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OSCILLATORY CRITERIA FOR NONLINEAR nTH-ORDER DIFFERENTIAL EQUATIONS WITH QUASIDERIVATIVES

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ABSTRACT. Sufficient conditions are given for the existence of oscillatory proper solutions of a differential equation with quasiderivatives $L_n y = f(t, L_0 y, \dots, L_{n-1} y)$ under the validity of the sign condition $f(t, x_1, \dots, x_n) x_1 \le 0$, $f(t, 0, x_2, \dots, x_n) = 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$.

1. INTRODUCTION

Consider the nth-order differential equation

$$L_n y(t) = f(t, L_0 y, L_1 y, \dots, L_{n-1} y) \quad \text{in} \quad \mathbb{D} = \mathbb{R}_+ \times \mathbb{R}^n, \tag{1}$$

where $n \ge 2$, $\mathbb{R}_+ = [0, \infty]$, $\mathbb{R} = (-\infty, \infty)$, $L_i y$ is the ith quasiderivative of y defined as

$$\begin{gathered} L_0 y(t) = \frac{y(t)}{a_0(t)}, \ L_i y(t) = \frac{(L_{i-1} y(t))'}{a_i(t)}, \quad i = 1, 2, \dots, n-1, \\ L_n y(t) = (L_{n-1} y(t))', \end{gathered} \tag{2}$$

functions $a_i \in C^\circ(\mathbb{R}_+)$ are positive, and $f : \mathbb{D} \to \mathbb{R}$ fulfills the local Carathéodory conditions.

Throughout the paper we assume that

$$f(t, x_1, \dots, x_n)\, x_1 \le 0, \ f(t, 0, x_2, \dots, x_n) = 0 \quad \text{in} \quad \mathbb{D}. \tag{3}$$

Definition. A function $y : [0, T) \to \mathbb{R}$, $T \in (0, \infty]$, is called a solution of (1) if (1) is valid for almost all $t \in [0, T)$. It is called noncontinuable if either $T = \infty$ or $T < \infty$, and

$$\limsup_{t \to T} \sum_{i=0}^{n-1} |L_i y(t)| = \infty .$$

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Let $y : [0, T) \to \mathbb{R}$, $T \le \infty$, be a noncontinuable solution of (1). It is said to be proper if $T = \infty$ and $\sup_{\tau \le t < \infty} |y(t)| > 0$ for all $\tau \in \mathbb{R}_+$. It is said to be singular of the first (second) kind if $t^* \in (0, \infty)$ exists such that

$$y \equiv 0 \text{ in } [t^*, \infty), \quad \sup_{0 \le t \le t^*} \sum_{i=0}^{n-1} |L_i y(t)| > 0$$

(if $T < \infty$). A proper solution y is said to be oscillatory if a sequence $\{t_k\}_0^\infty$ exists such that $t_k \in \mathbb{R}_+$, $\lim_{k\to\infty} t_k = \infty$ and $y(t_k) = 0$ holds. Otherwise, it is called nonoscillatory.

Many authors studied the problem of structure and properties of proper nonoscillatory solutions of (1) (see, e.g., [1]–[3]). But as regards proper oscillatory solutions, their existence is proved only in the cases where $n \ge 3$ and $a_i \equiv 1$ (see [4]–[6]), or $n = 3$ (see [1]).

Definition. Equation (1) has property A if every proper solution y is oscillatory for even n and it is either oscillatory or

$$\lim_{t\to\infty} L_i y(t) = 0 \quad \text{monotonically}, \quad i = 0, 1, \dots, n-1, \tag{4}$$

for odd n.

Similarly to a differential equation without quasiderivatives ($a_i \equiv 1$), it is possible to use the following way to prove the existence of proper oscillatory solutions: If

1° there exists no singular solution of the 1st kind;
2° there exists no singular solution of the 2st kind;
3° (1) has Property A;
4° the initial conditions of y at zero are choosen such that (4) is not valid,

then y is oscillatory proper.

Sufficient conditions for the validity of relations 1°, 2°, 4° can be easily obtained similarly to the case $a_i \equiv 1$ (see later). Very profound results concerning 3° are given in [7].

In our paper we generalize the results which could be obtained by this approach. Especially, we shall weaken conditions 1° and 3°.

Sometimes, we will suppose that

$$a_n(t)|x_1|^\lambda \leqq |f(t, x_1, \dots, x_n)| \quad \text{in} \quad \mathbb{D}, \tag{5}$$

where $0 < \lambda \le 1$, $a_n \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $a_n \ge 0$;

$$\int_0^\infty a_i(t)\,dt = \infty\,, \quad i = 1, 2, \ldots, n-1\,, \tag{6}$$

$$|f(t, x_1, \ldots, x_n)| \le h(t)\omega\Big(\sum_{i=1}^n |x_i|\Big) \quad \text{in} \quad \mathbb{D}, \tag{7}$$

where $h \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $\omega \in C^\circ(\mathbb{R}_+)$, $\omega(x) > 0$ for $x > 0$, $\int_0^\infty \frac{dt}{\omega(t)} = \infty$;

$$|f(t, x_1, \ldots, x_n)| \le A(t)g(|x_1|) \quad \text{in} \quad \mathbb{R}_+ \times [-\varepsilon, \varepsilon]^n, \tag{8}$$

where $\varepsilon > 0$, $A \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $g \in C^\circ[0, \varepsilon]$, $g(0) = 0$, $g(x) > 0$ for $x > 0$, $\displaystyle\int_0^\varepsilon \frac{dt}{g(t)} = \infty$;

$$\begin{cases} \text{let } \dfrac{a_1}{a_2} \in C^1(\mathbb{R}_+) \text{ for } n = 3\,, \\ a_1 \in C^1(\mathbb{R}_+),\ a_2 \in C^1(\mathbb{R}_+),\ \dfrac{a_3}{a_1} \in C^2(\mathbb{R}_+) \text{ for } \ n = 4 \\ \text{and let for } \ n > 4 \ \text{ an index } \ l \in \{1, 2, \ldots, n-4\} \text{ exist} \\ \text{such that } \ a_{l+j}, j = 1, 2, \ \text{ are absolutely continuous and} \\ a'_{l+j},\ \ j = 1, 2, \ \text{ are locally bounded from below.} \end{cases} \tag{9}$$

Notation. If $b_i \in C^\circ(I)$, then

$$I^\circ(t) \equiv 1,\ I^k(t, b_1, \ldots, b_k) = \int_0^t b_1(s) I^{k-1}(s, b_2, \ldots, b_k)\,ds,\ t \in I\,.$$

Put $a_{nj+i}(t) = a_i(t)$, $j \in \{\ldots, -1, 0, 1, \ldots\}$, $i \in \{0, 1, \ldots, n\}$, $N = \{1, 2, \ldots\}$.

2. Main results

Further, we shall investigate a solution y of (1) that satisfies the initial conditions

$$\begin{gathered} l \in \{0, 1, \ldots, n-1\}\,, \tau \in \{-1, 1\}\,,\ \tau L_i y(0) > 0,\ i = 0, 1, \ldots, l\,, \\ \tau L_j y(0) < 0,\ j = l+1, \ldots, n-1, \end{gathered} \tag{10}$$

and we shall prove that this solution is oscillatory proper under the validity of certain assumptions.

Theorem 1. *Let $\lambda \in (0,1)$ and let* (5), (7), *and* (9) *be valid. Let*

$$\int\limits_0^\infty a_{i+1}(\tau_{i+1}) \int\limits_0^{\tau_{i+1}} a_{i+2}(\tau_{i+2}) \int\limits_0^{\tau_{i+2}} \cdots \int\limits_0^{\tau_{n-1}} a_n(\tau_n) \Big[\int\limits_0^{\tau_n} a_{n+1}(\tau_{n+1}) \dots$$
$$\int\limits_0^{\tau_{i+n-1}} a_{i+n}(\tau_{i+n})\, d\tau_{i+n} \dots d\tau_{n+1} \Big]^\lambda \times d\tau_n \dots d\tau_{i+1} = \infty\,, \tag{11}$$
$$i = 0, 1, \dots, n-1\,.$$

Then any solution y of (1) *that fulfills the Cauchy initial conditions* (10) *is oscillatory proper.*

Theorem 2. *Let $\lambda = 1$,* (5), (6), *and* (7) *hold. Let*

$$\limsup_{t\to\infty} I^1(a_{n-1}) \int\limits_t^\infty \frac{I^{n-1}(s, a_1, \dots a_{n-1})}{I^1(s, a_{n-1})} a_n(s)\, ds > 1\,. \tag{12}$$

Further, let either (9) *or* (8) *hold.*

Then any solution y of (1), *that fulfills the Cauchy initial conditions* (10) *is oscillatory proper.*

Theorem 3. *Let* (6), (7) *be valid and let functions $a_n \in L_{loc}(\mathbb{R}_+)$, $b \in C^\circ(\mathbb{R}_+)$ exist such that $\int_0^\infty a_n(t)\, dt = \infty$, $b(0) = 0$, $b(x) > 0$ for $x > 0$, b is nondecreasing, and*

$$a_n(t) b(|x_1|) \le |f(t, x_1, \dots, x_n)| \quad \text{in} \quad \mathbb{D}\,.$$

Further, let either (9) *or* (8) *be valid. Then any solution y of* (1) *that fulfills* (10) *is oscillatory proper.*

3. Proof of main results

Let us define two special types of solutions of (1) that will be encountered later.

Type I ($\boldsymbol{\tau}$): $y : [0, \tau) \to \mathbb{R}$, $0 < \tau \le \infty$ and sequences $\{t_k^i\}$, $\{\bar{t}_k^{n-1}\}$, $k \in \mathbb{N}$, $i \in \{0, 1, \dots, n-1\}$ exist such that $\lim_{k\to\infty} t_k^\circ = \tau$,

$$0 \le t_k^0 < t_k^{n-1} \le \bar{t}_k^{n-1} < t_k^{n-2} \dots < t_k^1 < t_{k+1}^0\,,$$

$L_i y(t_k^i) = 0$, $i = 0, 1, \ldots, n-2$, $L_{n-1}y(t) = 0$ for $t \in [t_k^{n-1}, \bar{t}_k^{n-1}]$, $k \in \mathbb{N}$,

$$\begin{aligned} L_i y(t) L_0 y(t) &> 0 \quad \text{for} \quad t \in (t_k^0, t_k^i) \quad , i = 0, 1, \ldots, n-1, \\ &< 0 \quad \text{for} \quad t \in (t_k^i, t_{k+1}^0)\ , i = 0, 1, \ldots, n-2, \\ &< 0 \quad \text{for} \quad t \in (\bar{t}_k^{n-1}, t_{k+1}^0)\, , i = n-1,\ k \in \mathbb{N}\,. \end{aligned}$$

If $\tau < \infty$, then $\lim_{t \to \tau} L_i y(t) = 0$, $i = 0, 1, \ldots, n-1$.

Type II (s): $y : \mathbb{R}_+ \to \mathbb{R}$, $s \in \{0, 1, \ldots, n-1\}$, $\tau \in \mathbb{R}_+$,

$$\begin{aligned} L_j y(t) L_s y(t) &\geq 0 \quad \text{for} \quad j \in \{0, 1, \ldots, s\} \\ &\leq 0 \quad \text{for} \quad j \in \{s+1, \ldots, n-1\}\,, \\ L_m y(t) &\neq 0, m \in \{0, 1, \ldots, n-2\}, \quad t \in [\tau, \infty)\,. \end{aligned}$$

Remark. Any solution y of Type I (∞) (of Type II (s)) is oscillatory proper (nonoscillatory proper). If we define $y \equiv 0$ on $[\tau, \infty)$, then any solution y of Type I(τ), $\tau < \infty$ is singular of the first kind.

Lemma 1. *Let $J = [t_1, t_2] \subset \mathbb{R}_+$, $t_1 < t_2$ and $y : J \to \mathbb{R}$ be a solution of* (1)*.*

(a) *If $j \in \{1, 2, \ldots, n\}$, $L_j y(t) \geq 0$ (≤ 0) in J, then $L_{j-1}y$ is nondecreasing (nonincreasing) in J;*

(b) *if $j \in \{1, 2, \ldots, n\}$, $L_j y(t) > 0$ (< 0) in J, then $L_{j-1}\, y$ is increasing (decreasing) in J;*

(c) *if $L_0 y(t) \geq 0$ (≤ 0) in J, then $L_{n-1}\, y$ is nonincreasing (nondecreasing) in J.*

Proof.

(a) Let $L_j y(t) \geq 0$ in J. Then according to (2) either $(L_{j-1}y(t))' = a_j(t) L_j y(t) \geq 0$, $j < n$ or $(L_{n-1}y(t))' = L_n y(t) \geq 0$ holds.

(b), (c) The proof is similar, only (3) must be used instead of (2) in (c). $\square$

Lemma 2. *Let $y : \mathbb{R}_+ \to \mathbb{R}$ be a solution of* (1) *which satisfies* (10)*. Then one of the following possibilities holds:*

(a) *y is of Type* I *(∞)*

(b) *there exists $\tau \in (0, \infty)$ such that y is of Type* I*(τ) in $[0, \tau)$.*

(c) *there exists $i \in \{0, \ldots, n-1\}$ such that y is of Type* II *(i).*

Proof. First suppose that y satisfies the Cauchy initial conditions

$$\sigma L_i y(0) > 0, \quad i = 0, 1, \ldots, n-1\,. \tag{13}$$

According to Lemma 1 $\sigma L_i y > 0$, $i = 0, 1, \ldots, n-1$, in some right neighborhood of $t = 0$, and $\sigma L_j y$, $j = 0, 1, \ldots, n-2$, are nondecreasing ($\sigma L_{n-1} y$

is nondecreasing) until $\sigma L_{j+1}y \geq 0$ ($\sigma L_0 y \geq 0$). Thus either y is of Type II $(n-1)$ or numbers t^n, $\bar{t}^n$ exist such that

$$\begin{gathered} 0 < t^n \leq \bar{t}^n, \sigma L_j y(t) > 0 \quad \text{in} \quad [0, \bar{t}^n],\ j \in \{0,1,\dots,n-2\}, \\ \sigma L_{n-1}y(t) > 0 \quad \text{in} \quad [0, t^n), \sigma L_{n-1}y(t) \equiv 0 \quad \text{in} \quad [t^n, \bar{t}^n]\,, \\ \sigma L_j y(t) > 0, \sigma L_{n-1}y < 0 \quad \text{ in some right neighborhood of } \quad t = \bar{t}^n\,. \end{gathered}$$

By the same procedure it can be proved that either y is of Type II (s), $s \in \{0,\dots,n-2\}$, or numbers t^j, $j \in \{0,1,\dots,n-2\}$, exist such that

$$\begin{gathered} \bar{t}^{n-1} < t^{n-2} < \dots < t^0, \quad \sigma L_i y(t^i) = 0, \sigma L_i y > 0 \quad \text{in} \quad (t^{i+1}, t^i)\,, \\ \sigma L_m y > 0, \sigma L_k y < 0 \quad \text{in} \quad (t^{i+1}, t^i], \\ m \in \{0,1,\dots i-1\}, k \in \{i+1,\dots n-1\}, \end{gathered}$$

and

$$\sigma L_i y < 0,\ \ i \in \{0,1,\dots,n-1\} \ \text{ in some right neighborhood of } \ t^0. \tag{14}$$

Thus (13) is valid in this neighborhood and the statement follows by repeating the considerations in the case (13). Note that in the case Type I(τ), $\tau < \infty$, the relations $\lim_{t\to\tau} L_i y(t) = 0$, $i = 0,1,\dots,n-1$, must be valid because y is defined in $\mathbb{R}_+$.

Further, let (10) be valid. By the use of (13), (14) we see that the same initial conditions are valid in some t^*, $t^* \in [0, t^0]$, in the previous part of the proof. Thus the statement of the lemma can be proved similarly. □

Remark. Let $y : [0,\tau) \to \mathbb{R}$, $\tau < \infty$, be a noncontinuable solution. Then the statement of Lemma 2 is valid, too, if (a) is changed into

(a′) y is of Type I(τ) with the exception of $\lim_{t\to\tau} L_i y(t) = 0$, $i = 0,1,\dots,n-1$, and if Type II (s) is defined only on $[0,\tau)$.

Lemma 3 ([6, Lemma 9.2]). *Let $c_0 \geq 0$, $t_0 \in I \subset \mathbb{R}_+$, $h \in L_{loc}(I)$, $h \geq 0$, $\omega \in C^0(\mathbb{R}_+)$, $\omega(x) > 0$ for $x > c_0$, $\int_{c_0}^{\infty} \frac{ds}{\omega(s)} < \infty$. Then for every continuous function $x(t) : I \to \mathbb{R}_+$ which satisfies*

$$x(t) \leq c_0 + \Big[\int_{t_0}^{t} h(\tau)\omega(x(\tau))\,d\tau\Big] \operatorname{sign}(t-t_0), \quad t \in I\,,$$

we have

$$x(t) \leq \Omega^{-1}\bigg(\Big|\int_{t_0}^{t} h(\tau)\,d\tau\Big|\bigg), \quad t \in I,$$

where Ω^{-1} is the inverse function of $\Omega(s) = \displaystyle\int_{c_0}^{s} \frac{d\tau}{\omega(\tau)}$.

Lemma 4. *Let* (7) *hold. Then there exists no singular solution of* (1) *of the second kind.*

The lemma can be proved analogously to Lemma 4 in [7].

Lemma 5 (see [7], Lemma 1.5 and Consequence 1.2). *Let* $\omega : (0,\infty) \to \mathbb{R}_+$ *be continuous, nondecreasing and* $h \in L_{loc}(\mathbb{R}_+)$, $h \geq 0$, *such that*

$$\int_0^\infty h(t)\,dt = \infty, \quad \int_0^1 \frac{dx}{\omega(x)} < \infty\,.$$

Then the differential inequality $u' + a(t)\omega(u) \leq 0$ *has no proper positive solution in* $\mathbb{R}_+$.

Lemma 6. *Let* (5) *be valid and one of the following conditions hold:*
(a) $\lambda = 1$, (6) *and* (12) *hold*
(b) $\lambda \in (0,1)$, (11) *holds.*
Then there exists no solution of (1) *of Type* II(i), $i = 0,1,\dots,n-1$.

Proof. (a) With respect to (6) no solution of (1) of Type II(i), $i = 0,1,\dots,n-2$, exists (see [3]). The fact that there exists no solution of Type II $(n-1)$ is proved by Chanturia [7] in the proof of Theorem 3.5.

(b) We prove indirectly that a solution of Type II(s), $s \in \{0,1,\dots,n-1\}$, does not exist. Thus suppose, without loss of generality, that a solution of (1) $y : \mathbb{R}_+ \to \mathbb{R}$ exists such that $T \in \mathbb{R}_+$,

$$\begin{gathered} L_i y(t) \geq 0\,, \quad i = 0,1,\dots,s\,;\ L_j y(t) \leq 0\,, \quad j = s+1,\dots n-1\,, \\ L_m y(t) \neq 0\,, \quad m = 0,1,\dots,n-2\,,\ t \geq T\,. \end{gathered} \tag{15}$$

Then according to Lemma 1 and (3)

$$\begin{aligned} &|L_i y| \quad \text{is nondecreasing for} \quad i \in \{0,1,\dots,n-1\}\,,\ i \neq s\,, \\ &L_s y \quad \text{is nonincreasing in} \quad [T,\infty)\,. \end{aligned} \tag{16}$$

Further, by the use of (2), (5), (15), (16)

$$\begin{aligned} &|L_i y(t)| \geq \int_T^\infty a_{i+1}(s)|L_{i+1}y(s)|\,ds\,, \quad i = 0,1,\dots,n-2\,, \\ &|L_{n-1}y(t)| \geqq \int_T^\infty |L_n y(s)|\,ds \geqq \int_T^\infty a_n(s)|L_0 y(s)|^\lambda\,ds\,, \\ &-(L_s y(t))' = a_{s+1}(t)|L_{s+1}y(t)| \quad \text{for} \quad s \in \{0,1,\dots,n-2\}\,, \\ &-(L_s y(t))' = -L_n y(t) \geq a_n(t)(L_0 y(t))^\lambda \quad \text{for} \quad s = n-1\,. \end{aligned} \tag{17}$$

From this and (17) we have for $t \in [T, \infty)$

$$
\begin{gathered}
|L_{s+1}y(t)| \geqq \\
\geqq \int_T^t a_{s+2}(\tau_{s+2}) \int_T^{\tau_{s+2}} a_{s+3}(\tau_{s+3}) \cdots \int_T^{\tau_{n-2}} a_{n-1}(\tau_{n-1}) |L_{n-1}y(\tau_{n-1})| \geqq \\
\geqq \int_T^t a_{s+2}(\tau_{i+2}) \cdots \int_T^{\tau_{n-2}} a_{n-1}(\tau_{n-1}) \int_T^{\tau_{n-1}} a_n(\tau_n) \times \\
\times \Big[\int_T^{\tau_n} a_1(s_1) \int_T^{\tau_1} \cdots \int_T^{\tau_{s-1}} a_s(\tau_s) L_s y(\tau_s) \Big]^\lambda d\tau_s \dots d\tau_1 d\tau_n \dots d\tau_{s+2} \leq \\
\leq Z_s(t,T)(L_s y(t))^\lambda, \quad s = 0, 1, \dots, n-2, \\
|L_0 y(t)| \geq Z_{n-1}(t,T) L_{n-1} y(t) \quad (\text{for} \quad s = n-1),
\end{gathered}
$$

where

$$
\begin{gathered}
Z_s(t,T) = \int_T^t a_{s+2}(\tau_{i+2}) \cdots \int_T^{\tau_{n-1}} a_n(\tau_n) \Big[\int_T^{\tau_n} a_1(\tau_1) \dots \\
\cdots \int_T^{\tau_{s-1}} a_s(\tau_s) d\tau_s \dots d\tau_1 \Big]^\lambda d\tau_n \, d\tau_{s+2}, \quad s = 0, 1, \dots, n-2\,, \\
Z_{n-1}(t,T) = \int_T^t a_1(\tau_1) \int_T^{\tau_1} a_2(\tau_2) \cdots \int_T^{\tau_{n-2}} a_{n-1}(\tau_{n-1}) \, d\tau_{n-1} \dots d\tau_1 \\
(\text{for} \quad s = n-1)\,.
\end{gathered}
$$

It follows from (17) that

$$
(L_s y(t))' + a_{s+1}(t) Z_s^\beta(t,T)(L_s y(t))^\lambda \leq 0, \quad t \in [T, \infty),
$$

where $\beta = 1$ for $s \in \{0, 1, \dots, n-2\}$, $\beta = \lambda$ for $s = n-1$.
As according to (11)

$$
Z_s(\infty, T) = Z_s(\infty, 0) = \infty,
$$

we get the contradiction to Lemma 5 if $L_s y(t) > 0$ in $[T, \infty)$. Thus with respect to (17)

$$
s = n-1, \quad L_{n-1} y(t) \equiv 0 \quad \text{on} \quad [\tau, \infty), \quad \tau \in [T, \infty),
$$

is the last case which has to be considered. In that case, according to (17), (16),

$$0 = -(L_{n-1}y(t))' \geqq a_n(t)(L_0 y(t))^\lambda \,,$$
$$a_n(t) = 0 \quad \text{for almost all} \quad t \in [\tau, \infty)\,.$$

The contradiction to (11), $i = n-1$, proves the statement of the lemma. □

Remark.

(a) The idea of the proof (b) is due to Kiguradze [5] (for the nth-order differential equation); see [7], too.

(b) In [7] sufficient conditions for equation (1) to have Property A are given. For example, (1) has Property A if (5), (6), $\lambda = 1$,

$$\limsup_{t\to\infty} \frac{I^{n-i}(t, a_{n-1}, \dots, a_i)}{I^{n-i-1}(t, a_{n-1}, \dots, a_{i+1})} \times$$
$$\times \int_t^\infty \frac{I^{n-i-1}(s, a_{n-1}, \dots, a_{i+1}) I^i(s, a_1, \dots, a_i))}{I^1(s, a_i)} a_n(s)\, ds > 1 \tag{18}$$

for $i = 1, 2, \dots, n-1$, $2|(i+n)$ and $\int_0^\infty I^{n-1}(t, a_{n-1}, \dots, a_1) a_n(t)\, dt = \infty$ holds.

It is evident that if (1) has Property A then solutions of Type II (i), $i = 0, 1, \dots, n-1$, do not exist. Condition (12) is the same as (19) for $i = n-1$. Assumptions of Lemma 6 are weaker see the following example. A similar situation exists for $0 < \lambda < 1$. Moreover, in [7] an extra assumption is made in this case.

Example. Consider equation (1) with (5) where $n = 6$, $a_0 = a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = \frac{1}{t+1}$, $a_6 = \frac{1}{(t+1)^5}$. Then condition (11) is true, but (19) is not true for $i = 3$. Thus solutions of Type II(i), $i = 0, 1, \dots, 5$, do not exist; at the same time the above results of (5) do not guarantee Property A for (1).

Lemma 7. *Let* (6) *hold and functions* $a_n \in L_{loc}(\mathbb{R}_+)$, $g \in C^0(\mathbb{R}_+)$ *exist such that* $g(0) = 0$, $g(x) > 0$ *for* $x > 0$, g *is nondecreasing,* $\int_0^\infty a_n(t)\, dt = \infty$, *and*

$$a_n(t) g(|x_1|) \le |f(t, x_1, \dots, x_n)| \quad \text{in} \quad \mathbb{D}\,.$$

Then there exists no solution of (1) *of Type* II *(i),* $i = 0, 1, \dots, n-1$.

Proof. According to [3] and (6) no solution of Type II (i), $i = 0, 1, \dots, n-2$, exists. Let y be a solution of (1) of Type II $(n-1)$. Then according to Lemma 1 $|L_{n-1}y|$ is nonincreasing and

$$\infty > |L_{n-1}y(\infty) - L_{n-1}y(T)| = \int_T^\infty |L_n y(s)|\, ds \geqq$$

$$\geqq \int_T^\infty a_n(t) g(|L_0 y(s)|)\, ds \geq g(|L_0 y(T)|) \int_T^\infty a_n(s)\, ds = \infty\,.$$

The contradiction proves the lemma. $\square$

Lemma 8. *Let* (8) *be valid. Then there exists no singular solution of* (1) *of the first kind.*

Proof. Let on the contrary a solution y of (1) of the first kind exist. Then numbers $\tau, \tau_1 \in \mathbb{R}_+$, $\tau_1 < \tau$, exist such that

$$\begin{gathered} \varrho(\tau_1) > 0, L_i y \equiv 0 \quad \text{on} \quad [\tau, \infty),\ i = 0, 1, \dots, n-1, \\ \text{where} \quad \varrho(t) = \sum_{i=0}^{n-1} |L_i y(t)|\,. \end{gathered} \tag{19}$$

Then by the use of (2) and (8)

$$|L_i y(t)| \leq \int_t^\tau a_{i+1}(s) |L_{i+1} y(s)|\, ds, \quad i = 0, 1, \dots, n-2\,,$$

$$|L_{n-1} y(t)| \leq \int_t^\tau |L_n y(s)|\, ds\,,$$

$$|L_i y(t)| \leqq$$

$$\leqq \int_t^\tau a_{i+1}(s_{i+1}) \int_{s_{i+1}}^\tau a_{i+2} \cdots \int_{s_{n-2}}^\tau a_{n-1}(s_{n-1}) \int_{s_{n-1}}^\tau |L_n y(s_n)|\, ds_n \dots ds_{i+1} \leq$$

$$\leq \Big[\prod_{j=i+1}^{n-1} \int_{\tau_1}^\tau a_j(s)\, ds \Big] \int_t^\tau |L_n y(s)|\, ds, \quad i = 0, 1, \dots, n-2\,,$$

$$\varrho(t) \leq C \int_t^\tau |L_n y(s)|\, ds \leq C \int_t^\tau A(s) g(\varrho(s))\, ds\,, \quad t \in [\tau_1, \tau],$$

where

$$C = \sum_{i=0}^{n-2} \prod_{j=i+1}^{n-1} \int_{\tau_1}^{\tau} a_j(s)\, ds + 1\,.$$

Then it follows from Lemma 3 that

$$\int_0^{\varrho(\tau_1)} \frac{ds}{g(s)} \le C \int_{\tau_1}^{\tau} A(s)\, ds < \infty,$$

which contradicts (8) and (19). □

Lemma 9. *Let y be a solution of* (1) *defined in $\mathbb{R}_+$ that satisfies the initial conditions* (10). *Let* (9) *be valid. Then y is not of Type I(τ) for $\tau < \infty$.*

Proof. For $n = 3, 4$ the statement follows from [8] and [9]. Let $n > 4$. Let on the contrary a solution y of Type I(τ), $\tau < \infty$ exist. It follows from the assumptions of the lemma that an interval $\Lambda = [\tau_1, \tau]$, $\tau_1 < \tau$, exists, for which we have

$$\frac{\max\limits_{t\in\Lambda} a_e \cdot \max\limits_{t\in\Lambda} a_{e+1}}{\min\limits_{t\in\Lambda} a_e \cdot \min\limits_{t\in\Lambda} a_{e+1}} \le \frac{5}{4}\,, a_{e+1}(t)a_{e+2}(t) + [a'_{e+1}(t)]_- \int_\Lambda a_{e+2}(s)\, ds > 0\,,$$
$$a_{e+2}(t)a_{e+3}(t) + [a'_{e+2}(t)]_- \int_\Lambda a_{e+3}(s)\, ds > 0, \tag{20}$$

where $[g(t)]_- = \min(0, g(t))$.

Use the same notation as in the definition of Type I(τ).
According to $\lim_{t\to\tau} L_e y(t) = 0$ there exists $k_0 \in \mathbb{N}$ such that

$$|L_e y(t^{e+1}_{k_0})| > |L_e y(t^{e+1}_{k_0+1})| > 0\,, t^{e+1}_{k_0} > \tau_1\,. \tag{21}$$

Denote $t^{e+1}_{k_0} = t_1$, $t^{e}_{k_0} = t_2$, $t^{e-1}_{k_0} = t_3$, $\Lambda_1 = t_2 - t_1$, $\Lambda_2 = t_3 - t_2$. Then it follows from (21) and from the definition of Type I(τ) that (we choose

$L_{e-1}(t_2) > 0$ for simplicity)

$$\begin{aligned} & L_{e-1}y > 0 \text{ in } [t_1, t_3),\ L_{e-1}y(t_3) = 0\,, \\ & L_{e-1} \text{ is increasing (decreasing) in } [t_1, t_2] (\text{ in } [t_2, t_3])\,, \\ & L_e y > 0 \text{ in } [t_1, t_2)\,,\ L_e y(t_2) = 0\,,\ L_e y < 0 \text{ in } (t_2, t_3]\,, \\ & L_e \text{ is decreasing in } [t_0, t_3]\,, \\ & L_{e+1}y(t_1) = 0\,,\ L_{e+1}y < 0 \text{ in } (t_1, t_3], \\ & L_{e+1}y \text{ is decreasing in } [t_0, t_3] \\ & L_{e+j}y < 0 \text{ and } L_{e+j}y \text{ is decreasing in } [t_0, t_3],\ j = 2, 3\,. \end{aligned} \tag{22}$$

From this and (21), (22)

$$L_e y(t_1) > |L_e y(t_3)|, \tag{23}$$

$$L_{e+1}y(t) = \int_{t_1}^{t} a_{e+2}(s) L_{e+2}y(s)\, ds \ge L_{e+2}y(t) \int_{\Lambda} a_{e+2}(s)\, ds\,,\quad t \in [t_1, t_3],$$

$$\begin{gathered} [L_e y(t)]'' = [a_{e+1} L_{e+1} y(t)]' = a_{e+1}(t) a_{e+2}(t) L_{e+2} y(t) + \\ + a'_{e+1}(t) L_{e+1} y(t) \le a_{e+1}(t) a_{e+2}(t) L_{e+2} y(t) + \\ + [a'_{e+2}(t)]_- L_{e+1} y(t) \le L_{e+2} y(t) [a_{e+1}(t) a_{e+2}(t) + [a'_{e+2}(t)]_- \\ - \int_{\Lambda} a_{e+2}(s)\, ds] < 0\,,\ t \in [t_1, t_3]\,. \end{gathered} \tag{24}$$

Thus

$$L_e y \text{ is concave in } [t_1, t_3]\,. \tag{25}$$

We can prove similarly that

$$L_{e+1} y \text{ is concave in } [t_1, t_3]\,. \tag{26}$$

Further, by the use of (23), (25)

$$L_{e-1}y(t_2) = \int_{t_2}^{t_3} a_e(s) |L_e y(s)|\, ds \le \max_{s \in \Lambda} a_e(s) |L_e y(t_3)| \frac{\Lambda_2}{2}\,,$$

$$L_{e-1}y(t_2) \geqq L_{e-1}y(t_2) - L_{e-1}y(t_1) = \int_{t_1}^{t_2} a_e(s) L_e y(s)\, ds \ge$$

$$\ge \min_{s \in \Lambda} a_e(s) L_e y(t_1) \frac{\Lambda_1}{2}\,.$$

Thus, according to (24)

$$1 \leq \frac{|L_e y(t_3)|}{L_e y(t_1)} \frac{\max\limits_{s\in\Lambda} a_e(s)}{\min\limits_{s\in\Lambda} a_e(s)} \frac{\Lambda_2}{\Lambda_1} < \frac{\max\limits_{s\in\Lambda} a_e(s)}{\min\limits_{s\in\Lambda} a_e(s)} \frac{\Lambda_2}{\Lambda_1}. \tag{27}$$

According to (23), (26)

$$L_e(t_1) = \int_{t_1}^{t_2} a_{e+1}(s)|L_{e+1}y(s)|\,ds \leqq |L_{e+1}y(t_2)| \frac{\Lambda_1}{2} \max_{s\in\Lambda} a_{e+1}(s)\,,$$

$$|L_e(t_3)| = \int_{t_2}^{t_3} a_{e+1}(s)|l_{e+1}y(s)|\,ds \geqq |L_{e+1}y(t_2)|\Lambda_2 \min_{s\in\Lambda} a_{e+1}(s)\,.$$

Thus, according to (24), (27) and (23)

$$1 < \frac{\Lambda_1}{2\Lambda_2} \frac{\max\limits_{s\in\Lambda} a_{e+1}(s)}{\min\limits_{s\in\Lambda} a_{e+1}(s)} \leq \frac{1}{2} \frac{\max\limits_{s\in\Lambda} a_{e+1}(s) \max\limits_{s\in\Lambda} a_e(s)}{\min\limits_{s\in\Lambda} a_{e+1}(s) \min\limits_{s\in\Lambda} a_e(s)} \leq \frac{5}{8}.$$

The contradiction proves the statement of the lemma. □

Proof of Theorem 1. According to Lemmas 2, 6, and 9 y is of Type I(∞) and by the use of Lemma 4 it is proper. □

Proof of Theorem 2. The statement is a consequence of Lemmas 2, 4, 6, 8, and 9. □

Proof of Theorem 3. It follows from Lemmas 4, 8, and 9 that y is proper and according to Lemma 7 it is of Type I(∞). □

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