GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 1, 1996, 27-48

AN ALGEBRAIC MODEL OF FIBRATION WITH THE FIBER $K(\pi, n)$-SPACE

N. BERIKASHVILI

Abstract. For a fibration with the fiber $K(\pi, n)$-space, the algebraic model as a twisted tensor product of chains of the base with standard chains of $K(\pi, n)$-complex is given which preserves multiplicative structure as well. In terms of this model the action of the n-cohomology of the base with coefficients in π on the homology of fibration is described.

1. Introduction

For a fibration $F \to E \to B$ there is the Brown model [1], [2] as a twisted tensor product $C_*(B) \otimes_\varphi C_*(F)$. However, this model gives us no information about the multiplicative structure. The aim of this paper is to construct a model for the particular case of the fiber $F = K(\pi, n)$ which would inherit the multiplicative structure of $C^*(E)$ as well. The model is given as a twisted tensor product $C_*(B) \otimes_{z^{n+1}} C_*^{\square}(L(\pi, n))$ of the singular chain complex of the base with the chain complex of the cubical version of the complex $K(\pi, n)$ [3], [4] and it describes the cubical singular chain complex of the total space (Theorems 5.1 and 6.1 below). It turns out that this model is actually the chain complex of a cubical complex and hence, in addition, it carries a Serre cup product structure as well. In Section 7 the action of the group $H^n(B, \pi)$ on the model is discussed.

This paper is a reply to the needs of obstruction theory and is applied in [5]. The main result was announced in [6].

2. Preliminaries

Let X be a CW-complex and $F^pX = X^p$ be its filtration by skeletons. If H is the singular homology theory with coefficient group G, then for the

1991 *Mathematics Subject Classification.* 55N10, 55N45, 55R20, 55U99.

Key words and phrases. Simplicial set, cubical set, fibration, $K(\pi, n)$-space, cross section, twisted tensor product, perturbed differential, algebraic model, spectral sequence.

1072-947X/96/0100-0027$09.50/0

first term of the related spectral sequence we have $E^1_{p,q} = 0, \quad q > 0$, and $(E^1_{p,0}, d^1)$ is the chain complex

$$(H_p(X^p, X^{p-1}, G), d^1)$$

with the homology isomorphic to the singular homology of the space X. Of course $H_p(X^p, X^{p-1}, G)$ is isomorphic to ΣG_σ, where σ are p-cells of X and $G_\sigma = G$. The above chain complex will be referred to as the cell chain complex of the CW-complex.

In some cases the chain map of the cell-chain complex of the CW-complex to the singular or the cubical singular chain complex is defined which induces the sisomorphism of homology stated above.

Recall that the cubical set is a sequence of sets $Q = \{Q_0, Q_1, Q_2 \cdots\}$ together with the boundary and degeneracy operators

$$d_i^\epsilon : Q_n \to Q_{n-1}, \quad 1 \le i \le n, \quad \epsilon = 0, 1,$$
$$s_i : Q_n \to Q_{n+1}, \quad 1 \le i \le n+1,$$

subject to the standard equalities (see, e.g., [4], [7]).

Q_n are n-cubes of Q; $\sigma^n \in Q_n$ is said to be degenerate if

$$\sigma^n = s_i \tau^{n-1}.$$

The abbreviation for $d_i^\epsilon \sigma^n$ is $\sigma_i^{n,\epsilon}$, $\epsilon = 0, 1$.

The main example of a cubical set is the cubical singular complex $Q(X)$ of a space X.

Milnor's notion of the realization of a simplicial set [8] works for a cubical set as well and runs as follows.

Let I^n be the standard n-cube and let

$$e_i^\epsilon : I^{n-1} \to I^n, \quad 1 \le i \le n, \ \ \epsilon = 0, 1,$$
$$p_i : I^{n+1} \to I^n, \quad 1 \le i \le (n+1),$$

be i-face imbeddings and i-projections.

Let Q be a cubical set. Then the realization $|Q|$ is defined as a factor set of the space

$$\bigcup_n (Q_n \times I^n)$$

by the identifications

$$(\sigma^n, e_i^\epsilon x) = (d_i^\epsilon(\sigma^n), x), \quad \sigma \in Q_n, \ \ x \in I^{n-1},$$
$$(s_i(\sigma^n), x) = (\sigma^n, p_i x), \quad \sigma \in Q_n, \quad x \in I^{n+1}, \ \ 1 \le i \le (n+1).$$

$|Q|$ is a CW-complex with n-cells in 1–1 correspondence with the nondegenerate n-cubes of Q.

There is a standard continuous map

$$r : |Q(X)| \to X.$$

We define the Giever-Hu realization [9], [10] of the cubical set Q and denote it by $||Q||$, omitting in the above definition the equality with degeneracy operators. Then $||Q||$ has cells in 1–1 correspondence with all cubes of Q. We have the canonical continuous map

$$||Q|| \to |Q|.$$

We recall the notion of homology for the cubical set Q. $\tilde{C}_*^{\square}(Q)$ is the chain complex spanned in each dimension n by all n-cubes of Q, and the boundary operator d is defined by

$$d\sigma^p = \Sigma(-1)^i d_i^0 \sigma^p - \Sigma(-1)^i d_i^1 \sigma^p.$$

$\overline{C}{}_*^{\square}(Q)$ is a chain subcomplex of $\tilde{C}_*^{\square}(Q)$ spanned by the degenerate cubes. The quotient complex

$$C_*^{\square}(Q) = \tilde{C}_*^{\square}(Q)/\overline{C}{}_*^{\square}(Q)$$

is said to be the chain complex of Q and its homology is called the homology of Q.

If G is a group of coefficients, then by

$$C^n_{\square}(Q, G)$$

we understand the normalized cochains of Q, i.e., those which are zero on the degenerate cubes.

For the singular cubical set $Q(|Q|)$ of the space $|Q|$ consider the standard imbedding

$$Q \subset Q(|Q|).$$

It induces the following isomorphisms of homology and cohomology:

$$H_n^{\square}(Q, G) = H_n^{\square}(|Q|, G), \quad H^n_{\square}(Q, G) = H^n_{\square}(|Q|, G).$$

Hence we find that the map $r : |Q(X)| \to X$ induces the isomorphisms

$$H_n^{\square}(|Q(X)|, G) = H_n^{\square}(X, G)), \quad H^n_{\square}(|Q(X)|, G) = H^n_{\square}(X, G)).$$

The interaction of simplicial and cubical sets is as follows. Let Δ_n be the standard simplex and consider the map

$$\psi_n : I^n \to \Delta_n \tag{2.1}$$

from [11] defined by

$$
\begin{aligned}
y_0 &= 1 - x_1, \\
y_1 &= x_1(1 - x_2), \\
y_2 &= x_1 x_2 (1 - x_3), \\
&\vdots \\
y_{n-1} &= x_1 x_2 x_3 \cdots x_{n-1}(1 - x_n), \\
y_n &= x_1 x_2 x_3 \cdots x_n.
\end{aligned}
\tag{2.2}
$$

Clearly, ψ_n is the map of the pairs $\psi_n : (I^n, \partial I^n) \longrightarrow (\Delta_n, \partial\Delta_n)$, inducing the homeomorphism $I^n/\partial I^n \to \Delta_n/\partial\Delta_n$.

In [11], for a singular simplex $\sigma^n : \Delta_n \longrightarrow X$ J. P. Serre considers the composition $I^n \xrightarrow{\psi_n} \Delta_n \xrightarrow{\sigma^n} X$ as the singular cube $t\sigma^n$ of X, defining the chain map $C_*(S(X)) \longrightarrow C_*^{\square}(Q(X))$ from the simplicial singular chain complex to the singular normalized cubical chain complex (see the identities below). Hence for an abelian group G we have the cochain map

$$C^*_{\square}(Q(X), G) \longrightarrow C^*(X, G) \tag{2.3}$$

from the normalized singular cubical cochain complex to an ordinary singular cochain complex.

For map (2.3) at the geometrical level, $t : S(X) \longrightarrow Q(X)$, we have the identities

$$
\begin{aligned}
d_i^1 t(\sigma^n) &= t(\sigma^n_{i-1}), \quad 1 \le i \le n, \\
d_n^0 t(\sigma^n) &= t(\sigma^n_n), \\
d_i^0 t(\sigma^n) &= \underbrace{s_i s_i \cdots s_i}_{n-i} t(\sigma^n_{n,(n-1),\cdots,i}), \quad 1 \le i < n.
\end{aligned}
\tag{2.4}
$$

We see that $(t\sigma)_i^0$ is a degenerate cube if $i \neq n$.

The above equalities imply that the following lemma is valid.

Lemma 2.1. *Let $0 < k \le n$ and let P be the sum of closed k-cubes of I^n which are not degenerated by ψ_n. Then P is a deformation retract of I^n.*

One defines a chain map $C_*^{\square}(X) \longrightarrow C_*^N(X)$ of the normalized complexes as follows. Consider the standard triangulation of I^n and let $s(I^n)$ denote the basic n-dimensional chain of this triangulation as a singular chain of the space I^n. Define f by

$$f(\sigma^n) = \sigma^n_*(s(I^n)) \in C_n(X). \tag{2.5}$$

If σ^n is a degenerate cube, then all n-simplexes in $f(s(I^n))$ are degenerate and therefore f defines the map of the normalized complexes.

For cubical cochains J. P. Serre defines the $\smile$-product which is certainly valid for general cubical sets too [11]. The definition runs as follows. For the subset

$$K = (i_1 < i_2 < i_3 \cdots < i_k) \subset (1, 2, 3, \cdots, n)$$

and $\sigma^n \in Q_n$ we introduce the notation

$$d_K^\epsilon \sigma^n = d_{i_1}^\epsilon d_{i_2}^\epsilon d_{i_3}^\epsilon \cdots d_{i_k}^\epsilon (\sigma^n), \quad \epsilon = 0, 1.$$

Let $c^p \in C^p(Q, \Lambda)$, $\quad c^q \in C^q(Q, \Lambda)$ be the normalized p- and q-cochains of Q with coefficients in a commutative ring Λ. Define the product $c^{p+q} = c^p \smile c^q$ by

$$c^{p+q}(\tau^{p+q}) = \Sigma_{(H,K)}(-1)^{a(H,K)} c^p(d_K^0(\tau^{p+q})) c^q(d_H^1(\tau^{p+q})), \tag{2.6}$$

where (H, K) is the decomposition of $\{1, 2, 3, \cdots, (p+q)\}$ into two disjoint subsets.

Note that if τ^{p+q} is of the form $t(\sigma)$, then on the right side only one summand, precisely that of the decomposition $(1, 2, 3, \cdots, p) \bigcup (p+1, p+2, \cdots, p+q)$, is not zero, which shows that the cochain map (2.3) preserves multiplicative structure.

For any abelian group π and any positive integer n one introduces the cubical version of the $K(\pi, n)$-complex [3], [4]. $L(\pi, n)$ is a cubical set with p-cubes being n-cocycles, $Z^n(I^p, \pi)$, where $I^p = I \times I \times \cdots \times I$ is the standard cube with standard faces, and the cochain is understood as a cell cochain. The boundary and degeneracy operators are defined similarly to those in the c.s.s. case.

For the complex $L(\pi, n)$ we have, in $C_*^\square(L(\pi, n))$, the ring structure that converts $C_*^\square(L(\pi, n))$ into a graded differential ring and hence $H_*^\square(L(\pi, n))$ into a graded ring. The multiplication is defined as follows. Given cubes σ^p and σ^q in $L(\pi, n)$, i.e., $\sigma^p \in Z^n(I^p, \pi)$, $\sigma^q \in Z^n(I^q, \pi)$, we define the $(p+q)$-cube $\sigma^p \circ \sigma^q = \sigma^{p+q} \in Z^n(I^{p+q}, \pi)$ as $pr_1^* \sigma^p + pr_2^* \sigma^q$, where

$$pr_1 : I^{p+q} \to I^p, \quad pr_2 : I^{p+q} \to I^q$$

are standard projections of I^{p+q} on its first p-cube and its last q-cube. Under this multiplication $\tilde{C}_*^\square(L(\pi, n))$ is the differential graded ring and $\bar{C}_*^\square(L(\pi, n))$ is the ideal, and therefore $\tilde{C}_*^\square(L(\pi, n))/\overline{C}_*^\square(L(\pi, n))$ is the graded differential associative ring. We have $C_0^\square(L(\pi, n)) = Z$, $C_i^\square(L(\pi, n)) = 0$, $0 < i < n$, while $1 \in Z$ is the unitary element of the algebra $C_*^\square(L(\pi, n))$.

The Eilenberg–MacLane complex $K(\pi, n)$ is a simplicial set and $C_*(K(\pi, n))$ is the skew-commutative differential ring. The above map

$$\psi_n : I^n \to \Delta_n$$

enables us to define the chain map

$$C_*(K(\pi,n)) \to C_*^{\square}(L(\pi,n))$$

similarly to that of J. P. Serre. In an obvious manner similar to that used for (2.5) we define

$$C_*^{\square}(L(\pi,n)) \to C_*^{N}(K(\pi,n))$$

which is a homomorphism of differential rings. Hence $H_*^{\square}(L(\pi,n))$ and $H_*(K(\pi,n))$ are isomorphic as rings and thus $H_*^{\square}(L(\pi,n))$ is a commutative ring.

Recall some notions from the theory of perturbed differentials [12]. Let Y be a filtered left graded differential module over a filtered graded differential algebra A under the pairing

$$A \otimes Y \to Y.$$

Let $F^iA \supset F^{i+1}A$ and $F^iY \supset F^{i+1}Y$ be the above-mentioned filtrations.

Fix the integer n. Let

$$T^n(A) = \{a \in F^nA,\ |a| = +1,\ d_Aa = aa\};$$

a is known as a twisting element of the differential algebra.

For $a \in T^n(A)$ the perturbed differential in Y is defined as

$$d_ax = d_Yx + ax, \quad x \in Y.$$

The filtered complex with this perturbed differential will be denoted by Y_a. It is a 1-graded and filtered differential chain complex. Consider the set $G^n(A)$ of all elements in A of the form

$$g = 1 + p,\ p \in F^1A,\ d_Ap \in F^nA.$$

It is obvious that this set is the group under the product operation in the algebra A. The group $G^n(A)$ acts from the left on the set $T^n(A)$ by

$$\bar{a} = (1+p) \circ a = (1+p)^{-1}a(1+p) + (1+p)^{-1}d_A(1+p) \tag{2.7}$$

which is equivalent to the equality

$$d_A(1+p) = (1+p)\bar{a} - a(1+p). \tag{2.8}$$

The set of orbits under action (2.7) will be denoted by $D^n(A)$.

For $\overline{a} = (1+p) \circ a$ consider the map $\varphi_p : Y_{\bar{a}} \to Y_a$ given by

$$\varphi_p(x) = (1+p)x = x + px. \tag{2.9}$$

This is a chain map and, obviously, 1–1.

Proposition 2.1. *If $f : A_1 \to A$ is a morphism of filtered differential algebras inducing an isomorphism of the nth terms of spectral sequences, then the induced map $D^nf : D^n(A_1) \to D^n(A)$ is* 1–1 [12].

3. Auxiliary Complexes

Let B be a topological space and π an abelian group. Let $L(\pi, n)$ be the $K(\pi, n)$-complex of the preceding section and $C_*^{\square}(L(\pi, n))$ its normalized chain complex. Let G be an abelian group. Consider the chain and cochain complexes $C_*^{\square}(L(\pi, n), G)$ and $C^*_{\square}(L(\pi, n), G)$.

Consider the complexes

$$\begin{aligned} Y_{**}(B, \pi, n) &= C_*(B, C_*^{\square}(L(\pi, n))), \\ Y_{**}(B, \pi, n, G) &= C_*(B, C_*^{\square}(L(\pi, n), G)), \end{aligned} \tag{3.1}$$

i.e., the group of singular chains of B with coefficients in the normalized integral chains of the complex $L(\pi, n)$ and the group of singular chains of B with coefficients in the normalized chains of the complex $L(\pi, n)$ with coefficients in G. These complexes are bigraded differential modules, the first differential being that of B and the second one that of $L(\pi, n)$. Both complexes are covariant functors on the category of topological spaces. It is clear that

$$\begin{aligned} Y_{**}(B, \pi, n) &= C_*(B) \otimes C_*^{\square}(L(\pi, n)), \\ Y_{**}(B, \pi, n, G) &= C_*(B) \otimes C_*^{\square}(L(\pi, n), G), \\ Y_{**}(B, \pi, n, G) &= C_*(B, C_*^{\square}(L(\pi, n))) \otimes G. \end{aligned}$$

The cohomology version is

$$Y^{**}(B, \pi, n, G) = C^*(B, C^*_{\square}(L(\pi, n), G)).$$

We have

$$\begin{aligned} Y^{**}(B, \pi, n, G) &= Hom(C_*(B), C^*_{\square}(L(\pi, n), G)), \\ Y^{**}(B, \pi, n, G) &= Hom(Y_{**}(B, \pi, n), G)). \end{aligned}$$

Consider the graded differential algebra $C_*^{\square}(L(\pi, n))$ of the preceding section and the bigraded differential algebra

$$A^*{}_*(B, \pi, n) = C^*(B, C_*^{\square}(L(\pi, n))) \tag{3.2}$$

of singular cochains of B with coeficients in the normalized chains of $L(\pi, n)$, where the multiplication is defined as the $\smile$-product of B when the coefficients are multiplied by the product in the algebra $C_*^{\square}(L(\pi, n))$. We obtain

$$A^*{}_*(B, \pi, n) = Hom(C_*(B), C_*^{\square}(L(\pi, n)))$$

and the above multiplication is the same as that defined by the composition

$$C_*(B) \xrightarrow{\Delta} C_*(B) \otimes C_*(B) \xrightarrow{x \otimes y} C_*^{\square}(L(\pi, n)) \otimes C_*^{\square}(L(\pi, n)) \xrightarrow{\mu} C_*^{\square}(L(\pi, n))$$

for $x, y \in Hom(C_*(B), C_*^{\square}(L(\pi,n)))$, where Δ is the coproduct of B and μ is the product in $C_*^{\square}(L(\pi,n))$. By lifting the second index in (3.2) we obtain the bigraded algebra of the fourth quadrant with both differentials increasing the degree of elements by $+1$:

$$A^{*,-*}(B,\pi,n) = C^*(B, C_{\square}^{-*}(L(\pi,n))).$$

The total complex is the direct product of bihomogeneous components

$$A^m = \Pi_{p-q=m}^{p,-q} A^{p,-q}$$

and therefore every element x^n is given uniquely as the sum of its components

$$x^n = x^{n,o} + x^{n+1,-1} + x^{n+2,-2} + x^{n+3,-3} + \cdots .$$

The unitary element of the algebra $1 \in A^{0,0} \subset A^0$ is the cochain of B, $c^{0,0}$, equal to 1 of $C_*(L(\pi,n))$ for every 0-simplex of B. $A^{*,-*}(B,\pi,n)$ is a contravariant functor both on the category of topological spaces and on the category of simplicial sets. The filtration of bialgebra A with respect to the first degree is the decreasing one and is complete in the Eilenberg-Moore sense. The second term of the spectral sequence is

$$E_2^{p,-q} = H^p(B, H_q(\pi,n)),$$

the spectral sequence converging to

$$H^{p-q}(A^{*,-*}(B,\pi,n))$$

in the Eilenberg–Moore sense.

$Y_{**}(B,\pi,n,G) = Y^{-*,-*}(B,\pi,n,G)$ and $Y^{**}(B,\pi,n,G)$ are the left modules over the bigraded algebra $A^{*,-*}(B,\pi,n)$ to be understood as follows. The pairing

$$C_p^{\square}(L(\pi,n)) \otimes C_q^{\square}(L(\pi,n)) \to C_{p+q}^{\square}(L(\pi,n)) \tag{3.3}$$

induces the pairings

$$C_p^{\square}(L(\pi,n)) \otimes C_q^{\square}(L(\pi,n)) \otimes G \to C_{p+q}^{\square}(L(\pi,n)) \otimes G, \tag{3.4}$$

$$C_p^{\square}(L(\pi,n)) \otimes C_{\square}^{p+q}(L(\pi,n),G) \to C_{\square}^{q}(L(\pi,n),G). \tag{3.5}$$

The bicomplex $Y_{**}(B,\pi,n,G)$ is the module over the algebra $A^*{}_*(B,\pi,n)$ via the $\frown$-product of B and the product of coefficients (3.3) and (3.4). The bicomplex $Y^{**}(B,\pi,n,G)$ is the left module over the algebra $A^*{}_*(B,\pi,n)$ via the $\smile$-product in B and the product of coefficients (3.5).

The spectral sequence arguments show that if $f : B_1 \to B$ induces an isomorphism of homology, then the induced homomorphisms of complexes

$$A^{*,-*}(B,\pi,n) \to A^{*,-*}(B_1,\pi,n)$$

and

$$Y_{**}(B_1, \pi, n) \to Y_{**}(B, \pi, n)$$

induce an isomorphism of homology.

In particular, if K is the ordered simplicial complex then the imbedding $K \subset S(|K|)$ induces the isomorphism

$$H(A^{*,-*}(|K|, \pi, n)) \to H(A^{*,-*}(K, \pi, n)). \tag{3.6}$$

Consider the algebra $A^{*,-*}(B, \pi, n)$ as a filtered differential algebra with the filtration defined by the first degree and introduce the notation $D(A^{*,-*}(B, \pi, n)) = D^{n+1}(A^{*,-*}(B, \pi, n))$. Then

$$D(A^{*,-*}(B, \pi, n)) = T^{n+1}(A^{*,-*}(B, \pi, n))/G^{n+1}(A^{*,-*}(B, \pi, n)),$$

where $T^{n+1}(A^{*,-*}(B, \pi, n))$ is the set of all twisting elements of $A^{*,-*}(B, \pi, n)$ of the form

$$a = a^{n+1,-n} + a^{n+2,-n-1} + a^{n+3,-n-2} \cdots ,$$

and $G^{n+1}(A^{*,-*}(B, \pi, n))$ is the set of all elements in $A^{*,-*}(B, \pi, n)$ of the form

$$1 + p = 1 + p^{n,-n} + p^{n+1,-n-1} + p^{n+2,-n-2} + \cdots$$

(by virtue of $C_i^{\square}(L(\pi, n)) = 0,\ \ 0 < i < n$).

Proposition 3.1. *If $f : B_1 \to B$ induces an isomorphism of homology then $D(f) : D(A^{*,-*}(B, \pi, n)) \to D(A^{*,-*}(B_1, \pi, n))$ is the 1–1 map.*

Proof. We have $E_2^{p,-q} = H^p(B, H_q(\pi, n))$. By assumption, f induces an isomorphism of the second terms and hence of the $(n+1)$th terms of spectral sequences. To complete the proof apply Proposition 2.1. □

4. Transformation of the Functors $\beta : D(A^{*,-*}(B, \pi, n)) \to H^{n+1}(B, \pi)$ and $\gamma : H^{n+1}(B, \pi) \to D(A^{*,-*}(B, \pi, n))$

Consider the homomorphism $\alpha : C_n(L(\pi, n)) \to G$ defined on the n-cubes of $L(\pi, n)$, $\tau^n = [g],\ g \in G$, by $\alpha([g]) = g$. It induces the homomorphism of cochain complexes $\beta : C^*(B, C_n(L(\pi, n))) \to C^*(B, \pi)$. The latter homomorphism defines the map of sets $\beta : A^{*,-*}(B, \pi, n) \to C^*(B, \pi)$ by $\beta(a) = \beta(a^{n+k,-n})$ if $a = a^{n+k,-n} + a^{n+k+1,-n-1} + \cdots$. Let $a \in T(A(B, \pi, n))$. Then $|a| = 1,\ \ da = aa$, and hence $\delta_B a^{n+1,-n} + d_L a^{n+2,-n-1} = 0$. We see that $\beta(d_L a^{n+2,-n-1}) = 0$, so that $\beta(\delta_B a^{n+1,-n}) = 0$ and $\delta_B \beta(a^{n+1,-n}) = 0$. Hence, if $a \in T(A^{*,-*}(B, \pi, n))$, then $\beta(a) \in Z^{n+1}(B, \pi)$.

If $\bar{a} \sim a$, then $\beta(\bar{a}) - \beta(a) \in B^{n+1}(B, \pi)$. Indeed, by (2.8) we have $d_A p = \bar{a} - a + p\bar{a} - ap$, and therefore $\delta_B p^{n,-n} + d_L p^{n+1,-n-1} = \bar{a}^{n+1,n} -$

$a^{n+1,-n}$. Hence $\delta_B\beta(p^{n,-n}) = \beta(\bar{a}^{n+1,-n}) - \beta(a^{n+1,-n})$ which is equivalent to $\delta_B\beta(p) = \beta(\bar{a}) - \beta(a)$. The above shows that the map

$$\beta : D(A^{*,-*}(B,\pi,n)) \to H^{n+1}(B,\pi)$$

defined by $\beta(a) \in \beta(d), d \in D(A^{*,-*}(B,\pi,n)), a \in d$, is correct.

The second transformation

$$\gamma : H^{n+1}(B,\pi) \to D(A^{*,-*}(B,\pi,n))$$

is more difficult to define, since it requires the construction of the mappings

$$Z^{n+1}(B,\pi) \to T^{n+1}(A^{*,-*}(B,\pi,n)),$$
$$Z^{n+1}(B,\pi) \times C^n(B,\pi) \to G^{n+1}(A^{*,-*}(B,\pi,n)),$$

with suitable properties.

Lemma 4.1. *There is a rule which assigns a twisting cochain $a(z^{n+1}) \in A^{*,-*}(B,\pi,n)$ to every cocycle $z^{n+1} \in Z^{n+1}(B,\pi)$. If $f : B_1 \to B$ is a map, then $a(f^*(z^{n+1})) = f^*(a(z^{n+1}))$.*

Proof. For $z^{n+1} \in Z^{n+1}(B,\pi)$ we first construct the map

$$\kappa_{z^{n+1}} : S(B) \to L(\pi,n), \tag{4.1}$$

$\kappa_{z^{n+1}}(\sigma^{m+1})$ being the m-cube $\tau^m \in L(\pi,n)$ with the following properties:

(i) $\kappa_{z^{n+1}}(\sigma^{m+1})$ if $(m+1) \le n$ is the unique m-cube of $L(\pi,n)$ (which is degenerate for $m > 0$);

(ii) $\kappa_{z^{n+1}}(\sigma^{n+1}) = [g] \in \tilde{C}_n^{\square}(L(\pi,n))$, where $g = z^{n+1}(\sigma^{n+1})$;

(iii)
$$\begin{aligned}
d_i^1\kappa_{z^{n+1}}(\sigma^{m+1}) &= \kappa_{z^{n+1}}(\sigma_i^{m+1}), \quad 1 \le i \le m,\\
d_1^0\kappa_{z^{n+1}}(\sigma^{m+1}) &= \kappa_{z^{n+1}}(\sigma_0^{m+1}),\\
d_m^0\kappa_{z^{n+1}}(\sigma^{m+1}) &= \kappa_{z^{n+1}}(\sigma_{m+1}^{m+1}),\\
d_i^0\kappa_{z^{n+1}}(\sigma^{m+1}) &= \kappa_{z^{n+1}}(\sigma_{(1)}^{m+1})\kappa_{z^{n+1}}(\sigma_{(2)}^{m+1}), \;\; 1 < i < m,
\end{aligned} \tag{4.2}$$

where $\sigma_{(1)}^{m+1}$ and $\sigma_{(2)}^{m+1}$ are respectively the first i-dimensional and the last $(m+1-i)$-dimensional face of σ^{m+1}.

With such $\kappa_{z^{n+1}}$ constructed, the twisting element $\tilde{a}_z$ in the auxiliary algebra $\tilde{A}^*{}_*(B,\pi,n) = C^*(B,\tilde{C}_*(L(\pi,n)))$ is defined as $\tilde{a} = \tilde{a}^{2,-1} + \tilde{a}^{3,-2} + \cdots$, where $\tilde{a}^{m+1,-m}(\sigma^{m+1}) = \kappa_{z^{m+1}}(\sigma^{m+1}) \in \tilde{C}_m(L(\pi,n))$.

The above equalities imply that $d\tilde{a} = \tilde{a}\tilde{a}$. The image of $\tilde{a}$, $a(z^{n+1})$, with respect to the homomorphism $\tilde{A}^*{}_*(B,\pi,n) \to A^*{}_*(B,\pi,n)$ induced by $\tilde{C}_*^{\square}(L(\pi,n)) \to C_*^{\square}(L(\pi,n))$ is a twisting cochain. By virtue of the fact that all cubes of positive dimension $< n$ are degenerate, we find that

$$a(z^{n+1}) = a^{n+1,-n} + a^{n+2,-n-1} + \cdots$$

and

$$\beta(a) = \beta(a^{n+1,-n}) = z^{n+1}.$$

The rest of the proof consists in constructing $\kappa_{z^{n+1}}$.

The geometric background for the (algebraic) definition of $\kappa_{z^{n+1}}$ is as follows. Consider a cross section s of the fibration over the n-skeleton of $|S(B)|$ such that the obstruction cocycle $z^{n+1}(s)$ is equal to z^{n+1}, $z^{n+1} = z^{n+1}(s)$. For $\sigma^{m+1} \in S(B)$ consider the map $I^{m+1} \xrightarrow{\psi} \Delta_{m+1} \xrightarrow{\sigma^{m+1}} |S(B)|$. Consider a subcomplex of n-cubes of the cubical complex I^{m+1} which is sent by ψ in the n-simplex of Δ_{m+1}. By virtue of Lemma 2.1 this subcomplex is the retract of I^{m+1} and therefore the cross section over this subcomplex induced by the cross section s^n has an extension over I^{m+1}. Consider its restriction over the n-skeleton of I^{m+1}. On the m-cube $(0, x_2, x_3, \cdots, x_{m+1})$ it defines an n-dimensional cell-cochain with coefficients in $\pi_n(F) = \pi$. Obviously, this cochain is a cocycle. We can define $\kappa_z(\sigma^{m+1})$ to be this cocycle regarded as an m-cube of $L(\pi, n)$.

The reasoning above suggests the definition of $\kappa_{z^{n+1}}$ as follows. As above, we consider the map $\psi : I^{m+1} \to \Delta_{m+1}$ and also we consider I^m as the face $(0, x_2, x_3, \cdots, x_{m+1})$. The n-cochain $a^{m+1,-m}(\sigma^{m+1}) = \lambda^n$ of I^m is uniquely defined as follows: it is zero on the n-cubes of I^m which have a form differing from

$$u = (0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, x_{i_1}, 1, 1, 1, \ldots, 1, x_{i_2}, 1, 1, 1, \ldots, 1, x_{i_n}, \nu_1, \nu_2, \ldots, \nu_s).$$

For the above n-cube we assume ε_j to be the last ε equal to zero in the sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ and consider the $(n+1)$-cube in I^{m+1}

$$(1, \ldots, 1, x_{i_0}, 1, \ldots, 1, x_{i_1}, 1, \ldots, 1, x_{i_2}, 1, \ldots, 1, x_{i_n}, \nu_1, \nu_2, \ldots, \nu_s),$$

where ε_j is replaced by x_{i_0}. If all ε's are equal to 1, we consider the $(n+1)$-cube

$$(x_{i_0}, 1, \ldots, 1, x_{i_1}, 1, \ldots, 1, x_{i_2}, 1, \ldots, 1, x_{i_n}, \nu_1, \nu_2, \ldots, \nu_s).$$

This $(n+1)$-cube of I^{m+1} defines via the map ψ^{m+1} the nondegenerate (n+1)-simplex v in the simplex Δ^{m+1} and we set

$$\lambda(u) = (\sigma^{m+1})^*(z^{n+1})(v), \quad \sigma^{m+1} : \Delta^{m+1} \to B.$$

λ^n is the cocycle. This is proved directly.

Let $\kappa_{z^{n+1}}(\sigma^{m+1})$ be the cocycle λ^n regarded as the m-cube of $L(\pi, n)$.

It remains to show that equalities (4.2) are fulfilled. Each of the four equalities (4.2) is easy to check by using the algebraic definition of $\kappa_{z^{n+1}}$. It is obvious that $\tilde{a}$ and hence $a(z^{n+1})$ is functorial. $\square$

We next define

$$Z^{n+1}(B,\pi)\times C^n(B,\pi)\to G^{n+1}(A^{*,-*}(B,\pi,n)).$$

Assume that we are given a cocycle $z^{n+1}_{B\times I}\in Z^{n+1}(B\times I,\pi)$.

By $i_0:B\longrightarrow B\times I$ and $i_1:B\longrightarrow B\times I$ we denote the imbeddings of B as the lower and the upper bottom of $B\times I$. We introduce the notation

$$z^{n+1}_B=i_0^*z^{n+1}_{B\times I}\quad \bar z^{n+1}_B=i_1^*z^{n+1}_{B\times I}.$$

By virtue of Lemma 4.1

$$a(z^{n+1}_B)=i_0^*a(z^{n+1}_{B\times I})\quad a(\bar z^{n+1}_B)=i_1^*a(z^{n+1}_{B\times I}).$$

In the lemma below we shall show that $a(z^{n+1}_B)$ and $a(\bar z^{n+1}_B)$ are the equivalent twisting elements of the algebra $A^{*,-*}(B,\pi,n)$.

Let us consider the standard prism construction

$$w_1:C_*(B)\longrightarrow C_{*+1}(B\times I)$$

subject to the condition

$$dw_1-w_1d=i_1^*-i_0^*. \tag{4.3}$$

To every singular simplex $\sigma^m\in S(B)$ this construction assigns the singular $(m+1)$-chain $w_1(\sigma^m)$ which is the image of the main integral $(m+1)$-chain of the standard triangulation of $\Delta_m\times I$ by the map $\sigma^m\times id:\Delta_m\times I\longrightarrow B\times I$.

The map

$$w_1^*:A^{*,-*}(B\times I,\pi,n)\longrightarrow A^{*,-*}(B,\pi,n) \tag{4.4}$$

can be defined by the composition

$$C_B\xrightarrow{w_1}C_{B\times I}\xrightarrow{x}C_*^{\square}(L(\pi,n)),\ \ x\in A^{*,-*}(B\times I,\pi,n).$$

From (4.3) we obtain

$$w_1^*d=dw_1^*+i_1^*-i_0^*. \tag{4.5}$$

Let

$$u(z^{n+1}_{B\times I})=w_1^*(a(z^{n+1}_{B\times I})).$$

We see that $|u|=0,\ \ u\in F^n(A^{*,-*}(B,\pi,n))$.

Lemma 4.2. *In the algebra $A^{*,-*}(B,\pi,n)$ there holds the equality*

$$d_{A(B,\pi,n)}u(z^{n+1}_{B\times I})=a(\bar z^{n+1}_B)-a(z^{n+1}_B)+u(z^{n+1}_{B\times I})a(\bar z^{n+1}_B)-a(z^{n+1}_B)u(z^{n+1}_{B\times I}).$$

Proof. By Lemma 4.1 we have $d_{A(B\times I,\pi,n)}a_{z^{n+1}_{B\times I}} = a_{z^{n+1}_{B\times I}}a_{z^{n+1}_{B\times I}}$.

By virtue of the map w_1^* and (4.5) the left side of the equality becomes

$$d(u(z^{n+1}_{B\times I})) - a(\bar{z}^{n+1}_B) + a(z^{n+1}_B).$$

The right side becomes

$$u(z^{n+1}_{B\times I})a(\bar{z}^{n+1}_B) - a(z^{n+1}_B)u(z^{n+1}_{B\times I}) \tag{4.6}$$

which can be shown as follows. The standard triangulation of $\Delta_m \times I$ has as vertices of the lower bottom those of Δ_m, say, $b_0 < b_1 < b_2 < \cdots b_m$, while the copies of the vertices of Δ_m, say, $b'_0 < b'_1 < b'_2 < \cdots b'_m$, are the vertices of the upper bottom. It is assumed that $b(i) < b'(i)$. Only

$$(b_0 < b_1 < b_2 < \cdots b_i < b'_i < b'_{i+1} < \cdots b'_m), \quad i = 0, 1, \cdots m$$

are $(m+1)$-dimensional simplices.

By the above definition $w_1(\sigma^m)$ is the image of

$$\sum_{0\le i\le m} (-1)^i (b_0 < b_1 < b_2 < \cdots b_i < b'_i < b'_{i+1} < \cdots < b'_m).$$

Thus we see that the value of $\tilde{a}\tilde{a}$ on $w_1(\sigma^m)$ is equal to

$$\sum_{0\le i\le m} (-1)^i [\sum_{j\le i} a'(b_0 < b_1 < \cdots < b_j) \cdot$$
$$\cdot a'(b_j < b_{j+1} < \cdots < b_i < b'_i < b'_{i+1} < \cdots < b'_m) +$$
$$+ \sum_{i\le j\le m} a'(b_0 < b_1 < b_2 < \cdots < b_i < b'_i < b'_{i+1} < \cdots < b'_j) \cdot$$
$$\cdot a'(b'_j < b'_{j+1} < \cdots < b'_m)].$$

Here $a' = (\sigma^m \times id)^*\tilde{a}$ and the two summands coincide with the two summands in (4.6). □

Definition 4.1. Let $z^{n+1}_B \in Z^{n+1}(B,\pi)$ and $c^n_B \in C^n(B,\pi)$. On $B\times I$ consider the cocycle $z^{n+1}_{B\times I} = Pr^* z^{n+1}_B + \delta_{B\times I} c^n_{B\times I}$, where $c^n_{B\times I}$ is the cochain c^n_B imbedded in $B\times 0$. Denote $u(c^n_B, z^{n+1}_B) = u(z^{n+1}_{B\times I})$.

Lemma 4.3. *If $z^{n+1}_B \in Z^{n+1}(B,\pi)$ and $c^n_B \in C^n(B,\pi)$, then $a(z^{n+1}_B + \delta_B c^n_B)$ and $a(z^{n+1}_B)$ are the equivalent twisting elements, the equivalence being given by the element $1+u(c^n_B, z^{n+1}_B)$. Hence the map $\gamma : H^{n+1}(B,\pi) \to D(A^{*,-*}(B,\pi,n))$ defined by $a(z^{n+1}) \in \gamma(h)$ if $z^{n+1} \in h \in H^{n+1}(B,\pi)$ is correct.*

Proof. By the assumptions of the lemma the equality of Lemma 4.2 is identical to

$$a(z_B^{n+1}+\delta_B c_B^n)=(1+u(c_B^n,z_B^{n+1}))^{-1}a(z_B^{n+1})(1+u(c_B^n,z_B^{n+1}))+ \\ +(1+u(c_B^n,z_B^{n+1}))^{-1}d_A(1+u(c_B^n,z_B^{n+1})). \qquad \square$$

5. Algebraic Model

Recall that the bicomplexes $Y_{**}(B,\pi,n)=C_*(B,C_*^{\square}(L(\pi,n)))$, $Y_{**}(B,\pi,n,G)=C_*(B,C_*^{\square}(L(\pi,n),G))$ and $Y^{**}(B,\pi,n,G)=C^*(B,C^*_{\square}(L(\pi,n),G))$ are the modules over the bialgebra $A^{*,-*}(B,\pi,n)=C^*(B,C_*^{\square}(L(\pi,n)))$.

Definition 5.1. Let $F\to E\to B$ be a Serre fibration with the fiber F $K(\pi,n)$-space. Then $\pi_i(F)=0,\ i\neq n$ and $\pi_n(F)=\pi$. Let $h^{n+1}\in H^{n+1}(B,\pi)$ be the obstruction class of the fibration and let $z^{n+1}\in h^{n+1}$. Consider $a(z^{n+1})\in T(A^{*,-*}(B,\pi,n))$ and the perturbed differential

$$d_{a(z^{n+1})}(y)=d_Y(y)+a(z^{n+1})y, \\ y\in Y_{**}(B,\pi,n),\ Y_{**}(B,\pi,n,G),\ Y^{**}(B,\pi,n,G). \tag{5.1}$$

The complex $Y_{**}(B,\pi,n)$ with this perturbed differential $Y_{z^{n+1}}(B,\pi,n)$ $=Y_{a(z^{n+1})}(B,\pi,n)$ is the (integral) homology model of the fibration. The complex $Y_{**}(B,\pi,n,G)$ with this perturbed differential $Y_{z^{n+1}}(B,\pi,n,G)=Y_{a(z^{n+1})}(B,\pi,n,G)$ is the homology model with coefficients G. The complex $Y^{**}(B,\pi,n,G)$ with this perturbed differential $Y^{**}_{z^{n+1}}(B,\pi,n,G)=Y^{**}_{a(z^{n+1})}(B,\pi,n)$ is the cohomology model of the fibration.

We have

$$Y_{**z^{n+1}}(B,\pi,n,G)=Y_{**z^{n+1}}(B,\pi,n)\otimes G, \\ Y^{**}_{z^{n+1}}(B,\pi,n,G)=Hom(Y_{**z^{n+1}}(B,\pi,n),G).$$

The models of fibration depend on the choice of the cocycle in the obstruction class. However, they are isomorphic complexes: if z^{n+1}, $\bar z^{n+1}$ are two obstruction cocycles of the same fibration, then there is $c^n\in C^n(B,\pi)$ with $\delta c^n=\bar z^{n+1}-z^{n+1}$. By virtue of Lemma 4.3 $a(\bar z^{n+1})=(1+u(c^n,z^{n+1}))\circ a(z^{n+1})$. Hence the chain map (2.9)

$$\varphi_{u(c^n,z^{n+1})}:Y_{**\bar z^{n+1}}(B,\pi,n,G)\longrightarrow Y_{**z^{n+1}}(B,\pi,n,G), \\ \varphi_{u(c^n,z^{n+1})}(y)=(1+u(c^n,z^{n+1}))(y),\quad y\in Y_{**}(B,\pi,n,G)$$

is an isomorphism.

The same reasoning holds for the cohomology model.

Theorem 5.1. *Let $F \longrightarrow E \longrightarrow B$ be a Serre fibration with the fiber F $K(\pi, n)$-space and let $Y_{**z^{n+1}}(B, \pi, n, G)$, $Y^{**}_{z^{n+1}}(B, \pi, n, G), z^{n+1} \in Z^{n+1}(B, \pi_n(F))$, be its homology (resp., cohomology) model. Then:*

(i) *There are chain and cochain maps defined uniquely up to chain homotopy*

$$\begin{gathered} Y_{**z^{n+1}}(B, \pi, n, G) \longrightarrow C^{\square}_*(E, G), \\ C^*_{\square}(E, G) \longrightarrow Y^{**}_{z^{n+1}}(B, \pi, n, G) \end{gathered}$$

inducing an isomophism of homology groups.

(ii) *if $Y_{**\bar{z}^{n+1}}(B, \pi, n, G)$, $Y^{**}_{\bar{z}^{n+1}}(B, \pi, n, G)$ are other models of the fibration and $\delta c^n = \bar{z}^{n+1} - z^{n+1}$, $c^n \in C^n(B, \pi_n(F))$, then the triangles*

$$\begin{array}{ccc} C^{\square}_*(E) & \longleftarrow & Y_{**z^{n+1}}(B, \pi, n) \\ & \nwarrow & \uparrow \varphi_{u(c^n, z^{n+1})} \\ & & Y_{**\bar{z}^{n+1}}(B, \pi, n), \end{array} \qquad \begin{array}{ccc} C^*_{\square}(E, G) & \longrightarrow & Y^{**}_{z^{n+1}}(B, \pi, n, G) \\ & \searrow & \downarrow \varphi_{u(c^n, z^{n+1})} \\ & & Y^{**}_{\bar{z}^{n+1}}(B, \pi, n, G) \end{array}$$

are commutative up to chain homotopy.

The proof below uses the geometric interpretation of the model.

Let K be a simplicial set, $z^{n+1} \in Z^{n+1}(K, \pi)$, and define the cubical complex $K \times_{z^{n+1}} L(\pi, n)$ as follows. Consider as $(p+q)$-dimensional cubes of $K \times_{z^{n+1}} L(\pi, n)$ the pairs (σ^p, τ^q), where σ^p is a p-dimensional simplex of K and τ^q is a q-dimensional cube of the cubical set $L(\pi, n)$.

The face operators are defined by virtue of (2.4) and (4.2) as follows. Let $\kappa_{z^{n+1}} : K \longrightarrow L(\pi, n)$ be map (4.1).

Define

$$\begin{gathered} d^1_{p+i}(\sigma^p, \tau^q) = (\sigma^p, d^1_i \tau^q), \quad 1 \le i \le q, \\ d^0_{p+i}(\sigma^p, \tau^q) = (\sigma^p, d^0_i \tau^q), \quad 1 \le i \le q, \\ d^1_i(\sigma^p, \tau^q) = (\sigma^p_{i-1}, \tau^q), \quad 1 \le i \le p, \\ d^0_p(\sigma^p, \tau^q) = (\sigma^p_p, \tau^q), \\ d^0_i(\sigma^p, \tau^q) = (\sigma^p_1, \kappa_{z^{n+1}}(\sigma^p_2) \circ \tau^q) \quad 1 \le i \le (p-1), \end{gathered}$$

where σ^p_1 is the first $(i-1)$-face and σ^p_2 is the last $(p-i+1)$-face of σ^p, while $\circ$ is the product in the ring $\tilde{C}^{\square}_*(L(\pi, n))$.

The degeneracy operators are defined only partially:

$$s^{p+i}(\sigma^p, \tau^q) = (\sigma^p, s^i(\tau^q)), \quad 1 \le i \le (q+1).$$

The chain complexes $\tilde{C}_*^{\square}(K \times_{z^{n+1}} L(\pi,n))$, $\bar{C}_*^{\square}(K \times_{z^{n+1}} L(\pi,n))$, $C_*^{\square}(K \times_{z^{n+1}} L(\pi,n))$, $C^*_{\square}(K \times_{z^{n+1}} L(\pi,n), G)$ of $K \times_{z^{n+1}} L(\pi,n))$ are defined as for the case of cubical sets.

The obvious fact is

Lemma 5.1. *The integral chain complex of the cubical complex $S(B) \times_{z^{n+1}} L(\pi,n)$ is the complex $Y_{**z^{n+1}}(B,\pi,n)$.*

Let $|K \times_{z^{n+1}} L(\pi,n)|$ be the Milnor realization of this complex (Section 2). Let $||K||$ be the Giever–Hu realization of the simplicial set K (i.e., the degeneracy operators are passive). We have the map

$$\alpha : |K \times_{z^{n+1}} L(\pi,n)| \longrightarrow ||K|| \tag{5.2}$$

defined for every (σ^p, τ^q) by $I^{p+q} \xrightarrow{proj.} I^p \xrightarrow{\psi} \Delta_p$. The complex $|K \times_{z^{n+1}} L(\pi,n)|$ is filtered by subcomplexes $F^r = \bigcup_{p \le r} |(\sigma^p, \tau^q)|$. We have

Lemma 5.2. $H_{r+q}(F^r, F^{r-1}) = C_r(K, H_q^{\square}(L(\pi,n)))$.

Proof. F^r/F^{r-1} is a wedge of CW-complexes, one complex K_{σ^r} for every $\sigma^r \in K$. Hence $H_{r+q}(F^r, F^{r-1}) = \sum_{\sigma^r} H_{r+q}(K_{\sigma^r})$. The filtration of K_{σ^r} by its skeletons gives $H_{r+q}(K_{\sigma^r}) = H_q^{\square}(L(\pi,n))$. $\square$

Consider the standard map $||S(B)|| \longrightarrow B$ of the Giever–Hu realization of $S(B)$ and also consider the induced fibration $E' \xrightarrow{pr} ||S(B)||$. In the diagram

$$\begin{array}{ccc} E & \longleftarrow & E' \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle pr} \\ B & \longleftarrow & ||S(B)|| \end{array}$$

the horizontal maps induce an isomorphism of homology. In the induced fibration consider the filtration given by $pr^{-1}(||S(B)||^r)$.

Proposition 5.1. *There is a commutative diagram*

$$\begin{array}{ccc} E' & \longleftarrow & |S(B) \times_{z^{n+1}} L(\pi,n)| \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle \alpha} & \\ & \|S(B)\|, & \end{array}$$

where the upper map is the map of filtered spaces and induces an isomorphism of the first terms of the related spectral sequences.

Proof. The map $|S(B) \times_{z^{n+1}} {}_z L(\pi, n)| \longrightarrow E'$ is constructed by induction on degree of cells. Let s be a cross section over the n-skeleton of $||S(B)||$ whose obstruction cocycle is z^{n+1}. The induction steps are as follows:

(0). (σ^0, τ^0) is a vertex. Let $f|(\sigma^0, \tau^0) = s\alpha|(\sigma^0, \tau^0) = s(\sigma^0)$.

(i), $i<n$. (σ^j, τ^{i-j}) are i-dimensional cells. Let $f|(\sigma^j, \tau^{i-j}) = s\alpha|(\sigma^j, \tau^{i-j})$.

(n). (σ^j, τ^{n-j}) are n-cells. Let $f|(\sigma^j, \tau^{n-j}) = s\alpha|(\sigma^j, \tau^{n-j})$, $j < n$, and let $f|(\sigma^0, \tau^n)$ be the map of this "n-sphere" in a fiber over σ^0 as an element of $\pi_n(F, s(\sigma^0))$.

(n+1). For the cell (σ^{n+1}, τ^0) the map is already defined for its boundary and the image lies over σ^{n+1}. This n-sphere is homotopic to 0 over σ^{n+1} by virtue of the fact that $z^{n+1}(\sigma^{n+1}) \in \pi_n(F)$ is the class of the n-sphere defined by s on the boundary of σ^{n+1}. Hence f extends from the boundary onto the whole cell and the image lies over σ^{n+1}.

The map f of the boundary of (σ^1, τ^n) is homotopic to 0 and hence it extends onto the whole cell.

For the rest of the $(n+1)$-cells (σ^i, τ^{n+1-i}), $1 < i \leq n$, we assume that $f|(\sigma^i, \tau^{n+1-i}) = s\alpha|(\sigma^i, \tau^{n+1-i})$.

(m), $m > n+1$. (σ^i, τ^j), $i + j = m$, f already defined on its boundary is an $(m-1)$-sphere over σ^i and by the fact that $m - 1 > n$ it is homotopic to 0. f extends over the whole cell. The map is now constructed. By the inductive construction we see that it preserves filtrations.

The first term of both filtrations in the proposition under consideration is $C_p(S(B), H_q(F))$, and by virtue of the above map the homomorphism of the first terms of the spectral sequences is an isomorphism. □

Proof of Theorem 5.1*.* The standard imbedding

$$S(B) \times_{z^{n+1}} L(\pi, n) \longrightarrow Q(|S(B) \times_{z^{n+1}} L(\pi, n)|)$$

gives the chain map

$$C_*^{\square}(S(B) \times_{z^{n+1}} L(\pi, n)) \longrightarrow C_*^{\square}(|S(B) \times_{z^{n+1}} L(\pi, n)|)$$

inducing an isomorphism of cubical singular homologies. Then by Proposition 5.1 the composition

$$\begin{aligned} Y_{**z^{n+1}}(B, \pi, n, G) = C_*^{\square}(S(B) \times_{z^{n+1}} L(\pi, n), G) \longrightarrow \\ \longrightarrow C_*^{\square}(|S(B) \times_{z^{n+1}} L(\pi, n)|, G) \xrightarrow{f_{z_*^{n+1}}} C_*^{\square}(E, G) \end{aligned}$$

induces an isomorphism of homology. Let this chain map be the map of (i).

Consider the fibration $F \longrightarrow E'' \longrightarrow B \times I$ induced by projection pr : $B \times I \longrightarrow B$ and a map as in part (i) of the theorem:

$$E'' \xleftarrow{f_{z^{n+1}_{B\times I}}} |S(B \times I) \times_{z^{n+1}_{B\times I}} L(\pi, n)|$$

where $z^{n+1}_{B\times I} = pr{*}z^{n+1}_B - \delta\bar{c}^n$ ($\bar{c}^n$ is c^n placed on $B\times I$) is such that $i_0^* z^{n+1}_{B\times I} = z^{n+1}_B$, $i_1^* z^{n+1}_{B\times I} = \bar{z}^{n+1}_B$. We can choose this map so that on the bottoms of $B \times I$ it will coincide with $f_{z^{n+1}}$ and $f_{\bar{z}^{n+1}}$. Consider the imbeddings

$$C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|) \leftarrow C^{\square}_*(|S(B)\times_{z^{n+1}_B} L(\pi,n)|),$$
$$C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|) \leftarrow C^{\square}_*(|S(B)\times_{\bar{z}^{n+1}_B} L(\pi,n)|).$$

In view of the chain map

$$C^{\square}_*(E) \leftarrow C^{\square}_*(E'') \leftarrow C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|)$$

it is enough to show that the triangle

$$\begin{array}{ccc} C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|) & \longleftarrow & C^{\square}_*(|S(B)\times_{z^{n+1}_B} L(\pi,n)|) \\ & \nwarrow & \uparrow{\scriptstyle \varphi_{u(c^n, z^{n+1}_B)}} \\ & & C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|) \end{array}$$

is commutative up to chain homotopy. To this end consider

$$C^{\square}_*(|S(B\times I)\times_{z^{n+1}_{B\times I}} L(\pi,n)|) \xleftarrow{\varphi_{u(\bar{c}^n, pr^* z^{n+1}_B)}} C^{\square}_*(|S(B\times I)\times_{pr^* z^{n+1}_B} L(\pi,n)|).$$

The two imbeddings of $C^{\square}_*(|S(B)\times_{z^{n+1}_B} L(\pi,n)|)$ on the left side complex are chain homotopic and the composition gives the required homotopy. This proves (ii) for an integral homology. Hence we obtain (ii) for the coefficient group G and cohomology.

In particular, if $\bar{z}^{n+1} = z^{n+1}$ and $c^n = 0$, then we deduce that the map in (i) is defined uniquely up to chain homotopy. $\square$

6. Multiplicative Structure in the Algebraic Model

We have seen that the cohomology model $Y^{**}_{z^{n+1}}(B,\pi,n,G)$ of a fibration $F \to E \to B$ with F a $K(\pi,n)$-space can be identified with the cochain complex of the cubical complex $S(B)\times_{z^{n+1}} L(\pi,n)$. Each cubical complex is endowed with multiplicative structure via the Serre cup product. Hence $Y^{**}_{z^{n+1}}(B,\pi,n,\Lambda)$, where Λ is a commutative ring, has a multiplicative structure. It can evidently be described as follows.

Let x^s, $y^t \in Y^{**}_{z^{n+1}}(B,\pi,n,\Lambda)$. Being an element of the module $C^p(B, C^q(L(\pi,n)\Lambda))$, the (p,q)-component, where $p+q = s+t$, of the product $w^{s+t} = x^s y^t$ is a p-dimensional cochain of B with coefficients in $C^q(L(\pi,n),\Lambda)$

defined in the following manner: for the pair (σ^p, τ^q) consider all decompositions of $1, 2, \cdots, (p+q)$ in two disjoint sets (H, K) and let

$$w^{s+t}[(\sigma^p, \tau^q)] = \sum_{(H,K)} (-1)^{a(H,K)} x(d_K^0(\sigma^p, \tau^q)) y(d_H^1(\sigma^p, \tau^q)).$$

Theorem 6.1. *The cochain map of Theorem* 5.1

$$C^*_{\square}(E, \Lambda) \to Y^{**}_{z^{n+1}}(B, \pi, n, \Lambda)$$

is multiplicative.

Proof. The above cochain map is induced (see the proof of Theorem 5.1) by the map of cubical sets $S(B) \times_{z^{n+1}} L(\pi, n) \to Q(E)$. $\square$

7. Action of the Group $H^n(B, \pi)$ on the Homology of the Complexes $Y_{**z^{n+1}}(B, \pi, n, G)$ and $Y^{**}_{z^{n+1}}(B, \pi, n, G)$

Consider the space $B \times \Delta_2$ and assume that we are given a cocycle $z^{n+1}_{B\times\Delta^2} \in Z^{n+1}(B \times \Delta_2, \pi)$.

We have three imbeddings $i_0, i_1, i_2, : B \to B \times \Delta^2$ for three vertices of Δ_2 and three imbeddings $i_{01}, i_{12}, i_{02}, : B \times I \to B \times \Delta^2$ for three 1-faces of Δ_2. For a given cocycle $z^{n+1}_{B\times\Delta^2} \in Z^{n+1}(B \times \Delta^2)$ consider six (n+1)-cocycles (three on B and three on $B \times I$):

$$\begin{aligned} z^{n+1}_{B,0} &= i_0^*(z^{n+1}_{B\times\Delta^2}), & z^{n+1}_{B\times I,01} &= i_{01}^*(z^{n+1}_{B\times\Delta^2}),\\ z^{n+1}_{B,1} &= i_1^*(z^{n+1}_{B\times\Delta^2}), & z^{n+1}_{B\times I,12} &= i_{12}^*(z^{n+1}_{B\times\Delta^2}),\\ z^{n+1}_{B,2} &= i_2^*(z^{n+1}_{B\times\Delta^2}), & z^{n+1}_{B\times I,02} &= i_{02}^*(z^{n+1}_{B\times\Delta^2}). \end{aligned}$$

The left cocycles define on B three twisting elements $a(z^{n+1}_{B,0})$, $a(z^{n+1}_{B,1})$, $a(z^{n+1}_{B,2})$, while the right cocycles define on B three 0-elements as in Lemma 4.2:

$$\begin{aligned} u_{01}(z^{n+1}_{B\times I,01}), u_{12}(z^{n+1}_{B\times I,12}), u_{02}(z^{n+1}_{B\times I,02}) \in A^{*,-*}(B, \pi, n) =\\ = C^*(B, C_*^{\square}(L(\pi, n))). \end{aligned}$$

Consider the standard map

$$w_2 : C_*(B) \to C_{*+2}(B \times \Delta_2)$$

subject to the condition

$$d_{B\times\Delta_2} w_2 - w_2 d_B = (i_{01}^* + i_{12}^* - i_{02}^*) w_1. \tag{7.1}$$

To every singular simplex $\sigma^m \in S(B)$ this map assigns the singular $(m+2)$-chain $w_2(\sigma^m)$ which is the image of the main integral $(m+2)$-chain of the standard triangulation of $\Delta_m \times \Delta_2$ by the map $\sigma^m \times id : \Delta_m \times \Delta_2 \to B \times \Delta_2$.

Define the map

$$w_2^* : A^{*,-*}(B \times \Delta_2, \pi, n) \longrightarrow A^{*,-*}(B, \pi, n)$$

by the composition

$$C_*(B) \xrightarrow{w_2} C_*(B \times \Delta_2) \xrightarrow{x} C_*^{\square}(L(\pi, n)), \quad x \in A(B \times \Delta_2, \pi, n).$$

Let $v(z^{n+1}_{B\times\Delta_2}) = w_2^*(a(z^{n+1}_{B\times\Delta_2}))$. We see that $|v| = -1$ and $v = v^{0,-1} + v^{1,-2} + v^{2,-3} + \cdots$.

Lemma 7.1. *In the above notation we have the equality*

$$d_{A(B,\pi,n)} v = u_{01} + u_{12} + u_{01}u_{12} - u_{02} + va(z^{n+1}_{B,0}) + a(z^{n+1}_{B,2})v$$

in the algebra $A^{*,-*}(B, \pi, n)$.

Proof. The lemma is proved similarly to Lemma 4.2, using equality (7.1). □

In what follows we abbreviate $Y_{**z^{n+1}}(B, \pi, n)$ to $Y_{z^{n+1}}$.

In the situation we are considering we have the triangle of chain isomorphisms

$$\begin{array}{ccc} Y_{z^{n+1}_{B,0}} & \xrightarrow{\varphi_{u_{01}}} & Y_{z^{n+1}_{B,1}} \\ & \searrow^{\varphi_{u_{02}}} & \downarrow{\scriptstyle \varphi_{u_{12}}} \\ & & Y_{z^{n+1}_{B,2}} \, . \end{array}$$

Corollary 7.1. *The above triangle is commutative up to chain homotopy.*

Proof. By virtue of Lemma 7.1 the map $F : Y_{z^{n+1}_{B,0}} \to Y_{z^{n+1}_{B,2}}$ defined by $F(x) = vx$ yields a chain homotopy between the composition $\varphi_{u_{12}}\varphi_{u_{01}}$ and $\varphi_{u_{02}}$. □

Corollary 7.2. *Let* $z^{n+1}_B \in Z^{n+1}(B, \pi)$ *and* $c^n, c^n_1 \in C^n(B, \pi)$. *Then the triangle*

$$\begin{array}{ccc} Y_{z^{n+1}+\delta(c^n+c^n_1)} & \xrightarrow{\varphi_{u(c^n_1, z^{n+1}+\delta c^n)}} & Y_{z^{n+1}+\delta c^n} \\ & \searrow^{\varphi_{u(c^n_1+c^n, z^{n+1})}} & \downarrow{\scriptstyle \varphi_{u(c^n, z^{n+1})}} \\ & & Y_{zn+1} \end{array}$$

is commutative up to chain homotopy.

Proof. Let $Pr : B \times \Delta^2 \to B$ be the projection and consider the cocycle of $B \times \Delta^2$ $z^{n+1}_{B\times\Delta^2} = Pr^*(z^{n+1}_B) + \delta_{B\times\Delta^2}(\bar{c}^n) + \delta_{B\times\Delta^2}(\bar{\bar{c}}^n + \bar{\bar{c}}^n_1)$, $\bar{c}^n$ being c^n identified as the cochain of the subcomplex $B \times 1$ and zero otherwise, and $\bar{\bar{c}}^n + \bar{\bar{c}}^n_1$ being $c^n + c^n_1$ identified as the cochain of $B \times 2$ and zero otherwise. The rest follows from Corollary 7.1. □

Corollary 7.3. *Let* $z^{n+1}_B \in Z^{n+1}(B,\pi)$ *and* $0 \in C^n(B,\pi)$*; then* $\varphi_{u(0,z^{n+1})} : Y_{z^{n+1}_B} \to Y_{z^{n+1}_B}$ *is homotopic to the identity.*

Proof. From $0 = 0 + 0$ and Corollary 7.1 we see that $\varphi_{u(0,z^{n+1})}\varphi_{u(0,z^{n+1})} \sim \varphi_{u(0,z^{n+1})}$; hence by the fact that $\varphi_{u(0,z^{n+1})}$ is surjective, $\varphi_{u(0,z^{n+1})}$ is homotopic to the identity (in fact, $\varphi_{u(0,z^{n+1})}$ is the identity, but we do not need this). □

Corollary 7.4. *For* $z^{n+1}_B \in Z^{n+1}(B,\pi)$ *and* $c^n \in C^n(B,\pi)$ *the triangle*

$$\begin{array}{ccc} Y_{z^{n+1}} & \xrightarrow{\varphi_{u(-c^n,z^{n+1}-\delta c^n)}} & Y_{z^{n+1}-\delta c^n} \\ & \searrow^{id} & \downarrow{\scriptstyle \varphi_{u(c^n,z^{n+1})}} \\ & & Y_{zn+1} \end{array}$$

is commutative up to chain homotopy.

Proof. By virtue of Corollary 7.2 the composition in question is homotopic to $\varphi_{u(c^n-c^n,z^{n+1})} = \varphi_{u(0,z^{n+1})}$. Corollary 7.3 accomplishes the proof. □

Corollary 7.5. *For* $z^{n+1}_B \in Z^{n+1}(B,\pi)$ *and* $c^{n-1} \in C^{n-1}(B,\pi)$ *the map* $\varphi_{u(\delta c^{n-1},z^{n+1})} : Y_{z^{n+1}_B} \to Y_{z^{n+1}_B}$ *is homotopic to the identity.*

Proof. On the complex $B\times I$ consider an n-cochain $k^n = \delta_{B\times I}c^{n-1} - \delta_B c^{n-1}$; here c^{n-1} and $\delta_B c^{n-1}$ are identified as cochains in $B\times 0$ and zero otherwise. Then $\delta_{B\times I}k^n = \delta_{B\times I}(\delta_B c^{n-1})$. Identify k^n as a cochain of $B \times \Delta^2$ via $i_{01} : B \times I \to B \times \Delta^2$. Consider $z^{n+1}_{B\times\Delta^2} = Pr^* z^{n+1}_B - \delta_{B\times\Delta^2}k^n$. The restrictions of this cocycle on $(B\times I)_{02}$ and $(B\times I)_{12}$ are equal to $Pr^*(z^{n+1}_B)$ and the restriction on $(B\times I)_{01}$ is $Pr^* z^{n+1} - \delta_{B\times I}(\delta_B c^n)$. Hence Corollaries 7.2 and 7.4 carry the proof to the end. □

Theorem 7.1. *For every* $z^{n+1} \in Z^{n+1}(B,\pi)$ *and every abelian group* G *the cohomology group* $H^n(B,\pi)$ *acts from the left on the homology and* *cohomology groups* $H_{q+1}(Y_{z^{n+1}},G)$, $H^{q+1}(Y_{z^{n+1}},G)$. *The action is given by the chain map*

$$\varphi_{u(c^n,z^{n+1})}:Y_{z^{n+1}}\to Y_{z^{n+1}},\quad \varphi_{u(c^n,z^{n+1})}(y)=(1+u(c^n,z^{n+1}))(y),$$
$$c^n\in Z^n(B,\pi),\ \ y\in Y_{z^{n+1}}.$$

Proof. We readily prove the theorem, using the above corollaries. □

Acknowledgement

The research described in this publication was made possible in part by Grant No. RVC000 from the International Science Foundation.

For the improvement of the proof of Proposition 5.1 the author is obliged to S. Saneblidze.

References

1. E. H.Brown, Twisted tensor products. *Ann. Math.* **69**(1959), 223–246.

2. J. P. May, Simplicial objects in algebraic topology. *Van Nostrand, Princeton,* 1967.

3. S. MacLane, The homology product in $K(\Pi,n)$. *Proc. Amer. Math. Soc.* **5**(1954), 642–651.

4. M. M. Postnikov, Cubical resolutions. *Dokl. Akad. Nauk SSSR* **118**(1958), 1085–1087.

5. N. Berikashvili, On the obstruction functor. *Bull. Georgian Acad. Sci.* (*to appear*).

6. N. Berikashvili, An algebraic model of the Postnikov construction, *Bull. Georgian Acad. Sci.* **152**(1995).

7. D. M. Kan, Abstract homotopy. *Proc. Natl. Acad. Sci. USA* **41**(1955), 1092–1096.

8. J. Milnor, The geometric realization of a semi-simplicial complex. *Ann. Math.* **65**(1957), 357–362.

9. J. B. Giever, On the equivalence of two singular homology theories. *Ann. Math.* **51**(1950), 178–191.

10. S. T. Hu, On the realizability of homotopy groups and their operations. *Pacific J. Math.* **1**(1951), 583–602.

11. J. P. Serre, Homologie singuliere des éspaces fibrés, applications. *Ann. Math.* **54**(1951), 429–505.

12. N. Berikashvili, On the homology theory of fibrations. *Bull. Acad. Sci. Georgia* **139**(1990), 17–19.

(Received 21.09.1994)

Author's address:

A. Razmadze Mathematical Institute, Georgian Academy of Sciences

1, Z. Rukhadze St., Tbilisi 380093, Republic of Georgia