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# PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. Two integral operators $P^\alpha$ and $Q^\alpha_\beta$ for analytic functions in the open unit disk are introduced. The object of the present paper is to derive some properties of integral operators $P^\alpha$ and $Q^\alpha_\beta$.

## 1. Introduction

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Recently, Jung, Kim, and Srivastava [1] have introduced the following one-parameter families of integral operators:

$$P^\alpha f = P^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \Big(\log \frac{z}{t}\Big)^{\alpha-1} f(t)dt \quad (\alpha > 0), \tag{1.2}$$

$$Q^\alpha_\beta f = Q^\alpha_\beta f(z) = \Big(\frac{\alpha+\beta}{\beta}\Big)\frac{\alpha}{z^\beta} \int_0^z \Big(1 - \frac{t}{z}\Big)^{\alpha-1} t^{\beta-1} f(t)dt \tag{1.3}$$
$$(\alpha > 0,\ \beta > -1)$$

and

$$J_\alpha f = J_\alpha f(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)dt \quad (\alpha > -1), \tag{1.4}$$

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where $\Gamma(\alpha)$ is the familiar Gamma function, and (in general)

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)} = \begin{bmatrix}\alpha\\ \alpha-\beta\end{bmatrix}. \tag{1.5}$$

For $\alpha \in \mathbb{N} = \{1, 2, 3, \dots\}$, the operators $P^\alpha$, $Q_1^\alpha$, and $J_\alpha$ were considered by Bernardi ([2], [3]). Further, for a real number $\alpha > -1$, the operator $J_\alpha$ was used by Owa and Srivastava [4], and by Srivastava and Owa ([8], [6]).

*Remark* 1. For $f(z) \in A$ given by (1.1), Jung, Kim, and Srivastava [1] have shown that

$$P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \quad (\alpha > 0), \tag{1.6}$$

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \tag{1.7}$$
$$(\alpha > 0,\ \beta > -1)$$

and

$$J_\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \quad (\alpha > -1). \tag{1.8}$$

By virtue of (1.6) and (1.8), we see that

$$J_\alpha f(z) = Q_\alpha^1 f(z) \quad (\alpha > -1). \tag{1.9}$$

## 2. An Application of the Miller–Mocanu Lemma

To derive some properties of operators, we have to recall here the following lemma due to Miller and Mocanu [7].

**Lemma 1.** *Let $w(u, v)$ be a complex valued function,*

$$w : D \to \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \textit{ is the complex plane}),$$

*and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $w(u, v)$ satisfies the following conditions:*

(i) *$w(u, v)$ is continuous in $D$;*
(ii) *$(1, 0) \in D$ and $\mathrm{Re}\{w(1, 0)\} > 0$;*
(iii) *$\mathrm{Re}\{w(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$.*

*Let $p(z)$ be regular in $U$ and $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\mathrm{Re}\{w(p(z), zp'(z))\} > 0$ $(z \in U)$, then $\mathrm{Re}\{p(z)\} > 0$ $(z \in U)$.*

Applying the above lemma, we derive

**Theorem 1.** *If $f(z) \in A$ satisfies*

$$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \beta \quad (\alpha > 2;\ z \in U) \tag{2.1}$$

*for some $\beta$ ($\beta < 1$), then*

$$\mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \quad (z \in U). \tag{2.2}$$

*Proof.* Noting that

$$z(P^{\alpha}f(z))' = 2P^{\alpha-1}F(z) - P^{\alpha}f(z) \quad (\alpha > 1), \tag{2.3}$$

we have

$$\frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = 2\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} - 1 \ \ (\alpha > 1). \tag{2.4}$$

Define the function $p(z)$ by

$$\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} = \gamma + (1-\gamma)p(z) \tag{2.5}$$

with

$$\gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8}. \tag{2.6}$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in the open unit disk $U$. Since

$$\frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} - \frac{z(P^{\alpha}f(z))'}{P^{\alpha}f(z)} = \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)}, \tag{2.7}$$

or

$$\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}}, \tag{2.8}$$

we have

$$\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} - \beta\right\} =$$

$$= \mathrm{Re}\left\{\gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{2\{\gamma + (1-\gamma)p(z)\}} - \beta\right\} > 0. \tag{2.9}$$

Therefore, if we define the function $w(u, v)$ by

$$w(u, v) = \gamma - \beta + (1-\gamma)u + \frac{(1-\gamma)v}{2\{\gamma + (1-\gamma)u\}}, \tag{2.10}$$

then we see that

(i) $w(u,v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times C$;
(ii) $(1,0) \in D$ and $\mathrm{Re}\{w(1,0)\} = 1-\beta > 0$;
(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1+u_2^2)/2$,

$$\begin{aligned}\mathrm{Re}\{w(iu_2,v_1)\} &= \gamma - \beta + \frac{\gamma(1-\gamma)v_1}{2\{\gamma^2+(1-\gamma)^2u_2^2\}} \leq \\ &\leq \gamma - \beta - \frac{\gamma(1-\gamma)(1+u_2^2)}{4\{\gamma^2+(1-\gamma)^2u_2^2\}} = \\ &= -\frac{(1-\gamma)(\gamma+4(1-\gamma)\beta)}{4\{\gamma^2+(1-\gamma)^2u_2^2\}}u_2^2 \leq 0.\end{aligned}$$

This implies that the function $w(u,v)$ satisfies the conditions in Lemma 1. Thus, applying Lemma 1, we conclude that

$$\begin{aligned}&\mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \gamma = \\ &= \frac{4\beta-1+\sqrt{16\beta^2-8\beta+17}}{8} \quad (z \in U). \qquad \square \end{aligned} \tag{2.11}$$

Taking the special values for $\beta$ in Theorem 1, we have

**Corollary 1.** *Let $f(z)$ be in the class $A$. Then*

(i) $\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{2} \quad (z \in U)$

$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{4} \quad (z \in U)$,

(ii) $\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{4} \quad (z \in U)$

$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{5}-1}{4} \quad (z \in U)$,

(iii) $\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > 0 \quad (z \in U)$

$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{17}-1}{8} \quad (z \in U)$,

(iv) $\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{4} \quad (z \in U)$

$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{2} \quad (z \in U)$,

*and*

(v) $\mathrm{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{2} \quad (z \in U)$

$\Longrightarrow \mathrm{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{\sqrt{17}+1}{8} \quad (z \in U)$.

Next, we have

**Theorem 2.** *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2} f(z)}{Q_\beta^{\alpha-1} f(z)}\right\} > \gamma \quad (\alpha > 2,\ \beta > -1;\ z \in U) \tag{2.12}$$

*for some* $\gamma$ $((\alpha+\beta-3)/2(\alpha+\beta-1) \leq \gamma < 1)$, *then*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^{\alpha} f(z)}\right\} > \delta \quad (z \in U), \tag{2.13}$$

*where*

$$\delta = \frac{1 + 2\gamma(\alpha+\beta-1) + \sqrt{(1+2\gamma(\alpha+\beta-1))^2 + 8(\alpha+\beta)}}{4(\alpha+\beta)}. \tag{2.14}$$

*Proof.* By the definition of $Q_\beta^\alpha f(z)$, we know that

$$z(Q_\beta^\alpha f(z))' = (\alpha+\beta) Q_\beta^{\alpha-1} f(z) - (\alpha+\beta-1) Q_\beta^\alpha f(z) \tag{2.15}$$
$$(\alpha > 1,\ \beta > -1),$$

so that

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} = (\alpha+\beta)\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} - (\alpha+\beta-1). \tag{2.16}$$

We define the function $p(z)$ by

$$\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} = \delta + (1-\delta)p(z). \tag{2.17}$$

Then $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in $U$. Making use of the logarithmic differentiations in both sides of (2.17), we have

$$\frac{z(Q_\beta^{\alpha-1} f(z))'}{Q_\beta^{\alpha-1} f(z)} = \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{(1-\delta)zp'(z)}{\delta + (1-\delta)p(z)}. \tag{2.18}$$

Applying (2.15) to (2.18), we obtain that

$$\frac{Q_\beta^{\alpha-2} f(z)}{Q_\beta^{\alpha-1} f(z)} = \frac{1}{\alpha+\beta-1}\left\{(\alpha+\beta)\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} - 1 + \right.$$
$$\left. + \frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)}\right\} = \frac{1}{\alpha+\beta-1}\left\{\delta(\alpha+\beta) - 1 + \right.$$
$$\left. + (\alpha+\beta)(1-\delta)p(z) + \frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)}\right\}, \tag{2.19}$$

that is, that

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}-\gamma\right\}=\frac{1}{\alpha+\beta-1}\operatorname{Re}\Big\{\delta(\alpha+\beta)-1+$$

$$+(\alpha+\beta)(1-\delta)p(z)+\frac{(1-\delta)zp'(z)}{\delta+(1-\delta)p(z)}\Big\}-\gamma>0. \tag{2.20}$$

Now, we let

$$w(u,v)=\frac{1}{\alpha+\beta-1}\Big\{\delta(\alpha+\beta)-1+$$

$$+(\alpha+\beta)(1-\delta)u+\frac{(1-\delta)v}{\delta+(1-\delta)u}\Big\}-\gamma \tag{2.21}$$

with $u=u_1+iu_2$ and $v=v_1+iv_2$. Then $w(u,v)$ satisfies that

(i) $w(u,v)$ is continuous in $D=\left(\mathbb{C}-\left\{\frac{\delta}{\delta-1}\right\}\right)\times\mathbb{C}$;

(ii) $(1,0)\in D$ and $\operatorname{Re}\{w(1,0)\}=1-\gamma>0$;

(iii) for all $(iu_2,v_1)\in D$ and such that $v_1\le-(1+u_2^2)/2$,

$$\operatorname{Re}\{w(iu_2,v_1)\}=\frac{1}{\alpha+\beta-1}\Big\{\delta(\alpha+\beta)-1+$$

$$+\frac{\delta(1-\delta)v_1}{\delta^2+(1-\delta)^2u_2^2}\Big\}-\gamma\le\frac{1}{\alpha+\beta-1}\Big\{\delta(\alpha+\beta)-1-$$

$$-\gamma(\alpha+\beta-1)-\frac{\delta(1-\delta)(1+u_2^2)}{2\{\delta^2+(1-\delta)^2u_2^2\}}\Big\}\le 0.$$

Thus, the function $w(u,v)$ satisfies the conditions in Lemma 1. This shows that $\operatorname{Re}\{p(z)\}>0$, or

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\}>\delta\quad(z\in U).\qquad\square \tag{2.22}$$

If we take $\gamma=(\alpha+\beta-3)/2(\alpha+\beta-1)$ in Theorem 2, then we have

**Corollary 2.** *If $f(z)\in A$ satisfies*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}\right\}>\frac{\alpha+\beta-3}{2(\alpha+\beta-1)}$$
$$(\alpha>2,\ \beta>-1;\ z\in U),$$

*then*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\}>\frac{1}{2}\quad(z\in U).$$

Further, letting $\alpha=2-\beta$ and $\gamma=1/2$ in Theorem 2, we have

**Corollary 3.** *If $f(z) \in A$ satisfies*

$$\mathrm{Re}\left\{\frac{Q_\beta^{-\beta} f(z)}{Q_\beta^{1-\beta} f(z)}\right\} > \frac{1}{2} \quad (\beta > -1;\ z \in U),$$

*then*

$$\mathrm{Re}\left\{\frac{Q_\beta^{1-\beta} f(z)}{Q_\beta^{2-\beta} f(z)}\right\} > \frac{1+\sqrt{5}}{4} \quad (z \in U).$$

## 3. An Application of Jack's Lemma

We need the following lemma due to Jack [8], (also, due to Miller and Mocanu [7]).

**Lemma 2.** *Let $w(z)$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write*

$$z_0 w'(z_0) = k w(z_0), \tag{3.1}$$

*where $k$ is a real number and $k \geq 1$.*

Applying Lemma 2 for the operator $P^\alpha$, we have

**Theorem 3.** *If $f(z) \in A$ satisfies*

$$\mathrm{Re}\left\{\frac{P^{\alpha-2} f(z)}{P^{\alpha-1} f(z)}\right\} > \beta \ \ (\alpha > 2,\ \beta \leq 1/4;\ z \in U), \tag{3.2}$$

*then*

$$\mathrm{Re}\left\{\frac{P^{\alpha-1} f(z)}{P^{\alpha} f(z)}\right\} > \gamma \quad (z \in U), \tag{3.3}$$

*where*

$$\gamma = \frac{3 + 4\beta + \sqrt{16\beta^2 - 40\beta + 9}}{8}. \tag{3.4}$$

*Proof.* Defining the function $w(u, v)$ by

$$\frac{P^{\alpha-1} f(z)}{P^{\alpha} f(z)} = \frac{1 - (1-2\gamma) w(z)}{1 + w(z)}, \tag{3.5}$$

we see that $w(z)$ is analytic in $U$ and $w(0) = 0$. It follows from (3.5) that

$$\begin{aligned} \frac{z(P^{\alpha-1} f(z))'}{P^{\alpha-1} f(z)} &= \frac{z(P^{\alpha} f(z))'}{P^{\alpha} f(z)} - \\ &- \frac{(1-2\gamma) z w'(z)}{1 - (1-2\gamma) w(z)} - \frac{z w'(z)}{1 + w(z)}. \end{aligned} \tag{3.6}$$

Using (2.3), we obtain that

$$\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{1-(1-2\gamma)w(z)}{1+w(z)} - \\ -\frac{1}{2}\frac{zw'(z)}{w(z)}\Big(\frac{(1-2\gamma)w(z)}{1-(1-2\gamma)w(z)} + \frac{w(z)}{1+w(z)}\Big). \tag{3.7}$$

If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1 \ \ (w(z_0) \neq -1),$$

then Lemma 2 gives us that

$$z_0 w'(z_0) = kw(z_0) \quad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ $(\theta \neq \pi)$, we have

$$\begin{aligned} \operatorname{Re}\Big\{\frac{P^{\alpha-2}f(z_0)}{P^{\alpha-1}f(z_0)}\Big\} &= \operatorname{Re}\Big\{\frac{1-(1-2\gamma)w(z_0)}{1+w(z_0)} - \\ &-\frac{k}{2}\Big(\frac{(1-2\gamma)w(z_0)}{1-(1-2\gamma)w(z_0)} + \frac{w(z_0)}{1+w(z_0)}\Big)\Big\} = \\ = \operatorname{Re}\Big\{\frac{1-(1-2\gamma)e^{i\theta}}{1+e^{i\theta}} &- \frac{k}{2}\Big(\frac{(1-2\gamma)e^{i\theta}}{1-(1-2\gamma)e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}}\Big)\Big\} = \\ &= \gamma - \frac{k}{2}\Big(\frac{(1-2\gamma)(\cos\theta-(1-2\gamma))}{1+(1-2\gamma)^2-2(1-2\gamma)\cos\theta} + \frac{1}{2}\Big) \leq \\ &\leq \gamma - \frac{k}{2}\Big(\frac{1}{2} - \frac{1-2\gamma}{2(1-\gamma)}\Big) \leq \frac{\gamma(3-4\gamma)}{4(1-\gamma)} = \beta. \end{aligned} \tag{3.8}$$

This contradicts our condition (3.2). Therefore, we have

$$|w(z)| = \left| \frac{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} - 1}{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)} + (1-2\gamma)} \right| < 1 \ \ (z \in U), \tag{3.9}$$

which implies (3.3). □

Taking $\beta = 0$ and $\beta = 1/4$ in Theorem 3, we have

**Corollary 4.** *Let $f(z)$ be in the class $A$. Then*

$$\operatorname{Re}\Big\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\Big\} > 0 \quad (\alpha > 2;\ z \in U) \\ \Longrightarrow \operatorname{Re}\Big\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\Big\} > \frac{3}{4} \quad (z \in U)$$

*and*

$$\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{4} \quad (\alpha > 2;\ z \in U)$$
$$\Longrightarrow \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^{\alpha}f(z)}\right\} > \frac{1}{2} \quad (z \in U).$$

Finally, we prove

**Theorem 4.** *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{Q^{\alpha-2}f(z)}{Q^{\alpha-1}f(z)}\right\} > \gamma \quad (\alpha > 2;\ \beta > -1;\ z \in U) \tag{3.10}$$

*for some $\gamma$ $(\gamma < 1)$, then*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \delta \quad (z \in U), \tag{3.11}$$

*where $\delta$ $(0 \leq \delta < 1)$ is the smallest positive root of the equation*

$$\begin{aligned} &2(\alpha+\beta)\delta^2 - \{2(\alpha+\beta)(\gamma+1) - \\ &-(2\gamma-1)\}\delta + 2\{(\alpha+\beta-1)\gamma+1\} = 0. \end{aligned} \tag{3.12}$$

*Proof.* Defining the function $w(z)$ by

$$\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)} = \frac{1-(1-2\delta)w(z)}{1+w(z)}, \tag{3.13}$$

we obtain that

$$\begin{aligned} \frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)} = \frac{1}{\alpha+\beta-1}\Big\{(\alpha+\beta)\frac{1-(1-2\delta)w(z)}{1+w(z)} - 1 - \\ -\frac{zw'(z)}{w(z)}\Big(\frac{(1-2\delta)w(z)}{1-(1-2\delta)w(z)} + \frac{w(z)}{1+w(z)}\Big)\Big\}. \end{aligned} \tag{3.14}$$

Therefore, supposing that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1 \ \ (w(z_0) \neq -1),$$

we see that

$$\begin{aligned} \operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z_0)}{Q_\beta^{\alpha-1}f(z_0)}\right\} \leq \\ \leq \frac{1}{\alpha+\beta-1}\Big\{(\alpha+\beta)\delta - 1 - \frac{\delta}{2(1-\delta)}\Big\} = \gamma. \quad \square \end{aligned} \tag{3.15}$$

Making $\gamma = 0$ in Theorem 4, we have

**Corollary 5.** *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}\right\} > 0 \quad (\alpha > 2;\ \beta > -1;\ z \in U), \tag{3.16}$$

*then*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \frac{-1}{\alpha+\beta-1} \quad (z \in U). \tag{3.17}$$

Letting $\gamma = 1/2$ in Theorem 4, we have

**Corollary 6.** *If $f(z) \in A$ satisfies*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-2}f(z)}{Q_\beta^{\alpha-1}f(z)}\right\} > \frac{1}{2} \quad (\alpha > 2;\ \beta > -1;\ z \in U), \tag{3.18}$$

*then*

$$\operatorname{Re}\left\{\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}\right\} > \frac{\alpha+\beta-3}{\alpha+\beta-1} \quad (z \in U). \tag{3.19}$$

*Further, if $\alpha = 3-\beta$, we have*

$$\operatorname{Re}\left\{\frac{Q_\beta^{1-\beta}f(z)}{Q_\beta^{2-\beta}f(z)}\right\} > \frac{1}{2} \quad (\beta > -1;\ z \in U)$$

$$\Longrightarrow \operatorname{Re}\left\{\frac{Q_\beta^{2-\beta}f(z)}{Q_\beta^{3-\beta}f(z)}\right\} > 0 \quad (z \in U).$$

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