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ON STRONG DIFFERENTIABILITY OF INTEGRALS ALONG DIFFERENT DIRECTIONS

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Abstract. Theorems are proved as regards strong differentiability of integrals in different directions.

§ 1. Introduction

The well-known negative result in the theory of strong differentiability of integrals reads: there exists a summable function whose integral is differentiated in a strong sense in none of the directions.

Below we shall prove the theorems which in particular imply: for each pair of directions γ_1 and γ_2 differing from each other there exists a non-negative summable function whose integral is strongly differentiated in the directions γ_1 and is not strongly differentiated in the direction γ_2.

§ 2. Definitions and Formulation of the Problem

Let $B(x)$ be a differentiation basis at the point $x \in \mathbb{R}^n$, i.e., a family of bounded measurable sets with positive measure containing x and such that there is at least a sequence $\{B_k\} \subset B(x)$ with $\operatorname{diam}(B_k) \to 0$ as $k \to \infty$, (see, [1,Ch. II, Section 2]). A collection $B = \{B(x) : x \in \mathbb{R}^n\}$ is called a differentiation basis in $\mathbb{R}^n$.

For $f \in L_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by $M_B f(x)$, $\overline{D}_B(\int f, x)$, and $\underline{D}_B(\int f, x)$ are denoted, respectively, the maximal Hardy–Littlewood function with respect to B, and the upper and the lower derivatives of the integral $\int f$ with respect to B at x. If $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x)$, then this number will be denoted by $D_B(\int f, x)$; the basis B will be said to differentiate $\int f$ if the equality $D_B(\int f, x) = f(x)$ holds for almost all $x \in \mathbb{R}^n$. If the differentiation basis differentiates the integrals of all functions from some class M, then it is said to differentiate M [1, Ch. II and Ch. III].

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Let γ denote the set of n mutually orthogonal straight lines in $\mathbb{R}^n$ which intersect at the origin. The union of such sets will be denoted by $\Gamma(\mathbb{R}^n)$. Elements of $\Gamma(\mathbb{R}^n)$ will be called directions. If $\gamma \in \Gamma(\mathbb{R}^n)$, then B_2^γ will denote the differentiation basis consisting of all n-dimensional rectangles whose sides are parallel to straight lines from γ.

The standard direction in $\mathbb{R}^n$ will be denoted by γ^0 and, for simplicity, the basis $B_2^{\gamma^0}$ will be written as B_2.

We shall denote by B_3 the differentiation basis in $\mathbb{R}^n$ ($n \geq 2$) consisting of all n-dimensional rectangles and by B_1 the differentiation basis in $\mathbb{R}^n$ consisting of all n-dimensional cubic intervals whose sides are parallel to the coordinate axes.

The fact that B_2 differentiates $L(1 + \log^+ L)^{n-1}(\mathbb{R}^n)$ is well known [2], but in a class wider than $L(1+\log^+ L)^{n-1}(\mathbb{R}^n)$ there exists a function whose integral is not differentiated almost everywhere by the basis B_2 ([3], [4]). On the other hand, the basis B_3 with "freely" rotating constituent rectangles does not differentiate $L^\infty(\mathbb{R}^n)$ [1,Ch. V, Section 2]. The dependence of differentiation properties on the orientation of the sides of rectangles leads to the problem proposed by A. Zygmund ([1, Ch. IV, Section 2]): given a function $f \in L(\mathbb{R}^2)$, is it possible to choose a direction of $\gamma \in \Gamma(\mathbb{R}^2)$ such that B_2^γ would differentiate $\int f$?

A negative answer to A. Zygmund's question was given by D. Marstrand [5] who constructed an example of an integrable function whose integral is not differentiated almost everywhere by the basis B_2^γ for any fixed direction γ. Some generalizations of this result were later given in [6] and [7]. Hence we face the question whether the following hypothesis holds: if $f \in L(\mathbb{R}^2)$ and B_2 does not differentiate $\int f$, then B_2^γ will not differentiate $\int f$ either whatever γ is.

We shall show that there exists a function $f \in L(\mathbb{R}^2)$ for which this hypothesis does not hold.

In connection with this we have to answer the following questions: what is a set of those directions from $\Gamma(\mathbb{R}^2)$ that differentiate $\int f$? What are optimal conditions for such functions being integrable?

§ 3. Statement of the Main Results

Note that $\Gamma(\mathbb{R}^2)$ corresponds in a one-to-one manner to the interval $[0, \frac{\pi}{2})$. To each direction γ we put into correspondence a number $\alpha(\gamma)$, $0 \leq \alpha(\gamma) < \frac{\pi}{2}$, which is defined as the angle between the positive direction of the axis ox and the straight direction from γ lying in the first quadrant of the plane. Elements of the set $\Gamma(\mathbb{R}^2)$ will be identified with points from $[0, \frac{\pi}{2})$. By the neighborhood of the point 0 will be meant the union of intervals $[0, \varepsilon) \cup (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$, $0 < \varepsilon < \frac{\pi}{4}$.

Denote $S = [0, 1]^2$, and $[T]$ the closure of a set T.

Theorem 1. *Let $\Phi(t)$ be a nondecreasing continuous function on the interval $[0,\infty)$ and $\Phi(t) = o(t\log^+ t)$ for $t \to \infty$. Then there exists a nonnegative summable function $f \in \Phi(L)(S)$ such that*

(a) $\overline{D}_{B_2}(\int f, x) = +\infty$ *a.e. on S;*

(b) $D_{B_2^\gamma}(\int f, x) = f(x)$ *a.e. on S for each γ from $\Gamma(\mathbb{R}^2)\backslash\gamma^0$ and, moreover,*

$$\sup_{\gamma:\varepsilon<\alpha(\gamma)<\frac{\pi}{2}-\varepsilon} \Big\{M_{B_2^\gamma} f(x)\Big\} < \infty \ \ \text{a.e. on } S$$

for each number ε, $0 < \varepsilon < \frac{\pi}{2}$.

Theorem 2. *Let $\Phi(t)$ be a nondecreasing continuous function on the interval $[0,\infty)$ and $\Phi(t) = o(t\log^+ t)$ as $t \to \infty$. Let, moreover, a sequence $(\gamma_n)_{n=1}^\infty \subset \Gamma(\mathbb{R}^2)$ be given. Then there exists a nonnegative summable function $f \in \Phi(L)(S)$ such that*

(a) *for every $n = 1, 2, \dots$*

$$\overline{D}_{B_2^{\gamma_n}}(\textstyle\int f, x) = +\infty \ \ \text{a.e. on } S; \tag{1}$$

(b) *for almost every $\gamma \in \Gamma(\mathbb{R}^2)$*

$$D_{B_2^\gamma}(\textstyle\int f, x) = f(x) \ \ \text{a.e. on } S \tag{2}$$

and, moreover, for every set T for which $[T] \subset \Gamma(\mathbb{R}^2)\backslash(\gamma_n)_{n=1}^\infty$ a function f can bu chosen such that in addition to (1) and (2) the following relaton will be fullfiled:

$$\sup_{\gamma:\gamma\in T} \Big\{M_{B_2^\gamma} f(x)\Big\} < \infty \ \ \text{a.e. on } S.$$

§ 4. Auxiliary Statements

To prove Theorem 1 we shall make use of the following two lemmas in which $I = [0, l_1] \times [0, l_2]$; χ_A and $|A|$ will stand below for the characteristic function and Lebesgue measure of a set A, respectively.

Lemma 1. *Let $\gamma \in \Gamma(\mathbb{R}^2)\backslash\gamma^0$. There exists a constant $c(\gamma)$, $1 < c(\gamma) < \infty$, such that the inequality*

$$\Big|\Big\{x \in \mathbb{R}^2 : M_{B_2^\gamma}(\chi_I)(x) > \lambda\Big\}\Big| < 9c(\gamma)\lambda^{-1}|I|$$

holds for any λ, $0 < \lambda < 1$, satisfying the condition

$$c(\gamma)\lambda^{-1} l_1 \le l_2.$$

Lemma 2. *The inequality*

$$\big|\big\{x \in \mathbb{R}^2 : M_{B_2}(\chi_I)(x) > \lambda\big\}\big| > \frac{1}{\lambda}\log\Big(\frac{1}{\lambda}\Big)|I|$$

holds for any number λ, $0 < \lambda < 1$.

One can easily verify Lemma 2.

Proof of Lemma 1. Let $x \notin I$. It is easy to show that the maximal function $M_{B_2^\gamma}(\chi_I)$ at the point x can be estimated from above as follows:

$$M_{B_2^\gamma}(\chi_I)(x) \le c(\gamma)\,\frac{l_1}{\rho(x,I)}, \tag{3}$$

where

$$c(\gamma) = 2\max\Big\{\frac{1}{\cos(\alpha(\gamma))};\frac{1}{\sin(\alpha(\gamma))}\Big\} \tag{4}$$

and $\rho(x,I)$ denotes the distance between x and the interval I.

Hence it is obvious that

$$\Big\{x\in\mathbb{R}^2 : M_{B_2^\gamma}(\chi_I(x) > \lambda\Big\} \subset$$
$$\subset \Big\{x\in\mathbb{R}^2 : c(\gamma)\,\frac{l_1}{\rho(x,I)} > \lambda\Big\} \subset Q(I,\lambda^{-1},\gamma), \tag{5}$$

where

$$Q(I,\lambda^{-1},\gamma) = \big[-c(\gamma)\lambda^{-1}l_1, 2c(\gamma)\lambda^{-1}l_1\big] \times \big[-l_2, 2l_2\big]. \tag{6}$$

Clearly,

$$\big|Q(I,\lambda^{-1},\gamma)\big| = 9c(\gamma)\lambda^{-1}|I|.$$

We eventualy obtain

$$\Big|\Big\{x\in\mathbb{R}^2 : M_{B_2^\gamma}(\chi_I(x) > \lambda\Big\}\Big| \le \big|Q(I,\lambda^{-1},\gamma),\big| = 9c(\gamma)\lambda^{-1}|I|. \qquad \square$$

§ 5. Proof of the Main Results

Proof of Theorem 1. Let $(c_n)_{n=1}^\infty$ be an increasing sequence of natural numbers and the constants $c(\gamma)$ be defined by equality (4). Obviously, $1 < c < \infty$, where

$$c = \sup\Big\{c(\gamma) : \gamma\in\Gamma(\mathbb{R}^2),\ \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon\Big\}.$$

Let us construct a sequence of natural numbers $(\beta_n)_{n=1}^\infty$ so as to satisfy the conditions

$$\beta_n\log(\beta_n) \ge \max\big\{c_n\beta_n 2^{2n}; 2^n\Phi(\beta_n)\big\}. \tag{7}$$

The intervals $I^n = [0,l_1^n]\times[0,l_2^n]$ for $n = 1,2,\dots$ will be constructed so as to satisfy the equality

$$c_n\beta_n 2^n l_1^n = l_2^n. \tag{8}$$

In what follows, for $g \in L(\mathbb{R}^2)$, $0 < \lambda < \infty$, $\gamma \in \Gamma(\mathbb{R}^2)\setminus\gamma^0$, we shall use the following notation:

$$H_2(g,\lambda) = \{x \in \mathbb{R}^2 : M_{B_2}g(x) \geq \lambda\},$$
$$H_2^\gamma(g,\lambda) = \{x \in \mathbb{R}^2 : M_{B_2^\gamma}g(x) > \lambda\}.$$

Using (7) and Lemma 2 we obtain

$$\big|H_2(\beta_n\chi_I,1)\big| = \big|H_2(\chi_{I^n},\beta_n^{-1})\big| > \beta_n\log(\beta_n)|I^n| > 2^n\big(c_n\beta_n 2^n|I^n|\big). \quad (9)$$

Consider the interval

$$Q^n = \big[-c_n\beta_n 2^n l_1^n, c_n\beta_n 2^{n+1} l_1^n\big] \times \big[-l_2^n, 2l_2^n\big].$$

Note that if γ satisfies the condition $c(\gamma) < c_n$, then (see (5), (6), (8)) then we have the inclusions

$$H_2^\gamma(\beta_n\chi_{I^n},2^{-n}) \subset Q(I^n,\beta_n 2^n,\gamma) \subset Q^n. \quad (10)$$

Since Q^n is the cubic interval (see (8)), it is easy to ascertain that for each direction γ there exists an interval $E^{n,\gamma}$ such that $E^{n,\gamma} \in B_2^\gamma$ and the conditions

$$Q^n \subset E^{n,\gamma} \subset 2Q^n \quad (11)$$

are fulfilled (see Figure 1).

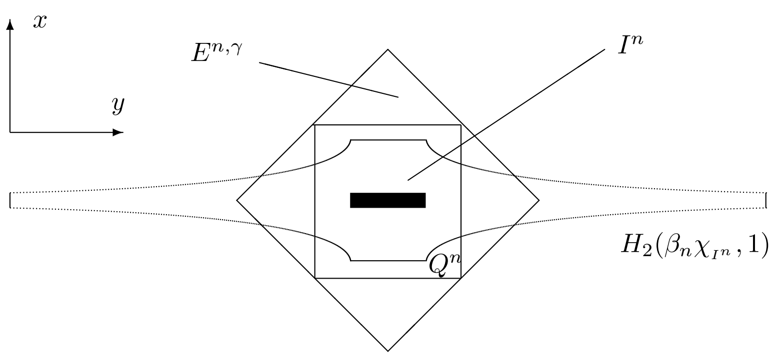

Figure 1

We have

$$|2Q^n| \leq 36c_n\beta_n 2^n|I^n|. \quad (12)$$

From (9) and (12) it follows that

$$\big|H_2(\beta_n\chi_{I^n},1)\big| > \frac{2^n}{36}\big(36c_n\beta_n 2^n|I^n|\big) \geq \frac{2^n}{36}|2Q^n|. \quad (13)$$

For each $n \in \mathbb{N}$ consider the set

$$H_2^n \equiv H_2(\beta_n \chi_{_{I^n}}, 1) \cup 2Q^n.$$

Since the set H_2^n is compact, almost the whole interval S can be represented as the union of nonintersecting sets that are homothetic to the set H_2^n and have a diameter not exceeding n^{-1} (see Lemma 1.3 from [1, Ch. III, Section 1]). Assuming that $H_2^{n,k}$ $(k = 1, 2, \dots)$ are such sets, we obtain

$$\operatorname{diam}\left(H_2^{n,k}\right) \le n^{-1}, \tag{14}$$

$$\left| S \setminus \bigcup_{k=1}^{\infty} H_2^{n,k} \right| = 0. \tag{15}$$

Let, moreover, $P^{n,k}$ denote a homothety transforming the set H_2^n to $H_2^{n,k}$. The images of the sets I^n, Q^n, $E^{n,\gamma}$ for the homothety $P^{n,k}$ will be denoted by $I^{n,k}$, $Q^{n,k}$ and $E^{n,\gamma,k}$.

Using one of the homothetic properties, from (13) we obtain

$$\left| \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| \le \frac{36}{2^n} \left| \bigcup_{k=1}^{\infty} H_2^{n,k} \right| = \frac{36}{2^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \left| \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| < \infty, \tag{16}$$

which implies

$$\left| \limsup_{n \to \infty} \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| = 0. \tag{17}$$

The function f_n on S is defined by

$$f_n(x) = \sum_{k=1}^{\infty} \beta_n \chi_{_{I^{n,k}}}(x), \quad n = 1, 2, \dots,$$

and f by

$$f(x) = \sup_{n \in \mathbb{N}} f_n(x).$$

Let us show that $f \in \Phi(L)(S)$. We have (see (15))

$$1 = |S| \ge \left| \bigcup_{k=1}^{\infty} H_2(\beta_n \chi_{_{I^{n,k}}}, 1) \right| > \sum_{k=1}^{\infty} \beta_n \log(\beta_n) |I^{n,k}|,$$

which by virtue of (7) gives us

$$1 \ge 2^n \sum_{k=1}^{\infty} \Phi(\beta_n) |I^{n,k}|.$$

Therefore

$$\int_S \Phi(f_n) = \sum_{k=1}^{\infty} \Phi(\beta_n)|I^{n,k}| \le 2^{-n}, \quad n = 1, 2, \dots .$$

Since $\Phi(t)$ is a continuous nondecreasing function, we obtain

$$\int_S \Phi(f) \le \sum_{n=1}^{\infty} \int_S \Phi(f_n) \le \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

To prove the theorem we have first to show that

$$\overline{D}_{B_2}(\textstyle\int f, x) = +\infty \text{ a.e. on } S.$$

We have (see (15))

$$1 = |S| = \Big|\limsup_{n\to\infty} \bigcup_{k=1}^{\infty} H_2^{n,k}\Big| = \Big|\limsup_{n\to\infty} \bigcup_{k=1}^{\infty} \big(H_2(\beta_n \chi_{I^{n,k}}, 1) \cup 2Q^{n,k}\big)\Big| \le$$
$$\le \Big|\limsup_{n\to\infty} \bigcup_{k=1}^{\infty} \big(H_2(\beta_n \chi_{I^{n,k}}, 1)\Big| + \Big|\limsup_{n\to\infty} \bigcup_{k=1}^{\infty} 2Q^{n,k}\big)\Big|.$$

Hence by virtue of (17) we conclude that

$$\Big|\limsup_{n\to\infty} \bigcup_{k=1}^{\infty} \big(H_2(\beta_n \chi_{I^{n,k}}, 1)\Big| = 1 \tag{18}$$

which clearly implies that for almost all $x \in S$ there exists a sequence $(n_i, k_i)_{i=1}^{\infty}$ (depending on x) such that

$$x \in H_2(\beta_{n_i} \chi_{I^{n_i,k_i}}, 1), \quad i = 1, 2, \dots .$$

Note further that

$$\operatorname{diam}\big(H_2(\beta_{n_i} \chi_{I^{n_i,k_i}}, 1)\big) < n_i^{-1}, \quad i = 1, 2, \dots .$$

From the inclusion $x \in H_2(\beta_{n_i} \chi_{I^{n_i,k_i}}, 1)$ and construction of sets $H_2(\beta_n \chi_{I^{n,k}}, 1)$ it follows that there exists an interval $R_i \in B_2(x)$ contained in the set $H_2(\beta_i \chi_{I^{n_i,k_i}}, 1)$ and

$$\frac{1}{|R_i|} \int_{R_i} \chi_{I^{n_i,k_i}}(y)\,dy \ge \beta_{n_i}^{-1}, \quad i = 1, 2, \dots . \tag{19}$$

We define f^n as

$$f^n(x) = \sup_{m\ge n} f_m(x).$$

By (19) the relations

$$\overline{D}_{B_2}(\textstyle\int f^n, x) \ge \displaystyle\lim_{i\to\infty} \frac{1}{|R_i|} \int_{R_i} f_{n_i}(y)dy \ge \lim_{i\to\infty} \frac{\beta_{n_i}}{|R_i|} \int_{R_i} \chi_{I^{n_i,k_i}}(y)dy \ge 1$$

a.e. on S.

We have

$$|\operatorname{supp}(f^n)| \le \sum_{m=n}^{\infty} |\operatorname{supp}(f_m)| \le \sum_{m=n}^{\infty} \Big| \bigcup_{k=1}^{\infty} 2Q^{m,k} \Big|.$$

Hence by virtue of (16) we find that for each number ε, $0 < \varepsilon < 1$, there exists a number $n = n(\varepsilon)$ such that $|\operatorname{supp}(f^n)| < \varepsilon$.

We have

$$\begin{aligned} &\big|\big\{x \in S : \overline{D}_{B_2}(\textstyle\int f^n, x) > f^n(x)\big\}\big| \ge \\ \ge &\big|\big\{x \in S : \overline{D}_{B_2}(\textstyle\int f^n, x) \ge 1,\ x \notin \operatorname{supp}(f^n)\big\}\big| = \\ = &\big|\big\{x \in S : x \notin \operatorname{supp}(f^n)\big\}\big| > 1 - \varepsilon \end{aligned}$$

which by the Besikovitch theorem (see [1, Ch. IV, Section 3]) implies

$$\big|\big\{x \in S : \overline{D}_{B_2}(\textstyle\int f^n, x) = +\infty\big\}\big| > 1 - \varepsilon.$$

Since ε is arbitrary and $f(x) \ge f^n(x)$, we have

$$\big|\big\{x \in S : \overline{D}_{B_2}(\textstyle\int f^n, x) = +\infty\big\}\big| = 1$$

thereby proving assertion (a) of Theorem 1.

Let us now show that if $\gamma \in \Gamma(\mathbb{R}^2)\backslash\gamma^0$ then

$$D_{B_2^\gamma}(\textstyle\int f, x) = f(x) \ \text{ a.e. on } \ S.$$

By virtue of the above-mentioned Besikovitch theorem it is sufficient to prove that

$$M_{B_2^\gamma} f(x) < \infty$$

at almost every point $x \in S$.

Fix $\gamma \ne \gamma^0$. Note that there exists a number $n(\gamma)$ such that $c(\gamma) < c_n$ for $n > n(\gamma)$. The inclusions (10) and (11) imply that the following inclusions are valid: if $n > n(\gamma)$ then

$$H_2^\gamma(\beta_n \chi_{I^{n,k}}, 2^{-n}) \subset Q(I^{n,k}, \beta_n 2^n, \gamma) \subset Q^{n,k} \subset E^{n,\gamma,k} \subset 2Q^{n,k} \quad (20)$$

for all $k = 1, 2 \dots$.

By (17) we find that for almost all $x \in S$ there exists a finite set $p(x) \subset \mathbb{N}$ with the properties

$$x \in \bigcup_{k=1}^{\infty} 2Q^{n,k} \quad \text{for} \quad n \in p(x)$$

and

$$x \notin \bigcup_{k=1}^{\infty} 2Q^{n,k} \quad \text{for} \quad n \notin p(x).$$

We obtain

$$M_{B_2^\gamma} f(x) \le \sum_{n=1}^{n(\gamma)} M_{B_2^\gamma} f_n(x) + \sum_{n\in p(x),\ n>n(\gamma)} M_{B_2^\gamma} f_n(x) +$$
$$+ \sum_{n\notin p(x),\ n>n(\gamma)} M_{B_2^\gamma} f_n(x) = I_1(x,\gamma) + I_2(x,\gamma) + I_3(x,\gamma).$$

Let us estimate from above the values $I_k(x,\gamma)$, $k = 1,2,3$. Note that

$$I_1(x,\gamma) \le \sum_{n=1}^{n(\gamma)} \beta_n < \infty$$

for all $x \in S$.

Similarly,

$$I_2(x,\gamma) \le \operatorname{card}\big(p(x)\big) \max_{n\in p(x)} \{\beta_n\} < \infty \ \text{ a.e. on } \ S.$$

We shall now prove that

$$M_{B_2^\gamma} f_n(x) \le 2^{-n}, \quad n \notin p(x), \quad n > n(\gamma).$$

Assume that $R^\gamma \in B_2^\gamma(x)$ and let $\{k_1^n, \dots, k_j^n, \dots\}$ denote a set of natural numbers for which

$$|R^\gamma \cap I^{n,k_j^n}| > 0, \quad j = 1,2,\dots .$$

Note that $R^\gamma \cap E^{n,\gamma,k_j^n} \in B_2^\gamma$, and if $n \notin p(x)$, $n > n(\gamma)$, then $x \notin E^{n,\gamma,k_j^n}$. Also taking into account the fact that the set $R^\gamma \cap E^{n,\gamma,k_j^n}$ contains at least one point from $(E^{n,\gamma,k_j^n})^c$, we obtain (see (20))

$$|R^\gamma \cap I^{n,k_j^n}| < \beta_n^{-1} 2^{-n} |R^\gamma \cap E^{n,\gamma,k_j^n}|, \quad j = 1,2,\dots,$$

for $n > n(\gamma)$, $n \notin p(x)$.

Hence it follows that if $n > n(\gamma)$, $n \notin p(x)$ then

$$\frac{1}{|R^\gamma|}\int_{R^\gamma} f_n(y)dy = \frac{\beta_n}{|R^\gamma|}\sum_{j=1}^{\infty} |R^\gamma \cap I^{n,k_j^n}| \le$$
$$\le \frac{\beta_n}{|R^\gamma|}\sum_{j=1}^{\infty} \beta_n^{-1} 2^{-n} |R^\gamma \cap E^{n,\gamma,k_j^n}|. \tag{21}$$

Since the sets E^{n,γ,k_j^n}, $j = 1,2,\dots$, do not intersect for $n > n(\gamma)$ (see (20)), we have

$$|R^\gamma| \ge \Big| \bigcup_{j=1}^{\infty} \big(R^\gamma \cap E^{n,\gamma,k_j^n}\big)\Big| = \sum_{j=1}^{\infty} |R^\gamma \cap E^{n,\gamma,k_j^n}|. \tag{22}$$

(21)–(22) imply that if $n \notin p(x)$, $n > n(\gamma)$ then

$$M_{B_2^\gamma} f_n(x) = \sup_{R^\gamma \in B_2^\gamma(x)} \frac{1}{|R^\gamma|} \int_{R^\gamma} f_n(y)dy \le 2^{-n}.$$

Therefore

$$I_3(x,\gamma) = \sum_{n \notin p(x),\ n > n(\gamma)} M_{B_2^\gamma} f_n(x) \le \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

Finally,

$$M_{B_2^\gamma} f(x) < \infty \tag{23}$$

for almost all $x \in S$, which proves the first part of assertion (b) of Theorem 1.

Let us prove the second part. There exists a number $n(\varepsilon)$ for which

$$c < c_n \quad \text{for} \quad n > n(\varepsilon). \tag{24}$$

We can write

$$M_{B_2^\gamma} f(x) \le \sum_{n=1}^{n(\varepsilon)} M_{B_2^\gamma} f_n(x) + \sum_{n \in p(x),\ n > n(\varepsilon)} M_{B_2^\gamma} f_n(x) + \\ + \sum_{n \notin p(x),\ n > n(\varepsilon)} M_{B_2^\gamma} f_n(x) = I_1(x) + I_2(x) + I_3(x).$$

We have

$$I_1(x) \le \sum_{n=1}^{n(\varepsilon)} \beta_n < \infty$$

and

$$I_2(x) \le \operatorname{card}\big(p(x)\big) \max_{n \in p(x)} \{\beta_n\} < \infty \quad \text{a.e. on } S.$$

The relations (10), (11), (24) imply that if $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$, then the inclusions

$$H_2^\gamma(\beta_n \chi_{I^{n,k}}, 2^{-n}) \subset Q(I^{n,k}, \beta_n^{-1} 2^{-n}, \gamma) \subset Q^{n,k} \subset E^{n,\gamma,k} \subset 2Q^{n,k} \tag{25}$$

hold for $n > n(\varepsilon)$, $k = 1, 2, \dots$.

From (25) it follows that relations (21) and (22) are valid for any direction γ for which $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$ when $n \notin p(x)$, $n > n(\varepsilon)$. This means that for such directions we have

$$M_{B_2^\gamma} f_n(x) \le 2^{-n} \quad \text{for} \quad n \notin p(x), \quad n > n(\varepsilon),$$

and thus

$$M_{B_2^\gamma} f(x) \le \sum_{n=1}^{n(\varepsilon)} \beta_n + \operatorname{card}\big(p(x)\big) \max_{n\in p(x)} \big\{\beta_n\big\} + 1 < \infty \quad \text{a.e. on } S.$$

Since the right-hand side does not depend on γ, we have

$$\sup_{\gamma:\varepsilon<\alpha(\gamma)<\frac{\pi}{2}-\varepsilon} \big\{M_{M_2^\gamma} f(x)\big\} < \infty \quad \text{a.e. on } S. \qquad \square$$

Remark. The first part of assertion (b) of Theorem 1 can be formulated as follows (see (23)):

$$M_{B_2^\gamma} f(x) < \infty \ \text{ a.e. on } S$$

for each direction $\gamma \in \Gamma(\mathbb{R}^2)\backslash\gamma^0$.

Proof of Theorem 2. Since $\gamma_n \notin \overline{T}$, the inclusion

$$K_n \equiv \big\{\gamma \in \Gamma(\mathbb{R}^2) : |\alpha(\gamma) - \alpha(\gamma_n)| > \pi\varepsilon_n^{-1}\big\} \supset T \tag{26}$$

holds for a sufficiently large number ε_n.

Note that if $\gamma_n = \gamma^0$ for some n, the set K_n will have the form

$$K_n \equiv \Big\{\gamma \in \Gamma(\mathbb{R}^2) : \pi\varepsilon_n^{-1} < \alpha(\gamma) < \frac{\pi}{2} - \pi\varepsilon_n^{-1}\Big\}.$$

Let, moreover,

$$\sum_{n=1}^{\infty} \varepsilon_n^{-1} < \infty. \tag{27}$$

Note that by virtue of the above remark Theorem 1 implies that for any number $n \in \mathbb{N}$ there exists a function $f_n \in \Phi(L)(S)$, $f \ge 0$, $\|\Phi(f_n)\|_{n(S)} < 2^{-n}$ such that the following three conditions hold:

$$\begin{cases} \overline{D}_{B_2^{\gamma_n}}(\textstyle\int f_n, x) = +\infty \ \text{ a.e. on } S, \\ M_{B_2^\gamma} f_n(x) < \infty, \ \ \forall\gamma \in \Gamma(\mathbb{R}^2)\backslash\gamma_n \ \text{ a.e. on } S, \\ \displaystyle\sup_{\gamma:|\alpha(\gamma)-\alpha(\gamma_n)|>\pi\varepsilon_n^{-1}} \big\{M_{B_2^\gamma} f_n(x)\big\} \le F_n(x), \end{cases}$$

where the function F_n is finite a.e. There exist a number P_n and a set $E_n \subset S$ such that $1 < P_n < \infty$, $|S\backslash E_n| < 2^{-n}$, and

$$F_n(x) \le P_n \quad \text{for} \quad x \in E_n, \quad n = 1, 2, \dots .$$

Let

$$g_n(x) = \frac{1}{P_n 2^n} f_n(x).$$

Clearly, the following conditions hold for g_n:

$$\begin{cases} \overline{D}_{B_2^{\gamma_n}}(\int g_n, x) = \dfrac{1}{P_n 2^n}\,\overline{D}_{B_2^{\gamma_n}}(\int f, x) = +\infty \ \text{ a.e. on } \ S, \\ M_{B_2^{\gamma}} g_n(x) < \infty, \ \ \forall \gamma \in \Gamma(\mathbb{R}^2) \backslash \gamma_n \ \text{ a.e. on } \ S, \\ \sup\limits_{\gamma : \gamma \in K_n \supset T} \big\{ M_{B_2^{\gamma}} g_n(x) \big\} \le \dfrac{1}{2^n} \ \text{ on } \ E_n. \end{cases}$$

We defined the function g as

$$g(x) = \sup_n g_n(x).$$

Note that $g \in \Phi(L)(S)$. Indeed, since $\Phi(t)$ is the nondecreasing continuous function,

$$\int_S \Phi(g) \le \sum_{n=1}^{\infty} \int_S \Phi(g_n) \le \sum_{n=1}^{\infty} \int_S \Phi(f_n) < \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

We have

$$\overline{D}_{B_2^{\gamma_n}}(\int g, x) \ge \overline{D}_{B_2^{\gamma_n}}(\int g_n, x) = +\infty \ \text{ a.e. on } \ S.$$

Relations (26) and (27) imply that

$$\sum_{n=1}^{\infty} |K_n^c| \le 2\pi \sum_{n=1}^{\infty} \varepsilon_n^{-1} < \infty.$$

Therefore

$$\big| \limsup_{n\to\infty} K_n^c \big| = 0. \tag{28}$$

Similarly, since

$$\sum_{n=1}^{\infty} |E_n^c| < \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

we have

$$\big| \limsup_{n\to\infty} E_n^c \big| = 0.$$

Define the sets

$$\begin{aligned} Z_n &\equiv \big\{ x \in S : F_n(x) = +\infty \big\}, \quad n = 1, 2, \dots, \\ K &\equiv \liminf_{n\to\infty} K_n, \\ E &\equiv \liminf_{n\to\infty} E_n \backslash \bigcup_{n=1}^{\infty} Z_n. \end{aligned}$$

For $\gamma \in K \backslash (\gamma_n)_{n=1}^{\infty}$ we set

$$G(\gamma) \equiv \bigcap_{n=1}^{\infty} \big\{ x \in S : M_{B_2^{\gamma}} g_n(x) < \infty \big\}.$$

Clearly,

$$|E| = |G(\gamma)| = 1, \quad |K| = \frac{\pi}{2}.$$

Let $\gamma \in K$. It follows from (28) that there exists a finite set $t(\gamma) \subset \mathbb{N}$ for which

$$\gamma \in K_n^c \quad \text{for} \quad n \in t(\gamma)$$

and

$$\gamma \in K_n \quad \text{for} \quad n \notin t(\gamma).$$

Similarly, if $x \in E$ then there exists a finite set $p(x) \subset \mathbb{N}$ for which

$$x \in E_n^c \quad \text{for} \quad n \in p(x),$$
$$x \in E_n \quad \text{for} \quad n \notin p(x).$$

Let us show that

$$M_{B_2^\gamma} g(x) < \infty$$

if $x \in G(\gamma) \cap E$ and $\gamma \in K \backslash (\gamma_n)_{n=1}^\infty$.

We can write

$$M_{B_2^\gamma} g(x) \leq \sum_{n \in p(x) \cup t(\gamma)} M_{B_2^\gamma} g_n(x) + \sum_{n \notin p(x),\, n \notin t(\gamma)} M_{B_2^\gamma} g_n(x) = I_1(x,\gamma) + I_2(x,\gamma).$$

For $x \in G(\gamma) \cap E$ we have

$$I_1(x,\gamma) \leq \operatorname{card}\big(p(x) \cup t(\gamma)\big) \max_{n \in p(x) \cup t(\gamma)} \big\{M_{B_2^\gamma} g_n(x)\big\} < \infty.$$

On the other hand,

$$I_2(x,\gamma) = \sum_{n : x \in E_n,\, \gamma \in K_n} M_{B_2^\gamma} g_n(x) \leq \sum_{n=1}^\infty 2^{-n} < \infty.$$

Therefore for $\gamma \in K \backslash (\gamma_n)_{n=1}^\infty$ we obtain ($|\sigma(\gamma) \cap E| = 1$)

$$M_{B_2^\gamma} g(x) < \infty \quad \text{a.e. on } S,$$

which by the Besikovitch theorem implies

$$D_{B_2^\gamma}(\textstyle\int g, x) = g(x) \quad \text{a.e. on } S.$$

Now let us prove the second part of condition (b) of Theorem 2. Since $K_n \supset T$, we have $K_n^c \cap T = \varnothing$, $n = 1, 2, \ldots$, and therefore

$$T \cap K_n^c = \varnothing, \quad n \in \mathbb{N}.$$

Thus if $\gamma \in T$, then $t(\gamma) = \varnothing$. We have

$$M_{B_2^\gamma} g(x) \leq \sum_{n \in p(x)} M_{B_2^\gamma} g_n(x) + \sum_{n \notin p(x)} M_{B_2^\gamma} g_n(x) = I_1(x) + I_2(x).$$

Note that if $\gamma \in T$, $x \in E$, then

$$I_1(x) \le \operatorname{card}\big(p(x)\big) \max_{n\in p(x)} \big\{F_n(x)\big\} < \infty$$

and

$$I_2(x) \le \sum_{n: x\in E_n} F_n(x) \le \sum_{n=1}^{\infty} 2^{-n} < 1.$$

Therefore if $\gamma \in T$ then we obtain ($|E| = 1$)

$$M_{B_2^\gamma} g(x) \le \operatorname{card}\big(p(x)\big) \max_{n\in p(x)} \big\{F_n(x)\big\} + 1 \ \text{ a.e. on } \ S,$$

and since the right-hand side of this inequality is independent of the direction γ, we have

$$\sup_{\gamma:\gamma\in T} \big\{M_{B_2^\gamma} g(x)\big\} < \infty \ \text{ a.e. on } \ S. \qquad \square$$

§ 6. Corollaries

Theorem 2 implies

Corollary 1. *There exists a nonnegative function $f \in L(S)$ such that the following conditions are fulfilled:*

(a) *if there is a direction γ such that $\alpha(\gamma)$ is a rational number, then*

$$\overline{D}_{B_2^\gamma}(\textstyle\int f, x) = +\infty \quad \textit{a.e. on } \ S;$$

(b) *for almost all directions γ*

$$D_{B_2^\gamma}(\textstyle\int f, x) = f(x) \quad \textit{a.e. on } \ S.$$

Definition. Assume that we are given a sequence of directions $(\gamma_n)_{n=1}^\infty$ and let $\gamma_n \nearrow \gamma$, $n \nearrow \infty$. Following [8], we shall say that the sequence of directions is exponential if there exists a constant $c > 0$ such that

$$|\alpha(\gamma_i) - \alpha(\gamma_j)| > c|\alpha(\gamma_i) - \alpha(\gamma)|, \quad i \ne j.$$

Corollary 2. *Assume that we are given two sequences of directions $(\gamma_n)_{n=1}^\infty$ and $(\gamma_n')_{n=1}^\infty$, the sequence $(\gamma_n)_{n=1}^\infty$ being exponential, and let*

$$\gamma_m' \in \Gamma(\mathbb{R}^2)\backslash\overline{\big(\gamma_n\big)_{n=1}^\infty}, \quad m = 1, 2, \dots .$$

There exists $f \in L(S)$, $f \ge 0$, such that

$$\overline{D}_{B_2^{\gamma_n'}}(\textstyle\int f, x) = +\infty \quad \textit{a.e. on } \ S, \ \ n = 1, 2, \dots,$$

and

$$D_B(\textstyle\int f, x) = f(x) \quad \textit{a.e. on } \ S,$$

where the differentiation basis B at the point x is defined as follows:

$$B(x) = \bigcup_n B_2^{\gamma_n}(x).$$

Proof. By virtue of Theorem 2 from [8] we find that the basis B with the exponential property differentiates the space $L^p(\mathbb{R}^2)$, $p > 2$. Therefore the basis with this property has the property of density. Using this fact and the fact that

$$M_B f(x) < \infty \quad \text{a.e. on} \quad S$$

(see Theorem 2, $T = (\gamma_n)_{n=1}^{\infty}$), by virtue of the de Guzmán and Menárguez theorem (see [1, Ch. IV, Section 3]), we obtain

$$D_B(\textstyle\int f, x) = f(x) \quad \text{a.e. on} \quad S. \qquad \square$$

§ 7. Remarks

1. A set of functions described by Theorems 1 and 2 forms a first-category set in $L(\mathbb{R}^2)$ (see Saks' theorem ([3]; [1, Ch. VII, Section 2]).

2. Let $\gamma_1,\ \gamma_2 \in \Gamma(\mathbb{R}^m)$, $m \geq 3$. Denote by $\alpha_k(\gamma_1, \gamma_2)$, $k = 1, 2, \dots, m$, the angle formed by the kth straight line of the direction γ_1 and by the kth straight line of the direction γ_2. If $\gamma \in \Gamma(\mathbb{R}^m)$ then we denote by $\overline{\gamma}$ the following subset from $\Gamma(\mathbb{R}^m)$:

$$\overline{\gamma} \equiv \left\{\gamma' \in \Gamma(\mathbb{R}^m) : \exists K \ (1 \leq k \leq m),\ \exists j \ (1 \leq j \leq 4),\ \alpha_k(\gamma, \gamma') = \frac{\pi}{2}(j-1)\right\}.$$

Without changing the essence of the proof of the main results, we can prove, for example,

Theorem 3. *Let $\Phi(t)$ be a nondecreasing continuous function on the interval $[0, \infty)$ and $\Phi(t) = o(t(\log^+ t)^{m-1})$ for $t \to \infty$ $(m \geq 3)$. For each pair of directions γ_1 and γ_2 for which $\gamma_2 \notin \overline{\gamma_1}$, there exists a nonnegative summable function $f \in \Phi(L)([0,1]^m)$ such that*

(a) $\overline{D}_{B_2^{\gamma_1}}(\int f, x) = +\infty$ *a.e. on* $[0,1]^m$;

(b) $D_{B_2^{\gamma_2}}(\int f, x) = f(x)$ *a.e. on* $[0,1]^m$.

3. We have ascertained that for one class of functions the so-called basis rotation changes the strong differentiability property of integrals. Note that there exist functions such that the basis rotation changes the integrability property of a strong maximal function. More exactly, for any number ε, $0 < \varepsilon < \frac{\pi}{2}$, there exists a function $f \in L(U)$, $U = [-1, 1]^2$, such that

(a) $$\int\limits_{\{M_{B_2} f > 1\}} M_{B_2} f(y)\, dy = +\infty;$$

(b) for any direction γ such that $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$ we have

$$\int\limits_{\{M_{B_2^\gamma} f > 1\}} M_{B_2^\gamma} f(y)\, dy < \infty.$$

Indeed, let the constant c, $1 < c < \infty$, be defined by the equality

$$c = \sup\big\{c(\gamma) : \gamma \in \Gamma(\mathbb{R}^2),\ \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon\big\}.$$

Consider the nonnegative function g for which the following three conditions are fulfilled:

$$g \in L \log^+ L(U) \setminus L(\log^+ L)^2(U), \tag{29}$$

$$\|g\|_1 > (2c)^{-1}, \quad \operatorname{supp}(g) \subset U. \tag{30}$$

Assuming that $0 \le \lambda < \infty$ and

$$E_\lambda = \big\{x \in U : g(x) > \lambda\big\},$$

we define the interval I_λ by

$$I_\lambda = [-l'_\lambda, l'_\lambda] \times [-1, 1],$$

where

$$l'_\lambda = 4^{-1}|E_\lambda|.$$

The function f is defined by

$$f(x) = \int_0^\infty \chi_{I_\lambda}(x)\, d\lambda.$$

It is clear that f and g are the equimeasurable functions. Let $x \in \mathbb{R}^2$, $\gamma \ne \gamma^0$, and $R \in B_2^\gamma(x)$. By using inequality (3) it is not difficult to show that there exists a cubic interval $Q_x \in B_1(x)$ such that for any λ we have the relation

$$\frac{1}{|R|}\, |R \cap I_\lambda| \le 4c(\gamma)\, \frac{1}{|Q_{x/3}|}\, |Q_{x/3} \cap I_\lambda| + c(\gamma)|I_\lambda|,$$

where $Q_{x/3}$ denotes the image of the interval Q_x under the homothety with center at the origin and coefficient $1/3$. Since R is arbitrary, we obtain (see [9, p. 649])

$$M_{B_2^\gamma} f(x) \le 4c(\gamma) M_{B_1} f(x/3) + c(\gamma) \int_S f.$$

Finally, for directions γ for which $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$ we have (see (30))

$$M_{B_2^\gamma} f(x) \le 4c(\gamma) M_{B_1} f(x/3) + 2^{-1}.$$

By virtue of Stein's theorem [10] and the fact that $f \in L\log^+ L(U)$ (see (29)) we obtain

$$\int\limits_{\{M_{B_2^\gamma}f>1\}} M_{B_2^\gamma} f(x)\,dx <$$

$$< 4c \int\limits_{\{x: M_{B_1}f(x/3)>1/8c\}} M_{B_1} f(x/3)\,dx + 2^{-1}\big|\{x : M_{B_1} f(x/3) > 1/8c\}\big| < \infty.$$

Now we shall prove assertion (a). Define the function $\Phi(x_1)$ as

$$\Phi(x_1) = f(x_1, 0).$$

Note that if $x_1 \in [-1, 1]$ and $x_2 \geq 1$ then

$$M_{B_2} f(x_1, x_2) \geq \frac{2}{x_2+1}\, M\Phi(x_1),$$

where $M\Phi(x_1)$ is the maximal Hardy–Littlewood function on the straight line. By performing transformations and using the fact that f does not belong to the class $L(\log^+ L)^2(U)$ we arrive at

$$\int\limits_{\{M_{B_2}f>1\}} M_{B_2} f(x_1, x_2)\,dx_1\,dx_2 \geq$$

$$\geq 2\int_1^\infty dx_2 \left(\int\limits_{\{x_1\in(-1,1): M\Phi(x_1)\geq \frac{x_2+1}{2}\}} \frac{1}{x_2+1}\, M\Phi(x_1)\,dx_1 \right) =$$

$$= 2 \int\limits_{\{x_1\in(-1,1): M\Phi(x_1)>1\}} M\Phi(x_1)\log^+\big(M\Phi(x_1)\big)\,dx_1 = +\infty.$$

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