x, y) \leq \eta(\rho(x, x_i) + \rho(x_i, y)) \leq 2\eta Nr_i < 4\eta \operatorname{dist}(x, F)$$

from which we conclude that

$$NB_i \subset B(x, 4\eta \operatorname{dist}(x, F)) \tag{5}$$

for any ball NB_i such that $NB_i \ni x$.

Taking now $y \in B(x_i, 2\operatorname{dist}(x_i, F))$, we obtain

$$\rho(x,y) \le \eta(\rho(x,x_i)+\rho(x_i,y)) \le \eta(Nr_i + 2\operatorname{dist}(x_i,F)) =$$
$$= \eta\Big(\frac{\operatorname{dist}(x_i,F)}{2\eta} + 2\operatorname{dist}(x_i,F)\Big) = \big(2\eta+\frac{1}{2}\big)\operatorname{dist}(x_i,F).$$

Therefore

$$B\Big(x, \big(2\eta+\frac{1}{2}\big)\operatorname{dist}(x_i,F)\Big) \supset B\big(x_i, 2\operatorname{dist}(x_i,F)\big).$$

Hence

$$B\Big(x, \big(2\eta+\frac{1}{2}\big)\operatorname{dist}(x_i,F)\Big) \cap F \neq \varnothing.$$

Thus

$$\operatorname{dist}(x,F) < \big(2\eta+\frac{1}{2}\big)\operatorname{dist}(x_i,F) = (4\eta^2+\eta)Nr_i.$$

Therefore

$$\operatorname{rad} NB_i > \frac{1}{4\eta^2+\eta}\operatorname{dist}(x,F). \tag{6}$$

From (5) and (6) we conclude that balls NB_i containing the fixed point x are included in the fixed ball $B(x, 4\eta\operatorname{dist}(x,F))$ and their radii are bounded from below by the fixed positive value $\frac{1}{4\eta^2+\eta}\operatorname{dist}(x,F)$. Therefore, since $\frac{1}{N}B_i$ do not intersect pairwise, the number of such balls NB_i is bounded from above by some absolute constant θ. As a result,

$$\sum_i \chi_{\widehat{NB_i}}(x,t) \le \theta\chi_\Omega(x,t). \qquad \square$$

§ 3. Proof of the Main Results

Proof of Theorem 1. Let us show that (1) ⇒ (2).

Take an arbitrary ball $B \subset NB_0$, $B \cap B_0 \neq \varnothing$. Let $y \notin NB = B(x, Nr)$ and some ball $B' = B(x', r')$ contain the point y and intersect with B. We shall prove that then $r' > r$.

Assume the opposite: $r' \le r$. Let $z \in B \cap B'$. Then

$$\rho(x,y) \le \eta(\rho(x,z)+\rho(z,y)) < \eta(r + \eta(\rho(z,x')+\rho(x',y)) <$$
$$< \eta(r + 2\eta r') \le \eta(1+2\eta)r = Nr,$$

which leads to the contradiction. Therefore $r' > r$.

If now $y \in B$ and $z \in B \cap B'$, then

$$\rho(x',y) \le \eta(\rho(x',z)+\rho(z,y)) < \eta(r' + \eta(\rho(z,x)+\rho(x,y)) <$$
$$< \eta(r' + 2\eta r) < \eta(1+2\eta)r' = Nr'.$$

Therefore $B \subset NB'$.

Fix an arbitrary $\varepsilon > 0$. There is a set $E_\varepsilon \subset B \cap B_0$ such that $w(x) > \operatorname{ess\,sup}_{t \in B \cap B_0} w(t) - \varepsilon$ for any point $x \in E_\varepsilon$. By Lemma 2 it can be assumed that

$$0 < \mu E_\varepsilon < \frac{\mu B}{d_N^2}.$$

Let $f(x) = \chi_{E_\varepsilon}(x)$ and $\lambda = \frac{d_N^2 \mu E_\varepsilon}{\mu B}$. Then $\lambda < 1$ and

$$\lambda_0 = \frac{d_N}{\mu B_0} \int\limits_{B_0} |f|\, d\mu = d_N \frac{\mu E_\varepsilon}{\mu B_0} \leq d_N^2 \frac{\mu E_\varepsilon}{\mu N B_0} \leq d_N^2 \frac{\mu E_\varepsilon}{\mu B} = \lambda.$$

Let further $(y,t) \notin \widehat{NB}$. Consider two cases:

(a) $y \notin NB$; then

$$Mf(y,t) = \sup_{\substack{B' \ni y \\ \operatorname{rad} B' > \frac{t}{2}}} \frac{1}{\mu B'} \int\limits_{B'} |f|\, d\mu \leq \sup_{\substack{B' \ni y \\ B' \cap B \neq \varnothing}} \frac{\mu E_\varepsilon}{\mu B'} \leq$$
$$\leq \sup_{\substack{B' \cap B \neq \varnothing \\ r' > r}} d_N \frac{\mu E_\varepsilon}{\mu N B'} \leq d_N \frac{\mu E_\varepsilon}{\mu B} < d_N^2 \frac{\mu E_\varepsilon}{\mu B} = \lambda.$$

(b) $y \in NB$, $t \geq 2Nr$; then

$$Mf(y,t) = \sup_{\substack{B' \ni y \\ B' \cap B \neq \varnothing \\ \operatorname{rad} B' > \frac{t}{2}}} \frac{1}{\mu B'} \int\limits_{B'} |f|\, d\mu \leq \sup_{\substack{B' \ni y \\ r' > Nr \\ B' \cap B \neq \varnothing}} \frac{\mu E_\varepsilon}{\mu B'} \leq$$
$$\leq \sup_{\substack{r' > r \\ B' \cap B \neq \varnothing}} d_N \frac{\mu E_\varepsilon}{\mu N B'} \leq \lambda.$$

Thus

$$\widehat{NB} \supset \{(y,t) : Mf(y,t) > \lambda\}.$$

Now in view of the above reasoning condition (1) leads to

$$\beta(\widehat{B}_0 \cap \widehat{NB}) \geq \beta\{(x,t) \in \widehat{B}_0 : Mf(x,t) > \lambda\} \geq$$
$$\geq \frac{c_1}{d_N^2} \frac{\mu B}{\mu E_\varepsilon} \int\limits_{\{x \in B \cap B_0 : \chi_{E_\varepsilon}(x) > \lambda\}} \chi_{E_\varepsilon}(x) w(x)\, d\mu =$$
$$= c_2 \frac{\mu B}{\mu E_\varepsilon} \int\limits_{E_\varepsilon} w\, d\mu \geq c_2 \mu B (\operatorname*{ess\,sup}_{x \in B \cap B_0} w(x) - \varepsilon).$$

By making $\varepsilon \to 0$ we get (2).

Now we shall prove that (2) $\Rightarrow$ (1).

Fix f and assume that $\operatorname{supp} f \subset B_0$ and $\lambda \geq \lambda_0 = \frac{d_N}{\mu B_0} \int_{B_0} |f|\, d\mu$. Consider the sets

$$\Omega = \{(x,t) \in X \times [0,\infty) : Mf(x,t) > \lambda\},$$
$$\Omega_c = \{x \in X : M_c f(x) > \lambda\},$$

where

$$M_c f(x) = \sup_{r>0} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |f|\, d\mu.$$

The set Ω satisfies the conditions of Lemma 3. Indeed, if $(x,t) \in \Omega$, then it is obvious that $(x,\tau) \in \Omega$, $0 \leq \tau < t$. Moreover, by familiar arguments $\Omega \subset \widehat{NB}_0$. Therefore Ω_X is the bounded set.

Let $x \in \Omega_c$. Then there exists $r > 0$ such that

$$\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} |f|\, d\mu > \lambda.$$

Obviously, $Mf(y,t) > \lambda$ for any $(y,t) \in \widehat{B}(x,r)$ and therefore $\widehat{B}(x,r) \subset \Omega$. Thus $\Omega_c \subset \Omega_0$, where Ω_0 is the set mentioned in Lemma 3. By the latter lemma there exists a sequence of balls $(B_k)_{k\geq 1}$ satisfying the statements of the lemma. Since $B_k \subset \Omega_0 \subset NB_0$ for each $k \geq 1$, from the condition (2) we get

$$\beta(\Omega \cap \widehat{B}_0) = \int\limits_{\widehat{B}_0} \chi_\Omega(x,t) d\beta \geq \frac{1}{\theta} \sum_{k\geq 1} \int\limits_{\widehat{B}_0} \chi_{\widehat{NB_k}}(x,t) d\beta =$$
$$= \frac{1}{\theta} \sum_{k\geq 1} \beta(\widehat{NB}_k \cap \widehat{B}_0) \geq c \sum_{k\geq 1} \mu B_k \operatorname*{ess\,sup}_{x\in B_k\cap B_0} w(x). \tag{7}$$

Since $\widehat{3\eta NB_k} \cap (X \times [0,\infty) \backslash \Omega) \neq \varnothing$, there exists $(x,t) \in \widehat{3\eta NB_k}$ such that $Mf(x,t) \leq \lambda$. Therefore

$$\frac{1}{\mu B_k} \int\limits_{B_k} |f|\, d\mu \leq \frac{d_{3\eta N}}{(3\eta NB_k)} \int\limits_{3\eta NB_k} |f|\, d\mu < d_{3\eta N} \lambda.$$

Now (7) takes the form

$$\beta(\Omega \cap \widehat{B}_0) \geq \frac{c_1}{\lambda} \sum_{k\geq 1} \operatorname*{ess\,sup}_{x\in B_k\cap B_0} w(x) \int\limits_{B_k} |f|\, d\mu =$$
$$= \frac{c_1}{\lambda} \sum_{k\geq 1} \operatorname*{ess\,sup}_{x\in B_k\cap B_0} w(x) \int\limits_{B_k\cap B_0} |f|\, d\mu \geq \frac{c_1}{\lambda} \sum_{k\geq 1} \int\limits_{B_k\cap B_0} |f| w\, d\mu \geq$$

$$\geq \frac{c_1}{\lambda} \int\limits_{\cup_{k\geq 1}(B_k\cap B_0)} |f|w\,d\mu = \frac{c_1}{\lambda} \int\limits_{\Omega_0\cap B_0} |f|w\,d\mu \geq \frac{c_1}{\lambda} \int\limits_{\Omega_c\cap B_0} |f|w\,d\mu =$$

$$= \frac{c_1}{\lambda} \int\limits_{\{x\in B_0: M_c f(x)>\lambda\}} |f|w\,d\mu \geq \frac{c_1}{\lambda} \int\limits_{\{x\in B_0:|f(x)|>\lambda\}} |f|w\;d\mu. \qquad \square$$

Proof of Theorem 2. First of all note that the implication (3) ⇒ (4) can be proved in the same manner as the implication (1) ⇒ (2) in the preceding theorem. So we shall prove that (4) ⇒ (3).

Fix an arbitrary ball B' and assume that $f \in L^1(X, w\,d\mu)$. For $l > 0$ we introduce the function

$$f_l(x) = \begin{cases} f(x)\cdot \chi_{lB'}(x), & \text{if} \quad |f(x)| < l, \\ l\cdot \operatorname{sign} f(x)\cdot \chi_{lB'}(x), & \text{if} \quad |f(x)| \geq l, \\ 0\cdot \chi_{X\setminus lB'}(x). \end{cases}$$

Let $\lambda > 0$. Then there exists a number $R > Nl$ such that

$$\frac{d_N^2}{\mu(RB')} \int\limits_X |f_l|\,d\mu \leq \lambda.$$

Let $B_0 = NRB'$, $\beta_R E = \beta E$ for $E \subset \widehat{RB'}$, and $\beta_R\{(x,t)\} = \infty$ for any point $(x,t) \neq \widehat{RB'}$.

We shall show that if β and w satisfy (4), then B_0, β_R, and w satisfy condition (2) of Theorem 1.

Indeed, consider an arbitrary ball $B \subset NB_0$, $B \cap B_0 \neq \varnothing$. If $\widehat{NB} \subset \widehat{RB'}$, then

$$\frac{\beta_R(\widehat{NB}\cap \widehat{B}_0)}{\mu B} = \frac{\beta_R(\widehat{NB})}{\mu B} = \frac{\beta\widehat{NB}}{\mu B} \geq c_4 \operatorname*{ess\,sup}_{x\in B} w(x) = c_2 \operatorname*{ess\,sup}_{x\in B\cap B_0} w(x).$$

Let $\widehat{NB} \not\subset \widehat{RB'}$. If $\widehat{NB} \subset \widehat{B}_0$, then

$$\frac{\beta_R(\widehat{NB}\cap \widehat{B}_0)}{\mu B} = \frac{\beta_R(\widehat{NB})}{\mu B} = \infty \geq c_2 \operatorname*{ess\,sup}_{x\in B\cap B_0} w(x).$$

Thus it remains for us to consider the case with $\widehat{NB} \not\subset \widehat{B}_0$. We shall show that $\beta_R(\widehat{NB} \cap \widehat{B_0}) = \infty$ in that case, too. To this end we have to prove that

$$(\widehat{NB} \cap \widehat{B_0})\backslash \widehat{RB'} \neq \varnothing. \tag{8}$$

If there exists a point $z \in (NB \cap B_0)\backslash RB'$, then (8) holds. If such a point does not exists, i.e., $NB \cap (B_0\backslash RB') = \varnothing$, then, since $NB \cap B_0 \neq \varnothing$, there is a point $y \in NB \cap RB'$.

On the other hand, since $\widehat{NB} \not\subset \widehat{B}_0$, we have either $NB \subset B_0$ and then

$$\operatorname{rad}(NB) > \operatorname{rad} B_0 > \operatorname{rad}(NB')$$

or $NB \not\subset B_0$, which together with the condition $NB \cap \frac{1}{N} B_0 = NB \cap RB' \neq \varnothing$, by familiar arguments, gives

$$\operatorname{rad}(NB) > \operatorname{rad}\left(\frac{1}{N} B_0\right) = \operatorname{rad}(NB').$$

Therefore, if $\widehat{NB} \not\subset \widehat{B_0}$, there exists a point $y \in NB \cap RB'$ and $\operatorname{rad}(NB) > \operatorname{rad}(NB')$. Then

$$(y, 2R \operatorname{rad} B') \in \widehat{NB} \backslash \widehat{RB'}.$$

Since $(y, 2R \operatorname{rad} B') \in \widehat{B_0}$, we have (8).

We have thereby shown that B_0, β_R, and w satisfy the condition (2) of Theorem 1.

As to λ, we have

$$\lambda_0 = \frac{d_N}{\mu B_0} \int_{B_0} |f_l| \, d\mu < \frac{d_N^2}{\mu(RB')} \int_{B_0} |f_l| \, d\mu \leq \lambda.$$

Now according to Theorem 1 we have

$$\beta_R\{(x,t) \in \widehat{B}_0 : Mf_l(x,t) > \lambda\} \geq \frac{c_3}{\lambda} \int_{\{x \in B_0 : |f_l(x)| > \lambda\}} |f_l| w \, d\mu. \tag{9}$$

But since $\operatorname{supp} f_l \subset \frac{R}{N} B'$, for $(x,t) \notin \widehat{RB'}$ we shall have

$$Mf_l(x,t) \leq \frac{d_N}{\mu(\frac{R}{N} B')} \int_{\frac{R}{N} B'} |f_l| \, d\mu \leq \frac{d_N^2}{\mu(RB')} \int_X |f_l| \, d\mu \leq \lambda.$$

Hence (9) takes the form

$$\beta\{(x,t) \in X \times [0,\infty) : Mf_l(x,t) > \lambda\} \geq \frac{c_3}{\lambda} \int_{\{x \in X : |f_l(x)| > \lambda\}} |f_l| w \, d\mu.$$

The more so

$$\beta\{(x,t) \in X \times [0,\infty) : Mf(x,t) > \lambda\} \geq \frac{c_3}{\lambda} \int_{\{x \in X : |f_l(x)| > \lambda\}} |f_l| w \, d\mu.$$

By making l tend to infinity we obtain the required inequality (3). $\square$

Proof of Theorem 3. Let $w(x) > 0$ on some subset B_0 of positive measure (otherwise there is nothing to prove). Then from (2) we conclude that $\beta\widehat{B_0} > 0$. If $f \neq 0$ almost everywhere on B_0, then

$$Mf(x,t) \geq \frac{1}{\mu B_0} \int\limits_{B_0} |f|\, d\mu > 0$$

for each $(x,t) \in \widehat{B_0}$. Hence from the condition

$$\int\limits_{\widehat{B_0}} Mf(x,t) d\beta < \infty$$

we obtain $f \in L(B_0, d\mu)$ and $\beta\widehat{B_0} < \infty$. Therefore again from (2) we conclude that w is bounded on B_0 and $f \in L(B_0, w\, d\mu)$.

Now we have

$$\int\limits_{B_0} |f| \log^+ |f| w\, d\mu = \int\limits_{\{|f|>1\}} |f| \log |f| w\, d\mu =$$

$$= \int\limits_{\{|f|>\lambda_0\}} |f| \log \frac{|f|}{\lambda_0} w\, d\mu + \int\limits_{\{1<|f|\leq\lambda_0\}} |f| \log |f| w\, d\mu + \log \lambda_0 \int\limits_{\{|f|>\lambda_0\}} |f| w\, d\mu,$$

where λ_0 is taken from condition (1) of Theorem 1. (If $\lambda_0 < 1$, then the latter expansion is unnecessary.)

By virtue of the above reasoning we see that the last two integrals are finite. Applying Theorem 1, we shall show the finiteness of the first integral:

$$\int\limits_{\{|f|>\lambda_0\}} |f| \log \frac{|f|}{\lambda_0} w\, d\mu = \int\limits_{\{|f|>\lambda_0\}} |f| \int\limits_{\lambda_0}^{|f|} \frac{d\lambda}{\lambda} w\, d\mu = \int\limits_{\lambda_0}^{\infty} \frac{1}{\lambda} \int\limits_{\{|f|>\lambda\}} |f| w\, d\mu\, d\lambda \leq$$

$$\leq c \int\limits_{\lambda_0}^{\infty} \beta\{(x,t) \in \widehat{B_0} : Mf(x,t) > \lambda\} d\lambda \leq$$

$$\leq c \int\limits_{0}^{\infty} \beta\{(x,t) \in \widehat{B_0} : Mf(x,t) > \lambda\} d\lambda = c \int\limits_{\widehat{B_0}} Mf(x,t)\, d\mu < \infty. \qquad \square$$

Proof of Theorem 4. The proof follows from Theorem 2 and the estimate

$$\int_X |f|\log^+|f|w\,d\mu = \int_{\{|f|>1\}} |f|\log|f|w\,d\mu = \int_{\{|f|>1\}} |f|\int_1^{|f|}\frac{d\lambda}{\lambda}w\,d\mu =$$

$$= \int_1^\infty \frac{1}{\lambda}\int_{\{|f|>\lambda\}} |f|w\,d\mu d\lambda \le c\int_1^\infty \beta\{(x,t)\in X\times[0,\infty): Mf(x,t)>\lambda\}d\lambda =$$

$$= c\int_{\{(x,t):Mf(x,t)>1\}} Mf(x,t)d\beta < \infty. \qquad \square$$

Proof of the Corollary. Following the result of F. Ruiz and J. Torrea [4] and Theorem 2 of this paper, for the inequalities

$$\frac{c_1}{\lambda}\int_{\{|f|>\lambda\}} |f|w\,d\mu \le \beta\{(x,t): Mf(x,t)>\lambda\} \le \frac{c_2}{\lambda}\int_{\{f|>\frac{\lambda}{2}\}} |f|w\,d\mu$$

to hold, it is necessary and sufficient that the inequalities

$$\frac{\beta\widehat{B}}{\mu B} \le c_3 \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{and} \quad \frac{\beta(\widehat{NB})}{\mu B} \ge c_4 \operatorname*{ess\,sup}_{x\in B} w(x)$$

be fulfilled simultaneously. Hence for any ball B we have

$$c_5 \operatorname*{ess\,sup}_{x\in\frac{1}{N}B} w(x) \le \frac{\beta\widehat{B}}{\mu B} \le c_3 \operatorname*{ess\,inf}_{x\in B} w(x).$$

From here on the proof of the corollary is clear.

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