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ON THE CONVERGENCE AND SUMMABILITY OF SERIES WITH RESPECT TO BLOCK-ORTHONORMAL SYSTEMS

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ABSTRACT. Statements connected with the so-called block-orthonormalized systems are given. The convergence and summability almost everywhere by the $(c, 1)$ method with respect to such systems are considered. In particular, the well-known theorems of Menshov-Rademacher and Kacmarz on the convergence and (c,1)-summability almost everywhere of orthogonal series are generalized.

1. The so-called block-orthonormal systems were introduced by V. F. Gaposhkin who obtained the first results [1] for series with respect to such systems. In particular, he generalized the well-known Menshov–Rademacher theorem. This paper presents the results on the convergence and (c,1)-summability almost everywhere of series with respect to block-orthonormal systems. These results were announced in [2] and [3] but here some of them are formulated in a slightly different form.

Let $\{N_k\}$ be a strictly increasing sequence of natural numbers and $\Delta_k = (N_k, N_{k+1}]$, $k = 1, 2, \dots$.

Definition 1 ([1]). Let $\{\varphi_n\}$ be a system of functions from $L^2(0, 1)$. $\{\varphi_n\}$ will be called a Δ_k-orthonormal system (Δ_k-ONS) if:
(1) $\|\varphi_n\|_2 = 1, \quad n = 1, 2, \dots;$
(2) $(\varphi_i, \varphi_j) = 0$ for $i, j \in \Delta_k$, $i \neq j$, $k \geq 1$.

Definition 2. A positive nondecreasing sequence $\{\omega(n)\}$ will be called the Weyl multiplier for the convergence ($(c, 1)$-summability) a.e. of series with respect to the Δ_k-ONS $\{\varphi_n(x)\}$ if the convergence of the series

$$\sum_{n=1}^{\infty} a_n^2 \omega(n)$$

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guarantees the convergence ($(c,1)$-summability) a.e. of the series

$$\sum_{n=1}^{\infty} a_n \varphi(x). \tag{1}$$

2. Let the sequence $\{N_k\}$ be fixed and $\Delta_k = (N_k, N_{k+1}]$. Without loss of generality it can be assumed that

$$N_0 = 0, \quad N_1 = 1, \quad \omega(0) = 1.$$

We have

Theorem 1. *In order that a positive nondecreasing sequence $\{\omega(n)\}$ be the Weyl multiplier for the convergence a.e. of series with respect to any Δ_k-ONS, it is necessary and sufficient that the following two conditions be fulfilled:*

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty$;

(b) $\log_2^2 n = O(\omega(n))$ *for* $n \to \infty$.

Proof. Sufficiency. Let the conditions of the theorem be fulfilled and for the sequence $\{a_n\}$

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty.$$

We introduce

$$\psi_k(x) = \sum_{n=N_k+1}^{N_{k+1}} a_n \varphi_n(x), \quad k = 0, 1, 2, \dots .$$

Then

$$\sum_{k=0}^{\infty} \|\psi_k(x)\|_1 \le \sum_{k=0}^{\infty} \|\psi_k(x)\|_2 = \sum_{k=0}^{\infty} \|\psi_k(x)\|_2 \big(\omega(N_k)\big)^{\frac{1}{2}} \big(\omega(N_k)\big)^{-\frac{1}{2}} \le$$
$$\le \sum_{k=0}^{\infty} \|\psi_k(x)\|_2^2 \omega(N_k) \sum_{k=0}^{\infty} \frac{1}{\omega(N_k)} = \sum_{k=0}^{\infty} \left(\sum_{n=N_k+1}^{N_{k+1}} a_n^2 \right) \omega(N_k) \sum_{k=0}^{\infty} \frac{1}{\omega(N_k)} \le$$
$$\le \sum_{n=1}^{\infty} a_n^2 \omega(n) \sum_{k=0}^{\infty} \frac{1}{\omega(N_k)} < \infty,$$

which by the Levy theorem implies that

$$\sum_{k=0}^{\infty} |\psi_k(x)| < \infty \quad \text{a.e.}$$

Therefore the sequence $S_{N_k}(x)$, where

$$S_k(x) = \sum_{n=1}^{k} a_n \varphi_n(x),$$

converges a.e.

Let

$$\delta_k(x) = \max_{N_k < j \le N_{k+1}} \Big| \sum_{n=N_k+1}^{j} a_n \varphi(x) \Big|, \quad k \ge 1.$$

Using the Kantorovich inequality, we obtain

$$\|\delta_k(x)\|_2^2 \le c \sum_{n=N_k+1}^{N_{k+1}} a_n^2 \log_2^2 n, \quad k \ge 1.$$

Now

$$\sum_{k=0}^{\infty} \|\delta_k(x)\|_2^2 \le c \sum_{k=0}^{\infty} \sum_{n=N_k+1}^{N_{k+1}} a_n^2 \log_2^2 n \le c \sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty,\text{[1]}$$

from which it follows that $\lim_{k\to\infty} \delta_k(x) = 0$ for a.e. $x \in (0,1)$. This together with the proven convergence almost everywhere of the series $S_{N_k}(x)$ guarantees the convergence of series (1) a.e. on $(0,1)$.

Necessity.

(1) Let

$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} = \infty.$$

Then there exist numbers $c_k > 0$ such that

$$\sum_{k=1}^{\infty} c_k^2 \omega(N_k) < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} c_k = \infty.$$

Let $\Phi_{N_k}(x) = 1$ ($k = 1, 2, \dots$; $x \in (0,1)$) and choose as other functions $\Phi_n(x)$ ($n \in N$, $n \ne N_k$, $k = 1, 2, \dots$) an arbitrary ONS orthogonal to 1. The system $\{\Phi_n(x)\}$ is an Δ_k-ONS. Take $b_n = 0$ ($n \ne N_1, N_2, \dots$), $b_{N_k} = c_k$ ($k = 1, 2, \dots$). Then

$$\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{k=1}^{\infty} c_k = \infty, \quad x \in (0,1),$$

though

$$\sum_{n=1}^{\infty} b_n \omega(n) = \sum_{k=1}^{\infty} c_k^2 \omega(N_k) < \infty.$$

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[1] In what follows c will denote, generally speaking , various absolute constants.

The necessity of condition (1) is proved.

(2) If equality (b) is not fulfilled, then

$$\frac{\log_2^2 2^k}{\omega(2^k)} \geq \frac{1}{4}\,\frac{\log_2^2 2^{k+1}}{\omega(2^k)} \geq \frac{1}{4}\,\frac{\log_2^2 n}{\omega(n)}, \quad n \in (2^k, 2^{k+1}]\;\; k = 1, 2, \dots,$$

which implies that the equality

$$\log_2^2 2^k = O\big(\omega(2^k)\big) \quad \text{for} \quad k \to \infty$$

is not fulfilled either. Therefore we can find an increasing sequence of natural numbers q_j, $j = 1, 2, \dots$, such that

$$1 \leq \sqrt{\omega(2^{q_j+1})} < \frac{q_j}{j^3}, \quad j = 1, 2, \dots . \tag{2}$$

Inequality (2) makes it possible to construct an orthonormal system $\{\Phi_n(x)\}$ (which simultaneously will also be a Δ_k-ONS) and a sequence $\{b_n\}$ (see [4], p. 298, the proof of Menshov's theorem) such that

$$\sum_{n=1}^{\infty} b_n^2 \omega(n) < \infty,$$

but the series

$$\sum_{n=1}^{\infty} b_n \Phi_n(x)$$

diverges a.e. on $(0, 1)$. □

Remark 1. The application of the proven theorem to orthonormal systems allows us to formulate the Menshov-Rademacher theorem as follows:

In order that a positive nondecreasing sequence $\{\omega(n)\}$ be the Weyl multiplier for the convergence a.e. of series with respect to any orthonormal systems, it is necessary and sufficient that the equality

$$\log_2^2 n = O\big(\omega(n)\big) \quad \textit{as} \quad n \to \infty$$

be fulfilled.

Remark 2. If

$$\omega(n) = \log_2^2 n,$$

then condition (b) of Theorem 1 is fulfilled and we obtain Gaposhkin's theorem [1, Proposition 1].

Remark 3. If

$$N_k = \left[2^{k^\alpha}\right], \quad 0 < \alpha \le \frac{1}{2},$$ [1]

then $\log_2^2 n$ will be the Weyl multiplier for the convergence a.e. not for each Δ_k-ONS. From Theorem 1 it follows that in that case

$$\omega(n) = \log_2^{\frac{1}{\alpha}+\varepsilon} n, \quad \varepsilon > 0,$$

is the Weyl multiplier.

Analogously, if

$$N_k = \left[k^\alpha\right], \quad \alpha \ge 1,$$

then

$$\omega(n) = n^{\frac{1}{\alpha}} \log_2^{1+\varepsilon} n, \quad \varepsilon > 0.$$

Also note that in both cases one should not take $\varepsilon = 0$.

3. Here a necessary and sufficient condition is established to be imposed on the sequence $\{N_k\}$ so that the well-known Kacmarz theorem on the $(c,1)$-summability a.e. of series with respect to orthonormal systems (see [5], p. 223, theorem [5.8.6]) remains valid also with respect to block-orthonormal systems. Moreover, a generalization of the Kacmarz theorem is given for a Δ_k-ONS.

In what follows we shall use the notation

$$\sigma_n(x) = \frac{1}{n}\sum_{i=1}^{n} S_i(x), \quad k(n) = \max\left\{k : N_k < n\right\}.$$

Lemma 1. *Let the sequence $\{N_k\}$ be fixed, $\{\varphi_n\}$ be an arbitrary Δ_k-ONS and for a positive nondecreasing sequence $\{\omega(n)\}$ let there be given*

$$\min\left\{k : N_k \ge n\right\} + n^2 \sum_{k:N_k \ge n} \frac{1}{N_k^2} = O\big(\omega(n)\big) \quad \textit{for} \ \ n \to \infty. \tag{3}$$

Then the condition

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty \tag{4}$$

implies the convergence a.e. of the series

$$\sum_{n=2}^{\infty} n\big(\sigma_n(x) - \sigma_{n-1}(x)\big)^2,$$

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[1] $[p]$ denotes the integer part of the number p.

Proof. Let conditions (3) and (4) be fulfilled. Then

$$\int_0^1 n\big(\sigma_n(x)-\sigma_{n-1}(x)\big)^2dx=\frac{1}{n(n-1)^2}\int_0^1\left(\sum_{i=1}^n a_i(i-1)\varphi_i(x)\right)^2dx\le$$

$$\le\frac{4}{n^3}\int_0^1\left(\sum_{i=1}^{N_{k(n)}}a_i(i-1)\varphi_i(x)+\sum_{i=N_{k(n)}+1}^n a_i(i-1)\varphi(x)\right)^2dx\le$$

$$\le\frac{8}{n^3}\left[\int_0^1\left(\sum_{j=0}^{k(n)-1}\sum_{i=N_j+1}^{N_{j+1}}a_i(i-1)\varphi_i(x)\right)^2dx+\right.$$

$$\left.+\int_0^1\left(\sum_{i=N_{k(n)}+1}^n a_i(i-1)\varphi_i(x)\right)^2\right]\le$$

$$\le\frac{8}{n^3}\left[k(n)\sum_{j=0}^{k(n)-1}\int_0^1\left(\sum_{i=N_j+1}^{N_{j+1}}a_i(i-1)\varphi_i(x)\right)^2dx+\sum_{i=N_{k(n)}+1}^n a_i^2(i-1)^2\right]=$$

$$=\frac{8}{n^3}\left[k(n)\sum_{i=1}^{N_{k(n)}}a_i^2(i-1)^2+\sum_{i=N_{k(n)}+1}^n a_i^2(i-1)^2\right]\le$$

$$\le\frac{8}{n^3}\left[k(n)\sum_{i=1}^{N_{k(n)}}a_i^2i^2+\sum_{i=N_{k(n)}+1}^n a_i^2i^2\right],\quad n\ge 2.$$

Therefore

$$\sum_{n=2}^\infty\int_0^1 n\big(\sigma_n(x)-\sigma_{n-1}(x)\big)^2dx\le 8\sum_{k=0}^\infty\sum_{n=N_k+1}^{N_{k+1}}\frac{1}{n^3}\left(k(n)\sum_{i=1}^{N_{k(n)}}a_i^2i^2+\right.$$

$$\left.+\sum_{i=N_{k(n)}+1}^n a_i^2i^2\right)=8\sum_{k=0}^\infty\left(\sum_{n=N_k+1}^{N_{k+1}}\sum_{i=1}^{N_k}\frac{k}{n^3}a_i^2i^2+\sum_{n=N_k+1}^{N_{k+1}}\sum_{i=N_k+1}^n\frac{1}{n^3}a_i^2i^2\right)=$$

$$=8\sum_{i=1}^\infty a_i^2i^2\sum_{k:N_k\ge i}k\sum_{n=N_k+1}^{N_{k+1}}\frac{1}{n^3}+8\sum_{k=0}^\infty\sum_{i=N_k+1}^{N_{k+1}}a_i^2i^2\sum_{n=i}^{N_{k+1}}\frac{1}{n^3}\le$$

$$\le 8\sum_{i=1}^\infty a_i^2i^2\sum_{k=k(i)+1}k\Big(\frac{1}{N_k^2}-\frac{1}{N_{k+1}^2}\Big)+c\sum_{k=0}^\infty\sum_{i=N_k+1}^{N_{k+1}}a_i^2=$$

$$= 8\sum_{i=1}^{\infty} a_i^2 i^2 \left[\big(k(i)+1\big)\frac{1}{N_{k(i)+1}^2} + \sum_{k=k(i)+2}^{\infty} \frac{1}{N_k^2} \right] + c\sum_{i=1}^{\infty} a_i^2 <$$

$$< c\sum_{i=1}^{\infty} a_i^2 \left(\min\big\{k : N_k \geq i\big\} + i^2 \sum_{k:N_k\geq i} \frac{1}{N_k^2} \right) \leq c\sum_{i=1}^{\infty} a_i^2 \omega(i) < \infty,$$

from which by the Levy theorem we obtain

$$\sum_{n=2}^{\infty} n\big(\sigma_n(x) - \sigma_{n-1}(x)\big)^2 < \infty \quad \text{a.e.} \qquad \square$$

Lemma 2. *Let $\{N_k\}$ be a given sequence, $\{\varphi_n(x)\}$ be an arbitrary Δ_k-ONS, and conditions (3), (4) be fulfilled. Then for the corresponding series (1) the convegence a.e. of the sequence $\{S_{2^n}(x)\}$ is equivalent to the convergence a.e. of the sequence $\{\sigma_{2^n}(x)\}$.*

Proof. Let conditions (3) and (4) be fulfilled. We have

$$S_n(x) - \sigma_n(x) = \frac{1}{n}\sum_{i=1}^{n} a_i(i-1)\varphi_i(x).$$

Then

$$\int_0^1 \big(S_{2^n}(x) - \sigma_{2^n}(x)\big)^2 dx = \int_0^1 \frac{1}{4^n}\left(\sum_{i=1}^{N_{k(2^n)}} a_i(i-1)\varphi_i(x) + \right.$$

$$\left. + \sum_{i=N_{k(2^n)}+1}^{2^n} a_i(i-1)\varphi_i(x) \right)^2 dx \leq \frac{2}{4^n}\left[k(2^n) \sum_{i=1}^{N_{k(2^n)}} a_i^2 (i-1)^2 + \right.$$

$$\left. + \sum_{i=N_{k(2^n)}+1}^{2^n} a_i^2 (i-1)^2 \right] \leq \frac{2}{4^n}\left[k(2^n) \sum_{i=1}^{N_{k(2^n)}} a_i^2 i^2 + \sum_{i=N_{k(2^n)}+1}^{2^n} a_i^2 i^2 \right].$$

Therefore

$$\sum_{n=1}^{\infty}\int_0^1 \big(S_{2^n}(x) - \sigma_{2^n}(x)\big)^2 dx \leq 2\left(\sum_{n=1}^{\infty} \frac{k(2^n)}{4^n} \sum_{i=1}^{N_{k(2^n)}} a_i^2 i^2 + \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{1}{4^n} \sum_{i=N_{k(2^n)}+1}^{2^n} a_i^2 i^2 \right) = 2(J_1 + J_2).$$

We have

$$J_1=\sum_{n=1}^{\infty}\frac{k(2^n)}{4^n}\sum_{i=1}^{N_{k(2^n)}}a_i^2i^2=\sum_{k=1}^{\infty}\sum_{\log_2 N_k<n\le\log_2 N_{k+1}}\frac{k(2^n)}{4^n}\sum_{i=1}^{N_{k(2^n)}}a_i^2i^2=$$

$$=\sum_{k=1}^{\infty}\sum_{\log_2 N_k<n\le\log_2 N_{k+1}}\frac{k}{4^n}\sum_{i=1}^{N_k}a_i^2i^2=$$

$$=\sum_{k=1}^{\infty}\Bigg(\sum_{\log_2 N_k<n\le\log_2 N_{k+1}}\frac{k}{4^n}\Bigg)\sum_{i=1}^{N_k}a_i^2i^2=$$

$$=\sum_{i=1}^{\infty}a_i^2i^2\sum_{k=k(i)+1}^{\infty}\Bigg(\sum_{\log_2 N_k<n\le\log_2 N_{k+1}}\frac{k}{4^n}\Bigg)=$$

$$=\sum_{i=1}^{\infty}a_i^2i^2\Bigg[\big(k(i)+1\big)\sum_{n>\log_2 N_{k(i)+1}}\frac{1}{4^n}+\sum_{k=k(i)+2}^{\infty}\sum_{n>\log_2 N_k}\frac{1}{4^n}\Bigg]\le$$

$$\le\sum_{i=1}^{\infty}a_i^2i^2\Bigg[\big(k(i)+1\big)\frac{4}{3}\,\frac{1}{N_{k(i)+1}^2}+\frac{4}{3}\sum_{k=k(i)+2}^{\infty}\frac{1}{N_k^2}\Bigg]\le$$

$$\le\frac{4}{3}\sum_{i=1}^{\infty}a_i^2i^2\Bigg[\frac{1}{i^2}\min\big\{k:N_k\ge i\big\}+\sum_{k:N_k\ge i}\frac{1}{N_k^2}\Bigg]\le c\sum_{i=1}^{\infty}a_i^2\omega(i)<\infty$$

and

$$J_2=\sum_{n=1}^{\infty}\frac{1}{4^n}\sum_{i=N_{k(2^n)}+1}a_i^2i^2\le\sum_{n=1}^{\infty}\frac{1}{4^n}\sum_{i=1}^{2^n}a_i^2i^2=$$

$$=\sum_{i=1}^{\infty}a_i^2i^2\sum_{2^n\ge i}\frac{1}{4^n}\le c\sum_{i=1}^{\infty}a_i^2<\infty.$$

Therefore

$$\sum_{n=1}^{\infty}\int_0^1\big(S_{2^n}(x)-\sigma_{2^n}(x)\big)^2<\infty$$

from which it follows that

$$\sum_{n=1}^{\infty}\int_0^1\big(S_{2^n}(x)-\sigma_{2^n}(x)\big)^2<\infty\quad\text{a.e.}$$

and therefore

$$\lim_{n\to\infty}\int_0^1\big(S_{2^n}(x)-\sigma_{2^n}(x)\big)^2=0\quad\text{a.e.}\qquad\square$$

Theorem 2. *Let $\{N_k\}$ be a given sequence, $\{\varphi_n(x)\}$ be an arbitrary Δ_k-ONS, and conditions* (3), (4) *be fulfilled. Then for series* (1) *to be $(c,1)$-convergent a.e. it is necessary and sufficient that the subsequence of partial sums $\{S_{2^n}(x)\}$ of* (1) *be convergent a.e.*

Proof. Sufficiency. Let conditions (3), (4) be fulfilled and the subsequence $\{S_{2^n}(x)\}$ of the corresponding series (1) converge a.e. Then by Lemma 3 the subsequence $\{\sigma_{2^n}(x)\}$ also converges a.e. and we have

$$\sup_{k\in(2^n,2^{n+1}]}\bigl(\sigma_k(x)-\sigma_{2^n}(x)\bigr)^2=\left(\sup_{k\in(2^n,2^{n+1}]}\sum_{i=2^n+1}^{k}\bigl(\sigma_i(x)-\sigma_{i-1}(x)\bigr)\right)^2\leq$$
$$\leq\sum_{i=2^n+1}^{2^{n+1}} i\bigl(\sigma_i(x)-\sigma_{i-1}(x)\bigr)^2,$$

which by Lemma 1 implies that $\{\sigma_n(x)\}$ converges a.e., i.e., series (1) is $(c,1)$-summable a.e.

Necessity. Let conditions (3), (4) be fulfilled and series (1) be $(c,1)$-summable a.e. Then $\{\sigma_{2^n}(x)\}$ converges almost everywhere and by Lemma 2 $\{S_{2^n}(x)\}$, too, converges almost everywhere. □

Lemma 3. *If*

$$\sum_{k=3}^{\infty}\frac{1}{(\log_2\log_2 N_k)^2}<\infty,$$

then

$$\min\bigl\{k: N_k\geq n\bigr\}+n^2\sum_{k:N_k\geq n}\frac{1}{N_k^2}=O\bigl((\log_2\log_2 n)^2\bigr)\quad\textit{for}\ \ n\to\infty.$$

Proof. Let

$$\sum_{k=2}^{\infty}\frac{1}{(\log_2\log_2 N_k)^2}<\infty.$$

Then

$$\lim_{k\to\infty}\frac{k}{(\log_2\log_2 N_k)^2}=0$$

and therefore for sufficiently large k's we have

$$2^{2^{\sqrt{k}}}<N_k.$$

By definition, $n\in(N_{k(n)},N_{k(n)+1}]$. Putting

$$q(n)=\begin{cases}k(n)+1, & \text{if }\ 2^{2^{\sqrt{k(n)+1}}}\geq n,\\ m, & \text{if }\ 2^{2^{\sqrt{k(n)+1}}}<n\ \text{ and }\ 2^{2^{\sqrt{m-1}}}\leq n<2^{2^{\sqrt{m}}},\end{cases}$$

for sufficiently large n's we have

$$\sum_{k:N_k\ge n}\frac{1}{N_k^2}=\sum_{k=k(n)+1}\frac{1}{N_k^2}=\sum_{k=k(n)+1}^{q(n)-1}\frac{1}{N_k^2}+\sum_{k=q(n)}^{\infty}\frac{1}{N_k^2}\le$$
$$\le\frac{q(n)-k(n)-1}{N_{k(n)+1}}+\sum_{k=q(n)}^{\infty}\frac{1}{(2^{2^{\sqrt{k}}})^2}\le\frac{q(n)-k(n)-1}{n^2}+$$
$$+\frac{c}{(2^{2^{\sqrt{q(n)}}})^2}\le\frac{q(n)-k(n)-1}{n^2}+\frac{c}{n^2}\le c\,\frac{(\log_2\log_2 n)^2}{n^2}.$$

Therefore for sufficiently large n's

$$\min\big\{k:N_k\ge n\big\}+n^2\sum_{k:N_k\ge n}\frac{1}{N_k^2}\le k(n)+1+n^2c\,\frac{(\log_2\log_2 n)^2}{n^2}\le$$
$$\le c\big(\log_2\log_2 n\big)^2.\qquad\square$$

Theorem 3. *Let the sequence $\{N_k\}$ be fixed. In order that the condition*

$$\sum_{n=2}^{\infty}a_n^2\big(\log_2\log_2 n\big)^2<\infty \tag{5}$$

guarantee the convergence a.e. of the sequence $\{S_{2^k}(x)\}$ for series (1) with respect to any Δ_k-ONS $\{\varphi_n(x)\}$, it is necessary and sufficient that the condition

$$\sum_{k=3}^{\infty}\frac{1}{(log_2\log_2 N_k)^2}<\infty \tag{6}$$

be fulfilled.

Proof. Sufficiency. Let conditions (5) and (6) be fulfilled. Define the sequence of natural numbers $\{M_i\}$ by the recurrent formula

$$\begin{aligned}M_1&=N_1=1,\\ M_i&=\min\big\{\min\{N_k:M_k>M_{i-1},\ k\in N\},\\ &\qquad\min\{2^m:2^m>M_{i-1},\ m\in N\}\big\},\quad i\ge2,\end{aligned}\tag{7}$$

i.e., $\{M_i\}$ is the increasing sequence whose terms have the form N_k or 2^m, $k\ge1$, $m\ge1$.

Assume that $N_i=M_{k_i}$, $i\ge1$, and $k_0=0$. Clearly,

$$M_i<2^i,\quad i\ge1, \tag{8}$$

and

$$\log_2 M_p + i + 1 \geq p \quad \text{for} \quad p \in (k_i, k_{i+1}], \quad i \geq 0. \tag{9}$$

Now, applying condition (6) and inequality (9), for sufficiently large i's and $p \in (k_i, k_{i+1}]$ we have

$$\begin{aligned} p \leq \log_2 M_p + i + 1 \leq \log_2 M_p + \log_2 2^{2^{\sqrt{i}}} \leq \log_2 M_p + \log_2 N_i = \\ = \log_2 M_p + \log_2 M_{k_i} \leq 2 \log_2 M_p. \end{aligned} \tag{10}$$

Set

$$b_n = \left(\sum_{j=M_n+1}^{M_{n+1}} a_j^2 \right)^{\frac{1}{2}}, \quad \psi_n(x) = \begin{cases} \dfrac{1}{b_n} \displaystyle\sum_{j=M_n+1}^{M_{n+1}} a_j \varphi_j(x), & \text{for } b_n \neq 0, \\ \varphi_{M_n+1}(x), & \text{for } b_n = 0, \end{cases} \quad n \geq 1.$$

Clearly, $\{\psi_n(x)\}$ is a $(k_i, k_{i+1}]$-ONS. Moreover, by condition (6) and inequality (8) we have

$$\sum_{i=3}^{\infty} \frac{1}{\log_2^2 k_i} \leq \sum_{i=3}^{\infty} \frac{1}{(\log_2 \log_2 M_{k_i})^2} = \sum_{i=3}^{\infty} \frac{1}{(\log_2 \log_2 N_i)^2} < \infty$$

and by (5) and (10)

$$\begin{aligned} \sum_{n=1}^{\infty} b_n^2 \log_2^2 n = \sum_{n=1} \left(\sum_{j=M_n+1}^{M_{n+1}} a_j^2 \right) \log_2^2 n \leq c \sum_{n=1}^{\infty} \left(\sum_{j=M_n+1}^{M_{n+1}} a_j^2 \right) \times \\ \times \bigl(\log_2 \log_2 M_n \bigr)^2 \leq c \sum_{n=1}^{\infty} \sum_{j=M_n+1}^{M_{n+1}} a_j^2 \bigl(\log_2 \log_2 j \bigr)^2 < \infty. \end{aligned}$$

Thus the conditions of V. Gaposhkin's theorem (see [1], Proposition 1) are fulfilled for $(k_i, k_{i+1}]$-ONS $\{\psi_n(x)\}$ and the sequence $\{b_n\}$. Therefore the series

$$\sum_{n=1}^{\infty} b_n \psi_n(x)$$

converges almost everywhere, which, in particular, guarantees the convergence a.e. of the sequence $\{S_{2^k}(x)\}$ for the corresponding series (1).

Necessity. Let

$$\sum_{k=3}^{\infty} \frac{1}{(\log_2 \log_2 N_k)^2} = \infty.$$

Then there exist numbers $c_k > 0$ such that

$$\sum_{k=2}^{\infty} c_k^2 \bigl(\log_2 \log_2 N_k \bigr)^2 < \infty, \quad \sum_{k=1}^{\infty} c_k = \infty.$$

Take $\Phi_{N_k}(x) \equiv 1$ $(k \geq 1)$ and as other functions $\Phi_n(x)$ $(n \neq N_1, N_2, \dots)$ choose an arbitrary ONS orthogonal to 1. The system $\{\Phi_n(x)\}$ is a Δ_k-ONS. Let $b_{N_k} = c_k$ $(k \geq 1)$ and $b_n = 0$ $(n \neq N_1, N_2, \dots)$. Then

$$\sum_{n=2}^{\infty} b_n^2 \big(\log_2 \log_2 n\big)^2 = \sum_{k=2}^{\infty} c_k^2 \big(\log_2 \log_2 N_k\big)^2 < \infty,$$

but

$$\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{k=1}^{\infty} b_{N_k} = \sum_{k=1}^{\infty} c_k = \infty, \quad x \in (0,1),$$

i.e., for the series

$$\sum_{n=1}^{\infty} b_n \Phi_n(x)$$

the sequence $\{S_{2^k}(x)\}$ diverges everywhere. $\square$

Theorem 4. *Let the sequence $\{N_k\}$ be fixed. In order that the sequence $\{(\log_2 \log_2 n)^2\}$ be the Weyl multiplier for the $(c,1)$-summability a.e. of series with respect to any Δ_k-ONS, it is necessary and sufficient that condition* (6) *be fulfilled.*

Proof. Sufficiency. Let conditions (5) and (6) be fulfilled. Then by Theorem 3 the sequence $\{S_{2^k}(x)\}$ converges a.e. for series (1), while by Lemma 3

$$\min\big\{k : N_k \geq n\big\} + n^2 \sum_{k:N_k \geq n} \frac{1}{N_k^2} = O\big((\log_2 \log_2 n)^2\big), \quad n \to \infty,$$

holds and therefore series (1) is $(c,1)$-summable by Theorem 2.

Necessity. Let

$$\sum_{k=3}^{\infty} \frac{1}{(log_2 \log_2 N_k)^2} = \infty.$$

Construct the Δ_k-ONS $\{\Phi_n(x)\}$ and $\{b_n\}$ as we did when proving the necessity in Theorem 3. Then the series

$$\sum_{n=1}^{\infty} b_n \Phi_n(x)$$

will not be $(c,1)$-summable anywhere. $\square$

Remark 4. If

$$N_k = \Big[2^{2^{k^\alpha}}\Big], \quad \alpha > \frac{1}{2},$$

then the above-mentioned Kacmarz theorem will hold for all Δ_k-ONS $\{\varphi_n(x)\}$.

Theorem 5. *Let the sequence $\{N_k\}$ be fixed. In order that the condition*

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty \tag{11}$$

guarantee the convergence almost everywhere of the subsequence of partial sums $\{S_{2^k}(x)\}$ of series (1) with respect to any Δ_k-ONS $\{\varphi_n(x)\}$, it is necessary and sufficient that the following two conditions be fulfilled:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty$; (12)

(b) $\log_2^2 k = O\big(\omega(M_k)\big)$ *for* $k \to \infty$, (13) *where the sequence $\{M_k\}$ is defined by the recurrent formula* (7).

Proof. Sufficiency. Let conditions (11), (12), (13) be fulfilled. Construct the system $\{\psi_n(x)\}$ and the sequence $\{b_n\}$ as we did when proving the sufficiency in Theorem 3. Set

$$v(k) = \omega(M_k), \quad k \geq 1.$$

Then we obtain

$$\sum_{k=1}^{\infty} b_k^2 v(k) = \sum_{k=1}^{\infty} \left(\sum_{j=M_k+1}^{M_{k+1}} a_j^2 \right) v(k) = \sum_{k=1}^{\infty} \left(\sum_{j=M_k+1}^{M_{k+1}} a_j^2 \right) \omega(M_k) \leq$$

$$\leq \sum_{k=1}^{\infty} \sum_{j=M_k+1}^{M_{k+1}} a_j^2 \omega(j) < \infty,$$

$$\sum_{i=1}^{\infty} \frac{1}{v(k_i)} = \sum_{i=1}^{\infty} \frac{1}{\omega(M_{k_i})} = \sum_{i=1}^{\infty} \frac{1}{\omega(N_i)} < \infty.$$

By condition (b) of Theorem 5 we have

$$\log_2^2 k = O\big(\omega(M_k)\big) = O\big(v(k)\big) \quad \text{for} \quad k \to \infty.$$

Hence we conclude that $\{\psi_n(x)\}$ is an $(k_i, k_{i+1}]$-ONS and

$$\sum_{i=1}^{\infty} \frac{1}{v(k_i)} < \infty, \quad \sum_{k=1}^{\infty} b_k^2 v(k) < \infty, \quad \log_2^2 k = O\big(v(k)\big) \ \text{ for } \ k \to \infty.$$

Now by Theorem 1 the series

$$\sum_{n=1}^{\infty} b_n \psi_n(x)$$

converges a.e. and therefore, in particular, it follows that the subsequence of partial sums $\{S_{2^k}(x)\}$ of the corresponding series (1) converges a.e.

Necessity.

(1) Let
$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} = \infty.$$
Construct $\{\Phi_n(x)\}$ and $\{b_n\}$ as we did in proving the necessity of condition (a) of Theorem 1. Then the sequence $\{S_{2^k}(x)\}$ diverges a.e. for series (1).

(2) Let
$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty$$
but
$$\log_2^2 k = c_k \omega(M_k), \quad k \geq 1,$$
where
$$\overline{\lim_{k\to\infty}}\, c_k = \infty.$$
Let $v(k) = \omega(M_k)$. Then
$$\log_2^2 k = c_k v(k) \quad \text{and} \quad \overline{\lim_{k\to\infty}}\, c_k = \infty.$$
Therefore there exist a $\{\Phi_n(x)\}$-ONS and a sequence $\{b_k\}$ (see Remark 1) such that
$$\sum_{k=1}^{\infty} b_k^2 v(k) < \infty$$
but the series
$$\sum_{k=1}^{\infty} b_k \Phi_k(x)$$
diverges a.e.

Construct the system $\{\psi_n(x)\}$ and the sequence $\{a_n\}$. Namely, let
$$a_{M_i} = b_i, \quad \psi_{M_i}(x) = \Phi_i(x), \quad i = 1, 2, \dots .$$
For the rest of $n \in (N_i, N_{i+1}]$ assume that $a_n = 0$ and as $\psi_n(x)$ take anyone of the functions $\Phi_k(x)$, $k \notin (k_i, k_{i+1}]$, so that $\psi_i(x) \neq \psi_j(x)$ for $i \neq j$ and $i, j \in \Delta_k$. In that case we obtain an Δ_k-ONS $\{\psi_n(x)\}$ for which
$$\sum_{n=1}^{\infty} a_n^2 \omega(n) = \sum_{i=1}^{\infty} a_{M_i}^2 \omega(M_i) = \sum_{i=1}^{\infty} b_i^2 v(i) < \infty$$
but the series
$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

diverges a.e. Then, following the construction of the terms of this series, the subsequence of partial sums $\{S_{M_k}(x)\}$, where $\{M_k\}$ is defined by (7), diverges a.e. But since

$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty,$$

the subsequence of partial sums $\{S_{N_k}(x)\}$ of the series

$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

converges alsmost everywhere. Let the $\{S_{2^n}(x)\}$ converge on a set $E \subset (0,1)$, $m(E) > 0$.

It is clear that from the sequences $\{N_m\}$ and $\{2^n\}$ we must obtain subsequences $\{N_{m_k}\}$ and $\{2^{n_k}\}$ such that

$$S_{2^{n_k}}(x) - S_{N_{m_k}}(x) = a_{2^{n_k}} \psi_{2^{n_k}}(x), \quad k \geq 1.$$

Then

$$\sum_{k=1}^{\infty} \int_0^1 \left(S_{2^{n_k}}(x) - S_{N_{m_k}}(x)\right)^2 dx \leq \sum_{k=1}^{\infty} a_{2^{n_k}}^2 < \infty$$

and therefore

$$\lim_{k \to \infty} \left(S_{2^{n_k}}(x) - S_{N_{m_k}}(x)\right) = 0 \quad \text{a.e.},$$

i.e.,

$$\lim_{n \to \infty} S_{2^n}(x) = \lim_{k \to \infty} S_{2^{n_k}}(x) = \lim_{k \to \infty} S_{N_{m_k}}(x) = \lim_{m \to \infty} S_{N_m}(x)$$
$$\text{almost every} \quad x \in E,$$

which contradicts the divergence a.e. of the sequence $\{S_{N_k}(x)\}$. □

Theorem 6. *Let the sequence $\{N_k\}$ be given and the equality*

$$\sum_{k=n}^{\infty} \frac{1}{N_k^2} = O\Big(\frac{n}{N_n^2}\Big) \quad \textit{for} \quad n \to \infty \tag{14}$$

be fulfilled.

In order that the positive nondecreasing sequence $\{\omega(n)\}$ be the Weyl multiplier for the $(c,1)$-summability a.e. of series with respect to any Δ_k-ONS, it is necessary and sufficient that conditions (12), *and* (13) *be fulfilled.*

Proof. Let condition (14) be fulfilled.

Sufficiency. Let conditions (11), (12) and (13) be fulfilled. Then for sufficiently large k's we have

$$k < \omega(N_k)$$

and therefore for sufficiently large n's

$$\min\big\{k : N_k \geq n\big\} + n^2 \sum_{k:N_k \geq n} \frac{1}{N_k^2} = k(n) + 1 + n^2 \sum_{k=k(n)+1}^{\infty} \frac{1}{N_k^2} \leq$$
$$\leq 2k(n) + n^2 \frac{c \cdot k(n)}{N_{k(n)+1}^2} \leq ck(n) \leq c\omega\big(N_{k(n)}\big) < c\omega(n)$$

which yields

$$\min\big\{k : N_k \geq n\big\} + n^2 \sum_{k:N_k \geq n} \frac{1}{N_k^2} = O\big(\omega(n)\big) \quad \text{for} \;\; n \to \infty. \tag{15}$$

Then by Theorem 5 the sequence $\{S_{2^k}(x)\}$ converges a.e. for series (1), while by Theorem 2 series (1) is $(c,1)$-summable slmost everywhere.

Necessity.

(a) Let

$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} = \infty.$$

Construct $\{\Phi_n(x)\}$ and $\{b_n\}$ as we did when proving the necessity of condition (a) of Theorem 1. Then we have

$$\sum_{n=1}^{\infty} b_n^2 \omega(n) < \infty$$

and

$$\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{k=1}^{\infty} b_{N_k} = \infty, \qquad x \in (0,1),$$

which imply that the series

$$\sum_{n=1}^{\infty} b_n \Phi_n(x)$$

is nowhere $(c,1)$-summable.

(b) Let

$$\sum_{k=1}^{\infty} \frac{1}{\omega(N_k)} < \infty$$

but condition (13) be not fulfilled. Then by Theorem 5 there exist a Δ_k-ONS $\{\psi_n(x)\}$ and a sequence $\{a_n\}$ such that

$$\sum_{n=1}^{\infty} a_n^2 \omega(n) < \infty$$

but the corresponding subsequence of partial sums $\{S_{2^k}(x)\}$ diverges a.e. Moreover, if equality (15) is fulfilled, then by Theorem 2 the series

$$\sum_{n=1}^{\infty} a_n \psi_n(x)$$

is not $(c,1)$-summable almost everywhere. □

Remark 5. From the proof of Theorem 6 it is clear that condition (14) in this theorem can be replaced by condition (15). Then, assuming that $\omega(n) = (\log_2 \log_2 n)^2$ and condition (12) is fulfilled, by inequality (10) we have

$$\log_2^2 k = O\big((\log_2 \log_2 M_k)^2\big) \quad \text{for} \quad k \to \infty,$$

and by Lemma 3

$$\min\big\{k : N_k \geq n\big\} + n^2 \sum_{k:N_k \geq n} \frac{1}{N_k^2} = O\big((\log_2 \log_2 n)^2\big) \quad \text{for} \quad n \to \infty,$$

and we obtain Theorem 4 as a corollary.

Remark 6. Theorem 6 implies that in the typical cases given below the Weyl multipliers for the $(c,1)$-summability a.e. of series with respect to any Δ_k-ONS are:

(a) if

$$N_k = \left[2^{2^{k^\alpha}}\right], \quad 0 < \alpha \leq \frac{1}{2},$$

then

$$\omega(n) = \big(\log_2 \log_2 n\big)^{\frac{1}{\alpha}+\varepsilon}, \quad \varepsilon > 0;$$

(b) if

$$N_k = \left[2^{k^\alpha}\right], \quad \alpha > 0,$$

then

$$\omega(n) = \big(\log_2 n\big)^{\frac{1}{\alpha}+\varepsilon}, \quad \varepsilon > 0;$$

(c) if

$$N_k = [k^\alpha], \quad \alpha \geq 1,$$

then

$$\omega(n) = n^{\frac{1}{\alpha}}\big(\log_2 n\big)^{1+\varepsilon}, \quad \varepsilon > 0.$$

Note that if $\varepsilon = 0$, then in cases (a), (b) and (c) $\{\omega(n)\}$ will be the Weyl multiplier not for each Δ_k-ONS.

Remark 7. Condition (14) is fulfulled, in particular, if

$$N_k = k\Phi(k),$$

where $\Phi(k)$ does not decrease.

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