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MIXED INTERFACE PROBLEMS FOR ANISOTROPIC ELASTIC BODIES

D. NATROSHVILI

Abstract. Three-dimensional mathematical problems of the elasticity theory of anisotropic piecewise homogeneous bodies are discussed. A mixed type boundary contact problem is considered where on one part of the interface, rigid contact conditions are given (jumps of the displacement and the stress vectors are known), while on the remaining part screen or crack type boundary conditions are imposed. The investigation is carried out by means of the potential method and the theory of pseudodifferential equations on manifolds with boundary.

1. Introduction

The investigation deals with the problems of anisotropic elasticity for composite bodies which have piecewise homogeneous structure. From the mathematical point of view these problems can be considered as problems involving a system of partial differential equations with discontinuous coefficients.

The most general structure of the elastic body under consideration mathematically can be described as follows. In three-dimensional Euclidean space $\mathbb{R}^3$ we have some closed smooth two-dimensional connected surface S_1 which involves other closed smooth surfaces $S_2, \dots S_m$ $(S_j \cap S_k = \varnothing,\ j \neq k)$. By these surfaces the space $\mathbb{R}^3$ is divided into several connected domains $\Omega_1, \dots, \Omega_\mu$. Each domain Ω_l is assumed to be filled up by an anisotropic material with the corresponding (in general, different) elastic coefficients

$$\overset{l}{c}_{kjpq} = \overset{l}{c}_{pqkj} = \overset{l}{c}_{jkpq}, \quad l = 1, \dots, \mu, \quad k, j, p, q = 1, 2, 3. \tag{1.1}$$

The common boundaries of two different materials are called contact boundaries (surfaces) of the piecewise homogeneous body. If some domains

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represent empty inclusions, then the surfaces corresponding to them together with S_1 are called boundary surfaces of the piecewise homogeneous body in question.

Such piecewise homogeneous bodies are encountered in many physical, mechanical, and engineering applications.

Classical and nonclassical mathematical problems for isotropic piecewise homogeneous bodies are studied in [1, 2, 3] by means of potential methods, while similar problems for anisotropic piecewise homogeneous bodies are investigated in [4, 5].

By using the functional methods analogous problems have been considered in [6].

In the present paper we treat the mixed boundary-contact problems with discontinuous boundary conditions on contact surfaces. Such types of problems have not been investigated even for isotropic piecewise homogeneous bodies. Our study is based on the potential methods and on the theory of pseudodifferential equations on manifolds with boundary.

A displacement vector corresponding to the domain Ω_l will be denoted by $\overset{l}{u} = (\overset{l}{u}_1, \overset{l}{u}_2, \overset{l}{u}_3)^T$; $\overset{l}{T}(D_x, n)\overset{l}{u}(x)$ denotes the corresponding stress vector calculated on the surface element with the unit normal vector $n = (n_1, n_2, n_3)$:

$$[\overset{l}{T}(D, n)\overset{l}{u}(x)]_k = \overset{l}{c}_{kjpq} D_q \overset{l}{u}_p(x), \quad D_q = \partial/\partial x_q.$$

Here and in what follows, summation over repeated indices is from 1 to 3. The symbol $[\cdot]^T$ denotes transposition.

Components of the stress $\overset{l}{\tau}_{kj}$ and of the strain $\overset{l}{e}_{kj}$ tensors are related by Hooke's law

$$\overset{l}{\tau}_{kj} = \overset{l}{c}_{kjpq} \overset{l}{e}_{pq}, \quad \overset{l}{e}_{pq} = 2^{-1}(D_p \overset{l}{u}_q + D_q \overset{l}{u}_p).$$

The potential energy

$$\overset{l}{E}(\overset{l}{u}, \overset{l}{u}) = \overset{l}{e}_{kj} \overset{l}{\tau}_{kj} = \overset{l}{c}_{kjpq} \overset{l}{e}_{kj} \overset{l}{e}_{pq} \tag{1.2}$$

is assumed to be a positive-definite quadratic form in symmetric variables $\overset{l}{e}_{kj} = \overset{l}{e}_{jk}$. Therefore there exists positive $\delta_0 > 0$ such that

$$\overset{l}{E}(\overset{l}{u}, \overset{l}{u}) \geq \delta_0 \overset{l}{e}_{kj} \overset{l}{e}_{kj}. \tag{1.3}$$

The basic homogeneous equation of statics reads as (provided bulk forces are equal to zero)

$$\overset{l}{A}(D_x)\overset{l}{u}(x) = 0, \quad x \in \Omega_l, \tag{1.4}$$

where

$$\overset{l}{A} = \|\overset{l}{A}_{kp}\|, \quad \overset{l}{A}_{kp}(D) = \overset{l}{c}_{kjpq} D_j D_q.$$

It follows from (1.1)–(1.3) that $\overset{l}{A}$ is a formally self-adjoint strongly elliptic matrix differential operator (cf. [6]), and consequently for any $\xi \in \mathbb{R}^3$ and for arbitrary complex vector $\eta \in \mathbb{C}^3$ the inequality

$$\operatorname{Re}\big(\overset{l}{A}(\xi)\eta, \eta\big) = \overset{l}{A}(\xi)\eta \cdot \eta \geq \delta_1 |\xi|^2 |\eta|^2$$

holds with $\delta_1 = const > 0$. As usual $(a, b) = a \cdot b = a_k \overline{b}_k$ denotes a scalar product of two vectors.

2. Formulation of the Problems

For simplicity we consider the following model problems. The piecewise homogeneous anisotropic body is assumed to consist of two connected domains $\Omega_1 = \Omega^+$ and $\Omega_2 = \Omega^-$, provided Ω^+ is bounded ($\operatorname{diam}\Omega^+ < \infty$) and $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega}^+$; $\overline{\Omega}^+ = \Omega^+ \cup S$, $S = \partial\Omega^\pm$. Thus we have only one contact surface S and the whole space $\mathbb{R}^3 = \overline{\Omega}_1 \cup \overline{\Omega}_2$ can be considered as a piecewise homogeneous anisotropic body with the interface S. For domains of general structure described in Section 1, all problems can be investigated similarly with slight modifications.

Let a smooth, connected, non-self-intersecting curve $\gamma \subset S$ divide the surface S into two parts S_1 and $S_2 : S = S_1 \cup S_2 \cup \gamma$; $\overline{S}_l = S_l \cup \gamma$.

The basic mixed contact (interface) problems can be formulated in the following way.

C–D Problem. Find the vectors $\overset{1}{u}$ and $\overset{2}{u}$ satisfying equations (1.4) in Ω_1 and Ω_2, respectively, and the contact conditions

$$\left.\begin{aligned} &[\overset{1}{u}(x)]^+ - [\overset{2}{u}(x)]^- = f(x), &(2.1)\\ &[\overset{1}{T}(D_x, n(x))\overset{1}{u}(x)]^+ - [\overset{2}{T}(D_x, n(x))\overset{2}{u}(x)]^- = F(x), &(2.2)\end{aligned}\right\} x \in S_1,$$

$$\left.\begin{aligned} &[\overset{1}{u}(x)]^+ = f_+(x), &(2.3)\\ &[\overset{2}{u}(x)]^- = f_-(x), &(2.4)\end{aligned}\right\} x \in S_2;$$

C–N Problem. Find the vectors $\overset{1}{u}$ and $\overset{2}{u}$ satisfying equations (1.4) in Ω_1 and Ω_2, respectively, contact conditions (2.1), (2.2) on S_1, and

$$\left.\begin{aligned} &[\overset{1}{T}(D_x, n(x))\overset{1}{u}(x)]^+ = F_+(x), &(2.5)\\ &[\overset{2}{T}(D_x, n(x))\overset{2}{u}(x)]^- = F_-(x), &(2.6)\end{aligned}\right\} x \in S_2;$$

here the symbols $[\cdot]^\pm$ denote limiting values on S from $\Omega^\pm$, and f, $f_\pm$, F, and $F_\pm$ are the given vector functions. In addition, in both problems we suppose that

$$\overset{2}{u}(x) = o(1) \tag{2.7}$$

as $|x| \to +\infty$. Condition (2.7) implies (see [7,8])

$$D^{\alpha}\overset{2}{u}(x) = O(|x|^{-1-|\alpha|}) \quad \text{as} \quad |x| \to +\infty \tag{2.8}$$

for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

From particular problems of mathematical physics and mechanics (see [9]) it is well known that, in general, solutions to mixed boundary value problems or the derivatives of the solutions possess singularities in the vicinity of curves of discontinuity of boundary conditions (curve γ) and they do not belong to the class of regular functions $C^1(\overline{\Omega}^{\pm})$. Moreover, for C^{∞}-regular data of problems they do not possess even C^{α}-smoothness with $\alpha > 1/2$ in the neighbourhood of γ while being infinitely differentiable elsewhere (i.e., in $\overline{\Omega}^{\pm}\backslash\gamma$).

Because of this fact we consider both mixed problems formulated above in Sobolev spaces

$$\overset{1}{u} \in W_p^1(\Omega^+), \quad \overset{2}{u} \in W_{p,loc}^1(\Omega^-) \tag{2.9}$$

with a view to involve a wider class of boundary data.

For conditions (2.1)–(2.6) to make sense we need some functional spaces.

By $B_{p,q}^{\nu}(\Omega^+)$, $B_{p,q,loc}^{\nu}(\Omega^-)$, $B_{p,q}^{\nu}(S)$ and $H_p^{\nu}(\Omega^+)$, $H_{p,loc}^{\nu}(\Omega^-)$, $H_p^{\nu}(S)$ are meant the Besov and the Bessel potential spaces, respectively, with $\nu \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$ (see [10,11,12]).

The definition of regular function spaces $C^{k+\alpha}(\overline{\Omega}^{\pm})$ and $C^{k+\alpha}(S)$ with integer $k \ge 0$ and $0 < \alpha < 1$ can be found, e.g., in [1].

Let us introduce the following functional spaces on $S_j \subset S$ $(j = 1, 2)$:

$$\begin{aligned}
B_{p,q}^{\nu}(S_j) &= \big\{ f|_{S_j} : f \in B_{p,q}^{\nu}(S) \big\}, \\
\widetilde{B}_{p,q}^{\nu}(S_j) &= \big\{ f \in B_{p,q}^{\nu}(S) : \operatorname{supp} f \subset \overline{S}_j \big\} \subset B_{p,q}^{\nu}(S), \\
H_p^{\nu}(S_j) &= \big\{ f|_{S_j} : f \in H_p^{\nu}(S) \big\}, \\
\widetilde{H}_p^{\nu}(S_j) &= \big\{ f \in H_p^{\nu}(S) : \operatorname{supp} f \subset \overline{S}_j \big\} \subset H_p^{\nu}(S)
\end{aligned}$$

(cf. [13,14]).

For vector function (2.9) the boundary and contact conditions for the displacement vectors can be considered in terms of traces. The first-order derivatives of the functions from $W_p^1(\Omega^+)$ and $W_{p,loc}^1(\Omega^-)$ belong to $L_p(\Omega^+)$ and $L_{p,loc}(\Omega^-)$, and they have no traces on S. For the boundary and contact conditions (2.2), (2.5), and (2.6) to make sense, we proceed as follows.

For regular solutions $\overset{l}{u} \in C^1(\overline{\Omega}_l) \cap C^2(\Omega_l)$ of equation (1.4) and for arbitrary regular vectors $\overset{l}{v} \in C^1(\overline{\Omega}_l)$ (with $\operatorname{diam}\operatorname{supp} \overset{2}{v} < +\infty$) the following Green's formulas are valid (see, e.g., [15])

$$\int_{\Omega^+} \overset{1}{A}(D)\overset{1}{u} \cdot \overset{1}{v}\, dx = -\int_{\Omega^+} \overset{1}{E}(\overset{1}{u}, \overset{1}{v})\, dx + \int_S [\overset{1}{T}(D,n)\overset{1}{u}]^+ \cdot [\overset{1}{v}]^+ dS, \tag{2.10}$$

$$\int_{\Omega^-} \overset{2}{A}(D)\overset{2}{u} \cdot \overset{2}{v}\, dx = -\int_{\Omega^-} \overset{2}{E}(\overset{2}{u}, \overset{2}{v})\, dx - \int_S [\overset{2}{T}(D,n)\overset{2}{u}]^+ \cdot [\overset{2}{v}]^+ dS, \tag{2.11}$$

where n is an exterior unit normal vector on S,

$$\overset{l}{E}(\overset{l}{u}, \overset{l}{v}) = \overset{l}{c}_{kjpq} D_k \overset{l}{u}_j D_p \overset{l}{v}_q. \tag{2.12}$$

We can rewrite (2.10) and (2.11) as follows (provided that $\overset{1}{u}$ and $\overset{2}{u}$ satisfy (2.9) and equation (1.4) (in the distributional sense))

$$\langle [\overset{1}{T}(D,n)\overset{1}{u}]^+, [\overset{1}{v}]^+ \rangle_S = \int_{\Omega^+} \overset{1}{E}(\overset{1}{u}, \overset{1}{v}) dx, \tag{2.13}$$

$$\langle [\overset{2}{T}(D,n)\overset{2}{u}]^-, [\overset{2}{v}]^- \rangle_S = -\int_{\Omega^-} \overset{2}{E}(\overset{2}{u}, \overset{2}{v}) dx \tag{2.14}$$

with

$$\overset{1}{v} \in W^1_{p'}(\Omega^+), \quad \overset{2}{v} \in W^1_{p',\mathrm{comp}}(\Omega^-), \quad p' = \frac{p}{p-1}.$$

It is evident that

$$[\overset{l}{v}]^{\pm}_S \in B^{1-1/p}_{p',p'}(S) = B^{1/p}_{p',p'}(S), \quad l = 1, 2,$$

and the symbol $\langle \cdot, \cdot \rangle_S$ defines the duality between $B^{-1/p}_{p,p}(S)$ and $B^{1/p}_{p',p'}(S)$, which for the smooth functions f and g has the form

$$\langle f, g \rangle_S = (f, g)_{L^2(S)} = \int_S f \cdot g\, dS.$$

Due to (2.12), dualities (2.13) and (2.14) define $[\overset{1}{T}\overset{1}{u}]^+$ and $[\overset{2}{T}\overset{2}{u}]^-$ on S correctly, and

$$[\overset{1}{T}\overset{1}{u}]^+,\ [\overset{2}{T}\overset{2}{u}]^- \in B^{-1/p}_{p,p}(S).$$

The conditions on stresses in (2.2), (2.5), and (2.6) are to be understood in the sense just described.

Now we can write precisely conditions for boundary data in the above formulated problems: $\overset{1}{u}$ and $\overset{2}{u}$ satisfy (2.7), (2.9) and

$$\begin{aligned} &f \in B^{1-1/p}_{p,p}(S_1), \quad f_\pm \in B^{1-1/p}_{p,p}(S_2), \\ &F \in B^{-1/p}_{p,p}(S_1), \quad F_\pm \in B^{-1/p}_{p,p}(S_2). \end{aligned} \tag{2.15}$$

It is evident that the compatibility conditions are

$$f^0 = \begin{cases} f & \text{on } S_1 \\ f_+ - f_- & \text{on } S_2 \end{cases} \in B^{1-1/p}_{p,p}(S) \tag{2.16}$$

in the C–D Problem, and

$$F^0 = \begin{cases} F & \text{on } S_1 \\ F_+ - F_- & \text{on } S_2 \end{cases} \in B^{-1/p}_{p,p}(S) \tag{2.17}$$

in the C–N Problem.

In what follows these conditions are assumed to be fulfilled.

Moreover, for simplicity we assume that S and γ possess the C^∞-smoothness (in fact it suffices to have some finite regularity).

3. Potentials and Their Properties

Let $\overset{l}{\Gamma}$ be the fundamental matrix of the operator $\overset{l}{A}(D)$

$$\overset{l}{A}(D)\overset{l}{\Gamma}(x) = I\delta(x),$$

where $\delta(\cdot)$ is the Dirac distribution and I is the unit matrix $I = \|\delta_{kj}\|_{3\times 3}$.

It can be proved that (see [15])

$$\overset{l}{\Gamma}(x) = (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-ix\xi}A^{-1}(-i\xi)\,d\xi = -\frac{1}{8\pi|x|}\int_0^{2\pi}\overset{l}{A}{}^{-1}(a\widetilde{\eta})\,d\varphi, \tag{3.1}$$

where $\overset{l}{A}{}^{-1}(\xi)$ is the matrix reciprocal to $\overset{l}{A}(\xi)$ and $a = \|a_{kj}\|_{3\times 3}$ is an orthogonal matrix with the property $a^T x = (0,0,|x|)^T$, $\widetilde{\eta} = (\cos\varphi, \sin\varphi, 0)$.

Obviously, (3.1) implies

$$\overset{l}{\Gamma}(tx) = t^{-1}\overset{l}{\Gamma}(x), \quad \overset{l}{\Gamma}(x) = \overset{l}{\Gamma}(-x) = [\overset{l}{\Gamma}(x)]^T, \quad t > 0.$$

Let us introduce the single- and double-layer potentials

$$(\overset{l}{V}g)(x) = \int_S \overset{l}{\Gamma}(x-y)g(y)\,dS_y,$$
$$(\overset{l}{U})(x) = \int_S [\overset{l}{T}(D_y, n(y))\overset{l}{\Gamma}(y-x)]^T g(y)\,dS_y.$$

The superscript l indicates that the potential corresponds to the fundamental matrix $\overset{l}{\Gamma}$.

The properties of these potentials are studied in [4,15] in regular $C^{k+\alpha}$ spaces, and in [14] in the Bessel potential H^ν_p and the Besov $B^\nu_{p,q}$ spaces.

We need some results obtained in the above cited works. We shall formulate them in the form of the following theorems.

Theorem 3.1 ([4,15]). *Let $k \geq 0$ be an integer and $0 < \alpha < 1$. Then*

$$\overset{l}{V} : C^{k+\alpha}(S) \to C^{k+1+\alpha}(\overline{\Omega}^{\pm}), \tag{3.2}$$

$$\overset{l}{U} : C^{k+\alpha}(S) \to C^{k+\alpha}(\overline{\Omega}^{\pm}). \tag{3.3}$$

For any $g \in C^{k+\alpha}(S)$ and $x \in S$

$$[(\overset{l}{V} g)(x)]^{+} = [(\overset{l}{V} g)(x)]^{-} = (\overset{l}{V} g)(x) \equiv \overset{l}{H} g(x), \tag{3.4}$$

$$[\overset{l}{T}(D_x, n(x))(Vg)(x)]^{\pm} = \big(\mp \tfrac{1}{2}\, I + \overset{l}{K}\big) g(x), \tag{3.5}$$

$$[(\overset{l}{U} g)(x)]^{\pm} = \big(\pm \tfrac{1}{2}\, I + \overset{l}{K}{}^{*}\big) g(x), \tag{3.6}$$

$$[\overset{l}{T}(D_x, n(x))(Ug)(x)]^{+} = [\overset{l}{T}(D_x, n(x))(Ug)(x)]^{-} \equiv \overset{l}{L} g(x), \quad k \geq 1, \tag{3.7}$$

where $n(x)$ is an exterior to Ω^{+} unit normal vector at the point $x \in S$; I stands for the unit operator,

$$\overset{l}{H} g(x) = \int_S \overset{l}{\Gamma}(x-y) g(y)\, dS_y,$$

$$\overset{l}{K} g(x) = \int_S \overset{l}{T}(D_x, n(x)) \overset{l}{\Gamma}(x-y) g(y)\, dS_y,$$

$$\overset{l}{K}{}^{*} g(x) = \int_S [\overset{l}{T}(D_y, n(y)) \overset{l}{\Gamma}(y-x)]^{T} g(y)\, dS_y,$$

$$\overset{l}{L} g(x) = \lim_{\Omega^{\pm} \ni z \to x \in S} \overset{l}{T}(D_z, n(x)) \int_S [\overset{l}{T}(D_y, n(y)) \overset{l}{\Gamma}(y-z)]^{T} g(y)\, dS_y.$$

Theorem 3.2 ([4,15]). *Let $k \geq 0$ be an integer and $0 < \alpha < 1$. Then*

(i) $$\overset{l}{H} : C^{k+\alpha}(S) \to C^{k+1+\alpha}(S), \tag{3.8}$$

$$\overset{l}{K},\ \overset{l}{K}{}^{*} : C^{k+\alpha}(S) \to C^{k+\alpha}(S), \tag{3.9}$$

$$\overset{l}{L} : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S); \tag{3.10}$$

(ii) *operators* (3.5) *and* (3.6) *are mutually adjoint singular integral operators* (*SIO*) *of normal type* (*i.e., their symbol matrices are not degenerated*) *and their indices are equal to zero; operators* (3.8) *and*

$$\tfrac{1}{2} I + \overset{l}{K}, \ \ \tfrac{1}{2} I + \overset{l}{K}{}^{*} : C^{k+\alpha}(S) \to C^{k+\alpha}(S)$$

are invertible; moreover

$$\overset{l}{H}{}^{-1} : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S)$$

is a singular integro-differential operator;

(iii) *for $h \in C^{\alpha}(S)$ and $g \in C^{1+\alpha}(S)$*

$$\big(-\overset{l}{H}h,h\big)_{L^2(S)} \geq 0, \quad \big(\overset{l}{L}g,g\big) \geq 0$$

with equality only for $h=0$ and $g(x)=[a\times x]+b$, where a and b are arbitrary three-dimensional constant vectors and $[\cdot\times\cdot]$ stands for a vector product;

(iv) *the general solution for the homogeneous equation*

$$\big(-\tfrac{1}{2}I+K^{*}\big)g=0$$

is $g(x)=[a\times x]+b$, $x\in S$, with arbitrary three-dimensional constant vectors a and b.

Theorem 3.3 ([15]). *Operator $\overset{l}{L}$ is a singular integro-differential operator, and the equations*

$$\begin{aligned}
&\overset{l}{K}{}^{*}\overset{l}{H}=\overset{l}{H}\overset{l}{K}, \quad \overset{l}{K}\overset{l}{L}=\overset{l}{L}\overset{l}{K}{}^{*},\\
&\overset{l}{H}\overset{l}{L}=-\tfrac{1}{4}I+(\overset{l}{K}{}^{*})^{2}, \quad \overset{l}{L}\overset{l}{H}=-\tfrac{1}{4}I+(\overset{l}{K})^{2}
\end{aligned}$$

hold.

Theorem 3.4 ([14]). *Operators (3.8)–(3.10) can be extended by continuity to the following bounded operators:*

$$\begin{aligned}
&\overset{l}{H}: H_p^{\nu}(S)\to H_p^{\nu+1}(S) && \big[B_{p,q}^{\nu}(S)\to B_{p,q}^{\nu+1}(S)\big],\\
&\overset{l}{K},\overset{l}{K}{}^{*}: H_p^{\nu}(S)\to H_p^{\nu}(S) && \big[B_{p,q}^{\nu}(S)\to B_{p,q}^{\nu}(S)\big],\\
&\overset{l}{L}: H_p^{\nu+1}(S)\to H_p^{\nu}(S) && \big[B_{p,q}^{\nu+1}(S)\to B_{p,q}^{\nu}(S)\big],\\
&\quad 1<p<\infty, \quad 1\leq q\leq\infty, && \nu\in\mathbb{R}.
\end{aligned}\tag{3.11}$$

Operators $\overset{l}{H}$, $\overset{l}{K}$, $\overset{l}{K}{}^{*}$, and $\overset{l}{L}$ are pseudodifferential operators of order -1, 0, 0, and 1, respectively.

Operators (3.11) *and*

$$\tfrac{1}{2}I+\overset{l}{K}, \;\; \tfrac{1}{2}I+\overset{l}{K}{}^{*}: H_p^{\nu}(S)\to H_p^{\nu}(S) \;\; \big[B_{p,q}^{\nu}(S)\to B_{p,q}^{\nu}(S)\big]$$

are invertible.

Theorem 3.5 ([14]). *Operators* (3.2) *and* (3.3) *can be extended by continuity to the following bounded operators:*

$$\overset{l}{V} : B^{\nu}_{p,p}(S) \to H^{\nu+1+1/p}_{p}(\Omega^+) \ \left[B^{\nu}_{p,q}(S) \to B^{\nu+1+1/p}_{p,q}(\Omega^+)\right], \quad (3.12)$$

$$\overset{l}{U} : B^{\nu}_{p,p}(S) \to H^{\nu+1/p}_{p}(\Omega^+) \ \left[B^{\nu}_{p,q}(S) \to B^{\nu+1/p}_{p,q}(\Omega^+)\right], \quad (3.13)$$

$$\overset{l}{V} : B^{\nu}_{p,p}(S) \to H^{\nu+1+1/p}_{p,loc}(\Omega^-) \ \left[B^{\nu}_{p,q}(S) \to B^{\nu+1+1/p}_{p,q,loc}(\Omega^-)\right], \quad (3.14)$$

$$\overset{l}{U} : B^{\nu}_{p,p}(S) \to H^{\nu+1/p}_{p,loc}(\Omega^-) \ \left[B^{\nu}_{p,q}(S) \to B^{\nu+1/p}_{p,q,loc}(\Omega^-)\right], \quad (3.15)$$

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad \nu \in \mathbb{R}.$$

Jump relations (3.4)–(3.7) *on* S *remain valid for operators* (3.12)–(3.15), *respectively, in the corresponding spaces.*

Theorem 3.6 ([14]). *Operators* $-\overset{l}{H}$ *and* $\overset{l}{L}$ *are formally self-adjoint pseudodifferential operators with positive-definite principal symbol matrices whose entries are homogeneous functions of order* -1 *and 1, respectively. Inequalities* (iii) *of Theorem* 3.2 *hold for any* $h \in H^{-1/2}_2(S)$ *and any* $g \in H^{1/2}_2(S)$ *with the same conclusion* (*they are to be understood as dualities*).

Theorem 3.7 ([14]). *The operators*

$$\overset{l}{H} : \widetilde{B}^{\nu}_{p,q}(S_j) \to B^{\nu+1}_{p,q}(S_j), \quad (3.16)$$

$$\overset{l}{H} : \widetilde{H}^{\nu}_{p}(S_j) \to H^{\nu+1}_{p}(S_j), \quad (3.17)$$

$$\overset{l}{L} : \widetilde{B}^{\nu+1}_{p,q}(S_j) \to B^{\nu}_{p,q}(S_j), \quad (3.18)$$

$$\overset{l}{L} : \widetilde{H}^{\nu+1}_{p}(S_j) \to H^{\nu}_{p}(S_j), \quad j = 1, 2, \quad (3.19)$$

are bounded for any $1 < p < \infty$, $1 \le q \le \infty$, $\nu \in \mathbb{R}$.

Operators (3.16) *and* (3.18) *are Fredholm operators if*

$$1/p - 3/2 < \nu < 1/p - 1/2 \quad (3.20)$$

holds.

Operators (3.17) *and* (3.19) *are Fredholm operators if and only if condition* (3.20) *holds.*

Operators (3.16)–(3.19) *are invertible for all* ν *satisfying* (3.20).

4. One Auxiliary Contact Problem (C–Problem)

Let us consider the following contact (interface) problem for a piecewise homogeneous space $\mathbb{R}^3 = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with an interface $S = \partial\Omega_1 = \partial\Omega_2$ (see Section 2).

C–Problem. Find regular vectors $\overset{1}{u}$ and $\overset{2}{u}$ satisfying equations (1.4) in Ω_1 and Ω_2, respectively, the contact conditions on S

$$\big[\overset{1}{u}(x)\big]^+ - \big[\overset{2}{u}(x)\big]^- = \varphi(x), \tag{4.1}$$

$$\big[\overset{1}{T}(D_x, n(x))\overset{1}{u}(x)\big]^+ - \big[\overset{2}{T}(D_x, n(x))\overset{2}{u}(x)\big]^- = \Phi(x), \quad x \in S, \tag{4.2}$$

and conditions (2.7) at infinity; here φ and Φ are the given vector-functions on S

$$\varphi \in C^{1+\alpha}(S), \quad \Phi \in C^{\alpha}(S).$$

Formulas (2.10) and (2.11) together with (1.3) imply

Lemma 4.1. *The homogeneous C-Problem has only the trivial solution.*

Let us look for the vectors $\overset{1}{u}$ and $\overset{2}{u}$ in the form of single layer potentials with special densities

$$\overset{l}{u}(x) = V\big[\overset{l}{H}{}^{-1}\overset{l}{g}\big](x), \quad x \in \Omega_l, \tag{4.3}$$

where $\overset{l}{H}{}^{-1}$ is the operator inverse to $\overset{l}{H}$ (see Theorem 3.2.ii)).

Then by Theorem 3.1 due to (4.1) and (4.2) for the unknown vector densities $\overset{l}{g}$ we obtain the following equations on S:

$$\begin{cases} \overset{1}{g} - \overset{2}{g} = \varphi, & (4.4)\\ \big(-\tfrac{1}{2}I + \overset{1}{K}\big)\overset{1}{H}{}^{-1}\overset{1}{g} - \big(\tfrac{1}{2}I + \overset{2}{K}\big)\overset{2}{H}{}^{-1}\overset{2}{g} = \Phi. & (4.5)\end{cases}$$

Denote

$$\overset{1}{N} = \big(-\tfrac{1}{2}I + \overset{1}{K}\big)\overset{1}{H}{}^{-1}, \quad \overset{2}{N} = -\big(\tfrac{1}{2}I + \overset{2}{K}\big)\overset{2}{H}{}^{-1}, \quad N = \overset{1}{N} + \overset{2}{N}. \tag{4.6}$$

By (4.4), (4.5), and (4.6) we have

$$\begin{cases} \overset{1}{g} = \varphi + \overset{2}{g}, & (4.7)\\ N\overset{2}{g} = \Phi - \overset{1}{N}\varphi. & (4.8)\end{cases}$$

Lemma 4.2. *Operators* $\overset{1}{N}$, $\overset{2}{N}$, *and* N *are singular integro-differential operators with the following properties:*

(i) $\overset{1}{N},\ \overset{2}{N},\ N : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S)$; (4.9)

for $f,\ g,\ h \in C^{1+\alpha}(S)$

$$(\overset{1}{N}f, f)_{L^2(S)} \geq 0, \quad (\overset{2}{N}g, g)_{L^2(S)} \geq 0, \quad (Nh, h)_{L^2(S)} \geq 0 \tag{4.10}$$

with equality only for $f(x) = [a \times x] + b$, $g = 0$, *and* $h = 0$; a *and* b *are arbitrary three-dimensional constant vectors.*

(ii) *Operators* $\overset{1}{N}$, $\overset{2}{N}$, *and* N *are formally self-adjoint pseudodifferential operators of order 1 and their principal* (*homogeneous of order* 1) *symbol matrices are positive definite;*

(iii) *Operators* (4.9) *can be extended by continuity to the following bounded operators:*

$$\overset{1}{N}, \overset{2}{N}, N : H_p^{\nu+1}(S) \to H_p^{\nu}(S) \ \ \big[B_{p,q}^{\nu+1}(S) \to B_{p,q}^{\nu}(S)\big]. \tag{4.11}$$

Inequalities (4.10) *remain valid for any* $f, g, h \in H_2^{1/2}(S)$ *with the same conclusion.*

The operator N *defined by* (4.9) *and* (4.11) *is invertible.*

Proof. The fact that $\overset{1}{N}$, $\overset{2}{N}$ and N are singular integro-differential operators follows directly from representation (4.6), since $\overset{l}{H}{}^{-1}$ is a singular integro-differential operator and $(\pm\frac{1}{2}I + \overset{l}{K})$ are singular integral operators. Therefore (4.9) holds.

To prove inequalities (4.10) we proceed as follows. If f and g have the $C^{1+\alpha}$ smoothness, then the single-layer potentials $\overset{1}{V}[\overset{1}{H}{}^{-1}f]$ and $\overset{2}{V}[\overset{2}{H}{}^{-1}g]$ are regular solutions of equations (1.4) in Ω_1 and Ω_2, respectively, satisfying condition (2.8). Due to formulas (2.10) and (2.11) we have

$$\begin{aligned}\int_{\Omega^+} \overset{1}{E}(\overset{1}{V}[\overset{1}{H}{}^{-1}f], \overset{1}{V}[\overset{1}{H}{}^{-1}f])\,dx = \\ = \int_S \big[\big(-\tfrac{1}{2}I + \overset{1}{K}\big)\overset{1}{H}{}^{-1}f\big] \cdot f\,dS \geq 0,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\int_{\Omega^-} \overset{2}{E}(\overset{2}{V}[\overset{2}{H}{}^{-1}f], \overset{2}{V}[\overset{2}{H}{}^{-1}f])\,dx = \\ = -\int_S \big[\big(\tfrac{1}{2}I + \overset{2}{K}\big)\overset{2}{H}{}^{-1}g\big] \cdot g\,dS \geq 0,\end{aligned} \tag{4.13}$$

where $\overset{l}{E}$ is defined by (2.12) and satisfies condition (1.3).

If in (4.12) and (4.13) we have equalities, then (see [15])

$$\overset{1}{V}[\overset{1}{H}{}^{-1}f](x)=[\overset{1}{a}\times x]+\overset{1}{b},\quad x\in\Omega_1=\Omega^+,$$
$$\overset{2}{V}[\overset{2}{H}{}^{-1}g](x)=[\overset{2}{a}\times x]+\overset{2}{b},\quad x\in\Omega_2=\Omega^-;$$

and applying (2.8), we conclude

$$f(x)=[\overset{1}{a}\times x]+\overset{1}{b},\quad g=0,\quad x\in S.$$

Thus (4.10) holds and assertion (i) is proved.

Formally self-adjointness of the operators $\overset{1}{N}$, $\overset{2}{N}$, and N readily follows from Theorem 3.3. Indeed,

$$\begin{aligned}N^*&=\big(\overset{1}{H}{}^{-1}\big)^*\big(-\tfrac12 I+\overset{1}{K}{}^*\big)-\big(\overset{2}{H}{}^{-1}\big)^*\big(\tfrac12 I+\overset{2}{K}{}^*\big)=\\&-\tfrac12\overset{1}{H}{}^{-1}+\overset{1}{H}{}^{-1}\overset{1}{K}{}^*-\tfrac12\overset{2}{H}{}^{-1}-\overset{2}{H}{}^{-1}\overset{2}{K}{}^*=\\&=-\tfrac12\overset{1}{H}{}^{-1}+\overset{1}{K}\overset{1}{H}{}^{-1}-\tfrac12\overset{2}{H}{}^{-1}-\overset{2}{K}\overset{2}{H}{}^{-1}=N.\end{aligned}$$

The remaining part of assertion (ii) follows from Theorems 3.4, 3.2.(ii), 3.6, and inequality (4.10).

The proof of assertion (iii) is quite similar to that of Theorems 3.4 and 3.6.

Due to the general theory of elliptic pseudodifferential equations on manifolds without boundary, the invertibility of the operator N follows both from the properties of its symbol matrix and from the positiveness of N. $\square$

Remark 4.3. Note that the operator N^{-1} inverse to operators (4.9) and (4.11) is a pseudodifferential operator of order -1 with the following mapping properties:

$$\begin{aligned}&N^{-1}:C^{k+\alpha}(S)\to C^{k+1+\alpha}(S),\quad k\ge0,\quad 0<\alpha<1,\\&N^{-1}:H_p^{\nu}(S)\to H_p^{\nu+1}(S)\ \big[B_{p,q}^{\nu}(S)\to B_{p,q}^{\nu+1}(S)\big],\\&\qquad 1<p<\infty,\quad 1\le q\le\infty,\quad \nu\in\mathbb{R}.\end{aligned}$$

The principal symbol matrix of N^{-1} is a positive-definite symmetric matrix with homogeneous (of order -1) entries.

Now from (4.3), (4.7), (4.8) and Lemmas 4.1 and 4.2 we get the representation of the unique solution of the C–problem:

$$\overset{1}{u}(x)=\overset{1}{V}\big[\overset{1}{H}{}^{-1}N^{-1}(\Phi+\overset{2}{N}\varphi)\big](x),\quad x\in\Omega_1,\tag{4.14}$$
$$\overset{2}{u}(x)=\overset{2}{V}\big[\overset{2}{H}{}^{-1}N^{-1}(\Phi-\overset{1}{N}\varphi)\big](x),\quad x\in\Omega_2,\tag{4.15}$$

where φ and Φ are vector functions contained in (4.1) and (4.2).

These formulas imply that if

$$\varphi \in C^{k+1+\alpha}(S), \quad \Phi \in C^{k+\alpha}(S),$$

then

$$\overset{l}{u} \in C^{k+1+\alpha}(\overline{\Omega}_l), \quad l = 1, 2$$

(see Lemma 4.2 and Theorem 3.1).

Remark 4.4. Note that if

$$\varphi \in B^{\nu+1}_{p,p}(S) \ \left[B^{\nu+1}_{p,q}(S)\right], \quad \Phi \in B^{\nu}_{p,p}(S) \ \left[B^{\nu}_{p,q}(S)\right],$$

then by (4.14) and (4.15)

$$\begin{gathered}\overset{1}{u} \in H^{\nu+1+1/p}_{p}(\Omega_1) \quad \left[B^{\nu+1+1/p}_{p,q}(\Omega_1)\right], \\ \overset{2}{u} \in H^{\nu+1+1/p}_{p,loc}(\Omega_2) \quad \left[B^{\nu+1+1/p}_{p,q,loc}(\Omega_2)\right], \\ 1 < p < \infty, \quad 1 \le q \le \infty, \quad \nu \in \mathbb{R}\end{gathered}$$

(see Theorems 3.4 and 3.5 and Lemma 4.2).

5. Investigation of the C–D Problem

Let us consider the C–D problem (2.1)–(2.4), (2.9), (2.15), (2.16). First we substitute conditions (2.3) and (2.4) by the equivalent equations

$$\left.\begin{aligned} [\overset{1}{u}(x)]^+ - [\overset{2}{u}(x)]^- &= f_+(x) - f_-(x), \qquad (5.1)\\ [\overset{1}{u}(x)]^+ + [\overset{2}{u}(x)]^- &= f_+(x) + f_-(x), \qquad (5.2)\end{aligned}\right\} x \in S_2.$$

Due to (2.16) it is evident that the difference

$$[\overset{1}{u}(x)]^+ - [\overset{2}{u}(x)]^- = f^0(x) \in B^{1-1/p}_{p,p}(S)$$

is a known vector on S.

Let $\widetilde{F}$ be some fixed extension of the vector F from S_1 onto S_2 which preserves the space, i.e.,

$$\widetilde{F} \in B^{-1/p}_{p,p}(S), \quad \widetilde{F}|_{S_1} = F.$$

Any other extension Φ of the vector F from S_1 onto S_2 preserving the space can be represented in the form

$$\Phi = \widetilde{F} + \Psi \in B^{-1/p}_{p,p}(S)$$

with arbitrary

$$\Psi \in \widetilde{B}^{-1/p}_{p,p}(S_2).$$

Let us now look for the solution of the C–D problem in the form (cf. (4.14), (4.15))

$$\overset{1}{u}(x) = \overset{1}{V}\big[\overset{1}{H}{}^{-1}N^{-1}(\widetilde{F} + \Psi + \overset{2}{N} f^0)\big](x), \quad x \in \Omega_1, \tag{5.3}$$

$$\overset{2}{u}(x) = \overset{2}{V}\big[\overset{2}{H}{}^{-1}N^{-1}(\widetilde{F} + \Psi - \overset{1}{N} f^0)\big](x), \quad x \in \Omega_2, \tag{5.4}$$

with the known vector-functions f^0, $\widetilde{F}$ and the unknown vector-function Ψ.

It is evident that equation (1.4) and conditions (2.1), (2.2), and (5.1) are satisfied. Condition (5.2) leads to the following pseudodifferential equation for Ψ:

$$N^{-1}\Psi = \tfrac{1}{2}(f_+ + f_-) - N^{-1}\widetilde{F} - \tfrac{1}{2}N^{-1}(\overset{2}{N} - \overset{1}{N})f^0 \quad \text{on} \quad S_2. \tag{5.5}$$

Note that by virtue of Lemma 4.2 and Remark 4.3

$$Q \equiv -N^{-1}\widetilde{F} - \tfrac{1}{2}N^{-1}(\overset{2}{N} - \overset{1}{N})f^0 \in B^{1-1/p}_{p,p}(S) \tag{5.6}$$

and

$$q \equiv \tfrac{1}{2}[f_+ + f_-] + Q|_{S_2} \in B^{1-1/p}_{p,p}(S_2). \tag{5.7}$$

Let r_2 be the restriction operator to S_2. Then (5.5) can be rewritten in the form

$$r_2 N^{-1}\Psi = q \quad \text{on} \quad S_2 \tag{5.8}$$

with q defined by (5.7), (5.6).

We have to investigate the solvability of (5.8) in the Besov space $\widetilde{B}^{-1/p}_{p,p}(S_2)$.

To this end denote by $\sigma(x;\xi)$, $x \in S$, $\xi \in \mathbb{R}^2$ the principal symbol matrix of the operator N^{-1} whose entries are homogeneous functions of order -1 with respect to ξ.

To establish the Fredholm property of equation (5.8) due to the general theory of pseudodifferential equations on manifold with boundary, we must investigate eigenvalues of the matrix

$$M = [\sigma(x;0,-1)]^{-1}[\sigma(x;0,1)]$$

(see [16]–[20]).

Lemma 5.1. *Eigenvalues of the matrix M for any $x \in S$ are positive numbers.*

Proof of the Lemma follows from positive definiteness of the matrix $\sigma(x;\xi)$ for any $x \in S$ and $|\xi| = 1$. Indeed, if λ is an eigenvalue of the matrix M, then there exists $\eta \in \mathbb{C}^3 \setminus \{0\}$ such that $M\eta = \lambda\eta$, whence

$$\sigma(x;0,1)\eta = \lambda\sigma(x;0,-1)\eta,$$

i.e.,

$$\lambda = \big(\sigma(x;0,1)\eta\cdot\eta\big)\big[\big(\sigma(x;0,-1)\eta\cdot\eta\big)\big]^{-1} > 0. \qquad \square$$

Lemma 5.1 implies (see [16]–[18], 20])

Theorem 5.2. *Let $1 < p < \infty$, $1 \le q \le \infty$. Then the operators*

$$r_2 N^{-1} : \widetilde{B}^{\nu}_{p,q}(S_2) \to B^{\nu+1}_{p,q}(S_2), \tag{5.9}$$

$$r_2 N^{-1} : \widetilde{H}^{\nu}_{p}(S_2) \to H^{\nu+1}_{p}(S_2) \tag{5.10}$$

are bounded for all $\nu \in \mathbb{R}$.

Operator (5.9) *is a Fredholm operator if* (3.20) *holds.*

Operator (5.10) *is a Fredholm operator if and only if conditions* (3.20) *hold.*

Both operators (5.9) *and* (5.10) *are invertible if ν and p satisfy inequality* (3.20)*.*

It is evident that Theorem 5.2 yields unique solvability of equation (5.8) for $4/3 < p < 4$. Consequently, we obtain the following

Theorem 5.3. *Let $4/3 < p < 4$ and conditions* (2.15), (2.16) *be fulfilled. Then the C–D problem has a unique solution satisfying* (2.9)*. The solution is representable in the form* (5.3), (5.4) *with Ψ defined by the uniquely solvable pseudodifferential equation* (5.8)*.*

The method described above makes it possible to improve the regularity property of the solution by increasing the smoothness of boundary data.

Theorem 5.4. *Let $4/3 < p < 4$, $1 < t < \infty$, $1 \le q \le \infty$, $1/t - 3/2 < \nu < 1/t - 1/2$, and $\overset{1}{u}$, $\overset{2}{u}$ be the solution of the C–D problem satisfying conditions* (2.9)*. In addition, if*

$$f \in B^{\nu+1}_{t,t}(S_1), \quad f_\pm \in B^{\nu+1}_{t,t}(S_2), \quad F \in B^{\nu}_{t,t}(S_1),$$

$$f^0 = \begin{cases} f & \text{on } S_1 \\ f_+ - f_- & \text{on } S_2 \end{cases} \in \quad B^{\nu+1}_{t,t}(S),$$

then

$$\overset{1}{u} \in H^{\nu+1+1/t}_{t}(\Omega_1), \quad \overset{2}{u} \in H^{\nu+1+1/t}_{t,loc}(\Omega_2).$$

If

$$f \in B^{\nu+1}_{t,q}(S_1), \quad f_\pm \in B^{\nu+1}_{t,q}(S_2), \quad F \in B^{\nu}_{t,q}(S_1), \quad f^0 \in B^{\nu+1}_{t,q}(S),$$

then

$$\overset{1}{u} \in B^{\nu+1+1/t}_{t,q}(\Omega_1), \quad \overset{2}{u} \in B^{\nu+1+1/t}_{t,q,loc}(\Omega_2).$$

In particular, if

$$f \in C^{\alpha}(S_1), \quad f_{\pm} \in C^{\alpha}(S_2), \quad F \in B^{\alpha-1}_{\infty,\infty}(S_1), \quad f^0 \in C^{\alpha}(S),$$

then

$$\overset{1}{u} \in C^{\alpha'}(\overline{\Omega}_1), \quad \overset{2}{u} \in C^{\alpha'}(\overline{\Omega}_2)$$

for any $\alpha' \in (0, \alpha_0)$*,* $\alpha_0 = \min\{\alpha, 1/2\}$*.*

Proof of the theorem follows from Theorems 5.2, 3.5 and the well-known embedding theorems for the Besov and the Bessel potential spaces (cf. similar theorems in [14, 20]). $\square$

6. Investigation of the C–N Problem

Now let us consider the C–N problem (2.1), (2.2), (2.5), (2.6), (2.9), (2.15), (2.17). We shall study this problem by applying the approach described in Section 5. First we substitute conditions (2.5) and (2.6) by the equivalent ones

$$\big[\overset{1}{T}(D_x, n(x))\overset{1}{u}(x)\big]^+ - \big[\overset{2}{T}(D_x, n(x))\overset{2}{u}(x)\big]^- = F_+(x) - F_-(x), \quad (6.1)$$

$$\big[\overset{1}{T}(D_x, n(x))\overset{1}{u}(x)\big]^+ + \big[\overset{2}{T}(D_x, n(x))\overset{2}{u}(x)\big]^- = F_+(x) + F_-(x), \quad (6.2)$$

$$x \in S_2.$$

Condition (6.1) combined with (2.2) gives (on S)

$$\big[\overset{1}{T}(D_x, n(x))\overset{1}{u}\big]^+ - \big[\overset{2}{T}(D_x, n(x))\overset{2}{u}\big]^- = F^0(x) \in B^{-1/p}_{p,p}(S)$$

due to (2.17).

Denote by $\widetilde{f}$ some fixed extension of the vector f from S_1 onto S_2 preserving the functional space

$$\widetilde{f} \in B^{1-1/p}_{p,p}(S), \quad \widetilde{f}\big|_{S_1} = f.$$

Any other extension φ of the vector f from S_1 onto S_2 preserving the functional space can be represented in the form

$$\varphi = \widetilde{f} + \psi \in B^{1-1/p}_{p,p}(S)$$

with

$$\psi \in \widetilde{B}^{1-1/p}_{p,p}(S_2).$$

Now if we look for the solution of the C–N problem in the form

$$\overset{1}{u}(x) = \overset{1}{V}\big\{\overset{1}{H}{}^{-1}N^{-1}\big[F^0 + \overset{2}{N}(\widetilde{f}+\psi)\big]\big\}(x), \quad x \in \Omega_1, \tag{6.3}$$

$$\overset{2}{u}(x) = \overset{2}{V}\big\{\overset{2}{H}{}^{-1}N^{-1}\big[F^0 - \overset{1}{N}(\widetilde{f}+\psi)\big]\big\}(x), \quad x \in \Omega_2, \tag{6.4}$$

then equation (1.4) and conditions (2.1), (2.2), (6.1) are satisfied automatically. Condition (6.2) implies the following equation on S_2 for the unknown vector $\psi \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$

$$\overset{2}{N}N^{-1}\overset{1}{N}\psi = \tfrac{1}{2}(F_+ + F_-) + \tfrac{1}{2}(\overset{2}{N} - \overset{1}{N})N^{-1}F^0 - \overset{2}{N}N^{-1}\overset{1}{N}\widetilde{f}. \tag{6.5}$$

This equation is obtained with the help of the equality

$$\overset{1}{N}N^{-1}\overset{2}{N} = \overset{2}{N}N^{-1}\overset{1}{N} \tag{6.6}$$

which can be easily verified.

It is evident that

$$Q_1 \equiv \tfrac{1}{2}(\overset{2}{N} - \overset{1}{N})N^{-1}F^0 - \overset{2}{N}N^{-1}\overset{1}{N}\widetilde{f} \in \widetilde{B}^{-1/p}_{p,p}(S), \tag{6.7}$$

$$q_1 \equiv \tfrac{1}{2}(F_+ + F_-) + Q_1\big|_{S_2} \in B^{-1/p}_{p,p}(S_2). \tag{6.8}$$

Using the notation

$$P = \overset{2}{N}N^{-1}\overset{1}{N},$$

from (6.5), (6.7), and (6.8) we get

$$r_2 P\psi = q_1 \quad \text{on} \quad S_2, \tag{6.9}$$

where r_2 is again the restriction operator (to S_2).

Solvability of equation (6.9) will be studied by the method applied in the previous section.

To this end we have to examine the properties of the operator P.

Lemma 6.1. *Operator P is a formally self-adjoint and non-negative operator with positive-definite principal symbol matrix. For arbitrary $h \in C^{1+\alpha}(S)$*

$$(Ph, h)_{L^2(S)} \geq 0$$

with equality only for $h = [a \times x] + b$, where a and b are arbitrary constant vectors.

Operators

$$P : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S), \quad k \geq 0, \quad 0 < \alpha < 1, \tag{6.10}$$

$$P : B^{\nu+1}_{p,q}(S) \to B^{\nu}_{p,q}(S), \tag{6.11}$$

$$P : H^{\nu+1}_{p}(S) \to H^{\nu}_{p}(S), \tag{6.12}$$

$$1 < p < \infty, \quad 1 \leq q \leq \infty, \quad \nu \in \mathbb{R},$$

are bounded.

Proof. Mapping properties (6.10)–(6.12) follow from Lemma 4.2. Self-adjointness is the consequence of equality (6.6) and Lemma 4.2.ii).

Let $h \in C^{1+\alpha}(S)$ and consider single layer potentials

$$\overset{1}{v}(x) = \overset{1}{V}\big[\overset{1}{H}{}^{-1} N^{-1} \overset{2}{N} h\big](x), \quad x \in \Omega_1,$$
$$\overset{2}{v}(x) = -\overset{2}{V}\big[\overset{2}{H}{}^{-1} N^{-1} \overset{1}{N} h\big](x), \quad x \in \Omega_2.$$

It is easy to show that these potentials are regular vectors in Ω_1 and Ω_2, respectively, and relation (2.8) holds for them. Therefore, making use of Green's formulas (2.10) and (2.11), we get

$$\int_{\Omega^+} \overset{1}{E}(\overset{1}{v}, \overset{1}{v}) dx = \int_S \overset{1}{N} N^{-1} \overset{2}{N} h \cdot N^{-1} \overset{2}{N} h \, dS,$$
$$\int_{\Omega^-} \overset{2}{E}(\overset{2}{v}, \overset{2}{v}) dx = \int_S \overset{2}{N} N^{-1} \overset{1}{N} h \cdot N^{-1} \overset{1}{N} h \, dS.$$

Upon taking the sum and using (6.6) and (4.6), we obtain

$$\int_{\Omega^+} \overset{1}{E}(\overset{1}{v}, \overset{1}{v}) dx + \int_{\Omega^-} \overset{2}{E}(\overset{2}{v}, \overset{2}{v}) dx = \int_S \overset{1}{N} N^{-1} \overset{2}{N} h \cdot h \, dS \geq 0.$$

In the case of the equality to zero we conclude that $\overset{1}{v}$ and $\overset{2}{v}$ are rigid displacements

$$\overset{1}{v}(x) = [a' \times x] + b', \ \ x \in \Omega_1; \quad \overset{2}{v}(x) = [a'' \times x] + b'', \ \ x \in \Omega_2.$$

Condition (2.8) for $\overset{2}{v}$ implies: $\overset{2}{v}(x) = 0$, $x \in \Omega_2$. Therefore $\overset{2}{H}{}^{-1} N^{-1} \overset{1}{N} h = 0$. Invertibility of the operators $\overset{2}{H}{}^{-1}$ and N^{-1} (see Theorem 3.2.ii) and Lemma 4.2) yields $\overset{1}{N} h = 0$, whence by Theorem 3.2.iv) $h(x) = [a \times x] + b$. Now it is easy to verify that $[\overset{1}{v}(x)]^+_S = h(x)$, i.e., $a' = a$, $b' = b$.

Positive definiteness of the principal symbol matrix of the operator P can be shown in the following way.

Denoting by σ, σ_1, σ_2, and σ_0 the principal symbol matrices of the operators N^{-1}, $\overset{1}{N}$, $\overset{2}{N}$, and P, respectively, we have (see (6.6))

$$\sigma_0(x;\xi) = \sigma_2(x;\xi)\sigma(x;\xi)\sigma_1(x;\xi) = \\ = \sigma_1(x;\xi)\sigma(x;\xi)\sigma_2(x;\xi), \quad x \in S, \;\; \xi \in \mathbb{R}^2 \setminus \{0\}. \tag{6.13}$$

By Lemma 4.2.(ii) all matrices on the right-hand side of (6.13) (and their reciprocal matrices) are positive definite. Moreover, the matrix $\sigma(x;\xi)$ has the form (see (4.6))

$$\sigma(x;\xi) = \big[\sigma_1(x;\xi) + \sigma_2(x;\xi)\big]^{-1}.$$

Hence

$$\sigma_0^{-1}(x;\xi) = \sigma_1^{-1}(x;\xi)\big[\sigma_1(x;\xi) + \sigma_2(x;\xi)\big]\sigma_2^{-1}(x;\xi) : \mathbb{C}^3 \to \mathbb{C}^3$$

is the isomorphism on $\mathbb{C}^3$ for any $x \in S$ and $|\xi| = 1$.

Let

$$\eta = \sigma_0^{-1}(x;\xi)\zeta, \quad \zeta \in \mathbb{C}^3.$$

Due to the positive definiteness of the matrices $\sigma_l(x;\xi)$, $l = 1, 2$, we have

$$\big(\sigma_0(x;\xi)\eta, \eta\big) = \big(\zeta, \sigma_0^{-1}(x;\xi)\zeta\big) = \\ = \overline{\big(\sigma_1^{-1}(x;\xi)\zeta, \zeta\big)} + \overline{\big(\sigma_2^{-1}(x;\xi)\zeta, \zeta\big)} \ge \delta' |\zeta|^2 > 0$$

for any $x \in S$, $|\xi| = 1$ and $\eta \in \mathbb{C}^3 \setminus \{0\}$.

Therefore $\sigma_0(x;\xi)$ is a positive-definite matrix (since it is positive and nondegenerated). $\square$

Remark 6.2. From the proof of Lemma 6.1 it follows that the general solution of equation $Ph(x) = 0$, $x \in S$, is representable by the formula $h(x) = [a \times x] + b$, $x \in S$, with arbitrary tree-dimensional vectors a and b. This implies that if h solves the above homogeneous equation and $\operatorname{supp} h \neq S$, then $h(x) = 0$, $x \in S$.

Lemma 6.3. *Eigenvalues of the matrix*

$$\big[\sigma_0(x;0,-1)\big]^{-1}\big[\sigma_0(x;0,1)\big],$$

where $\sigma_0(x;\xi)$ is the principal symbol of P, are positive numbers.

Proof. It is quite similar to that of Lemma 5.1. $\square$

Thus the operator P belongs to the class of pseudodifferential operators for which the equations of type (6.9) on manifolds with boundary were studied in [16]–[18]. As in Section 5 the results obtained in the above-cited papers allow us to formulate the following theorems.

Theorem 6.4. *Let $1 < p < \infty$, $1 \le q \le \infty$. Then the operators*

$$r_2 P : \widetilde{B}^{\nu+1}_{p,q}(S_2) \to B^{\nu}_{p,q}(S_2), \tag{6.14}$$

$$r_2 P : \widetilde{H}^{\nu+1}_{p}(S_2) \to H^{\nu}_{p}(S_2) \tag{6.15}$$

are bounded for all $\nu \in \mathbb{R}$.

Operator (6.14) *is a Fredholm operator if* (3.20) *holds.*

Operator (6.15) *is a Fredholm operator if and only if* (3.20) *holds.*

Both operators (6.14) *and* (6.15) *are invertible if ν satisfies inequality* (3.20).

Theorem 6.5. *Let $4/3 < p < 4$ and conditions* (2.15), (2.17) *be fulfilled. Then the C–N problem has a unique solution of the class* (2.9). *The solution is representable in the form of* (6.3) *and* (6.4) *with ψ defined by the uniquely solvable pseudodifferential equation* (6.9).

Theorem 6.6. *Let $4/3 < p < 4$, $1 < t < \infty$, $1 \le q \le \infty$, $1/t - 3/2 < \nu < 1/t - 1/2$ and let $\overset{1}{u}$, $\overset{2}{u}$ be the solution of the C–N problem satisfying* (2.9). *In addition, if*

$$f \in B^{\nu+1}_{t,t}(S_1), \quad F \in B^{\nu}_{t,t}(S_1), \quad F_{\pm} \in B^{\nu}_{t,t}(S_2)$$

and F^0 defined by (2.17) *belongs to $B^{\nu}_{t,t}(S)$, then*

$$\overset{1}{u} \in H^{\nu+1+1/t}_{t}(\Omega_1), \quad \overset{2}{u} \in H^{\nu+1+1/t}_{t,loc}(\Omega_2).$$

If

$$f \in B^{\nu+1}_{t,q}(S_1), \quad F \in B^{\nu}_{t,q}(S_1), \quad F_{\pm} \in B^{\nu}_{t,q}(S_2), \quad F^0 \in B^{\nu}_{t,q}(S),$$

then

$$\overset{1}{u} \in B^{\nu+1+1/t}_{t,q}(\Omega_1), \quad \overset{2}{u} \in B^{\nu+1+1/t}_{t,q,loc}(\Omega_2).$$

In particular, if

$$f \in C^{\alpha}(S_1), \quad F \in B^{\alpha-1}_{\infty,\infty}(S_1), \quad F_{\pm} \in B^{\alpha-1}_{\infty,\infty}(S_2), \quad F^0 \in B^{\alpha-1}_{\infty,\infty}(S),$$

then

$$\overset{1}{u} \in C^{\alpha'}(\overline{\Omega}_1), \quad \overset{2}{u} \in C^{\alpha'}(\overline{\Omega}_2)$$

for any $\alpha' \in (0, \alpha_0)$, $\alpha_0 = \min\{\alpha, 1/2\}$.

Remark 6.7. The C–D and C–N problems involve as particular cases the screen and the crack type problems. In fact, if both elastic materials occupying Ω_1 and Ω_2 are the same and conditions (2.1) and (2.2) are homogeneous ($f = 0$, $F = 0$), then the surface S_1 becomes a formal interface; the displacement vector satisfies equation (1.4) for the points $x \in S_1$, and we obtain either the screen type or the crack type problems for $\mathbb{R}^3$ with a cut along the surface S_2.

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Author's address:
Department of Mathematics (99)
Georgian Technical University
77, M. Kostava St., Tbilisi 380075
Republic of Georgia