GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 5, 1996, 423-446

CRITERIA OF STRONG TYPE TWO-WEIGHTED INEQUALITIES FOR FRACTIONAL MAXIMAL FUNCTIONS

A. GOGATISHVILI AND V. KOKILASHVILI

Abstract. A strong type two-weight problem is solved for fractional maximal functions defined in homogeneous type general spaces. A similar problem is also solved for one-sided fractional maximal functions.

1. Introduction

The solution of a two-weight problem for maximal functions in Euclidean spaces was obtained for the first time in [1]. This paper gives a new complete description of pairs of weight functions which provides the validity of strong type two-weighted inequalities for fractional maximal functions defined in homogeneous type general spaces. A similar result was obtained in [2] for homogeneous type spaces having group structure. It should be noted that condition (1.2) below, which has turned out to be a criterion of strong type two-weight estimates for fractional maximal functions, appeared for the first time in [3], [4].

In this paper we also solve a strong type two-weight problem for one-sided maximal functions of fractional order on the real axis.

The homogeneous type space (X, d, μ) is a topological space with a complete measure μ such that compactly supported functions are dense in the space $L^1(X, \mu)$. Moreover, it is assumed that there is a nonnegative real-valued function $d : X \times X \to R^1$ satisfying the following conditions:

(i) $d(x, x) = 0$ for all $x \in X$;

(ii) $d(x, y) > 0$ for all $x \neq y$ in X;

(iii) there is a constant a_0 such that $d(x, y) \leq a_0 d(y, x)$ for all x, y in X;

(iv) there is a constant a_1 such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for all x, y and z in X;

1991 *Mathematics Subject Classification.* 42B20, 42B25.

Key words and phrases. Homogeneous type space, fractional maximal function, strong type two-weighted inequality, one-sided fractional maximal functions.

(v) for each neighborhood V of x in X there is $r > 0$ such that the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is contained in X.

The balls $B(x, r)$ are measurable for all x and $r > 0$.

There is a constant b such that $\mu B(x, 2r) \le b\mu B(x, r)$ for all nonempty $B(x, r)$ (see [5]).

For a locally summable function $f : X \to R^1$ the fractional maximal function is defined as follows:

$$M_\gamma(f)(x) = \sup(\mu B)^{\gamma-1} \int\limits_B |f(y)| d\mu, \quad 0 < \gamma < 1, \tag{1.0}$$

where the supremum is taken with respect to all balls B of positive measure containing the point x.

A measurable function $w : X \to R^1$, which is positive almost everywhere and locally summable, is called a weight function (weight). For a μ-measurable set E we put

$$wE = \int\limits_E w(x) d\mu.$$

Main Theorem. *Let $1 < p < q < \infty$, $0 < \gamma < 1$, v and w be the weight functions. For a constant $c > 0$ to exist so that the inequality*

$$\left(\int\limits_X \big(M_\gamma(f)(x)\big)^q v(x) d\mu \right)^{\frac{1}{q}} \le c \left(\int\limits_X |f(x)|^p w(x) d\mu \right)^{\frac{1}{p}} \tag{1.1}$$

would hold, it is necessary and sufficient that the condition

$$\sup_{\substack{x \in X \\ r > 0}} \big(w^{1-p'} B(x, 2N_0 r)\big)^{\frac{1}{p'}} \times$$

$$\times \left(\int\limits_{X \setminus B(x,r)} v(y) \big(\mu B(x, d(x, y))\big)^{(\gamma-1)q} d\mu \right)^{\frac{1}{q}} < \infty \tag{1.2}$$

be fulfilled.

This theorem was announced in [6].

2. Preliminary Results

We begin this section by generalizing the results of [1] for homogeneous type general spaces. After that we give the proof of the main theorem which is based on Theorem 2.1 to be proved below and our results obtained previously [7] (see also [8]) for fractional integrals defined in homogeneous type general spaces.

First we give a familiar covering lemma which is valid for arbitrary spaces with a quasi-metric.

Lemma A ([5]). *Let E be a bounded set in X and, for each point $x \in E$ let a ball $B_x = B(x, r_x)$ be given such that $\sup_{x\in X} \operatorname{rad} B_x < \infty$. Then from the family $\{B_x\}_{x\in E}$ we can choose a (finite or infinite) sequence of pairwise disjoint balls $(B_j)_j$ for which $E \subset \cup_{j\geq 1} N_0 B_j$, $N_0 = a_1(1+2a_0)$, and for each $B_x \in \{B_x\}_{x\in E}$ there exists a ball B_{j_0} such that $x \in N_0 B_{j_0}$ and* $\operatorname{rad} B_x \leq 2 \operatorname{rad} B_{j_0}$.

We have

Theorem 2.1. *Let $0 \leq \gamma < 1$, $1 < p \leq q < \infty$. Then the following two conditions are equivalent:*

(i) *there exists a constant $c_1 > 0$ such that*

$$\bigg(\int_X \big(M_\gamma(f)(x)\big)^q v(x)d\mu\bigg)^{\frac{1}{q}} \leq c_1 \bigg(\int_X |f(x)|^p w(x)d\mu\bigg)^{\frac{1}{p}} \tag{2.1}$$

for any $f \in L^p_w(X,\mu)$;

(ii) *there exists a constant $c_2 > 0$ such that*

$$\bigg(\int_B \Big(M_\gamma(\chi_B w^{1-p'})(x)\Big)^q v(x)d\mu\bigg)^{\frac{1}{q}} \leq c_2 \bigg(\int_B w^{1-p'}(x)d\mu\bigg)^{\frac{1}{p}} \tag{2.2}$$

for any ball $B \subset X$.

Proof. On substituting $f = \chi_B w^{1-p'}$ in (2.1), we obtain (2.2). Therefore the implication $(i) \Rightarrow (ii)$ is fulfilled. We will prove $(ii) \Rightarrow (i)$. Let $N = a_1(1+2a_1(1+a_0))$ and $N_1 = a_1(1+a_1^2(1+a_0)(N+a_0))$. Assume $b > 1$ to be a constant such that

$$\mu(N_1 B) \leq b\mu B$$

for an arbitrary ball B.

Let further B_0 be an arbitrarily fixed ball and f be an arbitrary integrable function which is positive almost everywhere and satisfies the condition $\operatorname{supp} f \subset B_0$.

We set

$$\Omega_k = \big\{x \in X : \ M_\gamma(f)(x) > b^k\big\}, \quad k \in \mathbb{Z}.$$

Obviously, for each $x \in \Omega_k$ there exists a ball $B(y,r) \ni x$ such that

$$\frac{1}{(\mu B(y,r))^{1-\gamma}} \int_{B(y,r)} f(z)d\mu > b^k.$$

The set of radius lengths of such balls will be bounded by virtue of the fact that $\operatorname{supp} f \subset B_0$.

Consider the values

$$\mathcal{R}_x^k = \sup\Big\{r:\ \exists B(y,r)\ni x,\ \frac{1}{(\mu B(y,r))^{1-\gamma}}\int\limits_{B(y,r)} f(z)d\mu > b^k\Big\}.$$

Obviously, for arbitrary $x\in\Omega_k$ there exist $y_x\in X$ and $r_x>\frac{\mathcal{R}_x^k}{2}$ such that

$$\frac{1}{(\mu B(y_x,r_x))^{1-\gamma}}\int\limits_{B(y_x,r_x)} f(z)d\mu > b^k.$$

Along with this, for each ball B' which contains the point x and for which $\operatorname{rad} B'\ge 2r_x$ we have

$$\frac{1}{(\mu B')^{1-\gamma}}\int\limits_{B'} f(z)d\mu \le b^k. \tag{2.3}$$

By Lemma A for each k there exists a sequence $(B_j^k)_{j\ge1}$ of nonintersecting balls such that $\cup_{j\ge1}NB_j^k\supset\Omega_k$ and

$$\frac{1}{(\mu B_j^k)^{1-\gamma}}\int\limits_{B_j^k} f(z)d\mu > b^k.$$

Let us show that if $B_j^k\cap B_i^n\ne\varnothing$ and $n>k$, then

$$NB_i^n\subset NB_j^k. \tag{2.4}$$

To this end it will first be shown that if $r_i=\operatorname{rad} B_i$, $r_j=\operatorname{rad} B_j$, and $n>k$, then

$$r_i^n<\frac{r_j^k}{a_1(a_0+N)}\,.$$

Assume the opposite, i.e., $r_j^k\le a_1(a_0+N)r_i^n$. Then for $y\in B_j^k$ and $x\in B_j^k\cap B_i^n$ we will have

$$\begin{aligned}
d(x_i^n,y) &\le a_1\big(d(x_i^n,x)+d(x,y)\big)\le\\
&\le a_1\Big(r_i^n+a_1\big(a_0d(x_j^k,x)+d(x_j^k,y)\big)\Big)<a_1\big(r_i^n+a_1(1+a_0)r_j^k\big)\le\\
&\le a_1\big(1+a_1^2(1+a_0)(N+a_0)\big)r_i^n=N_1r_i^n.
\end{aligned}$$

where x_j^k are the centers of B_j^k. Therefore $B_j^k\subset N_1B_i^n$. Along with this, $2r_j^k<N_1r_i^n$. Therefore by virtue of (2.3) we obtain

$$b^n<\frac{1}{(\mu B_i^n)^{1-\gamma}}\int\limits_{B_i^n} f(z)d\mu\le\frac{b}{\mu(N_1B_i^n)^{1-\gamma}}\int\limits_{N_1B_i^n} f(z)d\mu\le b^{k+1}.$$

Thus $n \le k$, which leads to a contradiction. Therefore

$$r_j^k > a_1(a_0+N)r_i^n.$$

Now for $x \in NB_i^n$ and $y \in B_i^n \cap B_j^k$ we derive

$$\begin{aligned} d(x_j^k, x) &\le a_1\Big(d(x_j^k, y) + a_1\big(a_0 d(x_i^n, y) + d(x_i^n, x)\big)\Big) \le \\ &\le a_1\big(r_j^k + a_1(a_0+N)r_i^n\big) \le 2a_1 r_j^k < N r_j^k. \end{aligned}$$

Thus $NB_i^n \subset NB_j^k$ provided that $B_j^k \cap B_i^n \ne \varnothing$ and $n > k$.

We introduce the sets

$$E_j^k = \Big(NB_j^k \backslash \bigcup_{i=1}^{j-1} NB_i^k\Big) \cap \big(\Omega_k \backslash \Omega_{k+1}\big), \quad k \in \mathbb{Z}, \ \ j \in \mathbb{N}.$$

As is easy to verify,

$$\bigcup_{j=1}^{\infty} E_j^k = \Omega_k \backslash \Omega_{k+1} \quad \text{and} \quad E_j^k \cap E_i^n = \varnothing \ \text{ for } \ k \ne n \ \text{ or } \ i \ne j.$$

Therefore we have

$$\begin{aligned} \int\limits_X \big(M_\gamma(f)(x)\big)^q v(x)d\mu &\le \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} b^{(k+1)q} vE_j^k \le \\ &\le b^q \sum_{j,k} \bigg(\frac{1}{(\mu B_j^k)^{1-\gamma}} \int\limits_{B_j^k} f(z)d\mu\bigg)^q vE_j^k = \\ &= c\sum_{j,k} \bigg(\frac{\sigma(NB_j^k)}{(\mu B_j^k)^{1-\gamma}}\bigg)^q vE_j^k \bigg(\frac{1}{\sigma(NB_j^k)} \int\limits_{B_j^k} \frac{f(z)}{\sigma(z)}\sigma(z)d\mu\bigg)^q, \end{aligned} \tag{2.5}$$

where $\sigma = w^{1-p'}$.

Let now ν be a discrete measure given on $\mathbb{Z}^2$ by

$$\nu\big\{(j,k)\big\} = \Big(\frac{\sigma(NB_j^k)}{(\mu B_j^k)^{1-\gamma}}\Big)^q vE_j^k.$$

Define the operator T acting from $L^1_{loc}(X, d\mu)$ into the set of ν-summable functions defined on $\mathbb{Z}^2$ as follows:

$$T(g)(j,k) = \frac{1}{\sigma(NB_j^k)} \int\limits_{B_j^k} |g(z)|\sigma(z)d\mu.$$

Equality (2.5) can now be rewritten as

$$\int\limits_X \big(M_\gamma(f)(x)\big)^q \nu(x)d\mu \le c \int\limits_{\mathbb{Z}^2} \Big(T\Big(\frac{f}{\sigma}\Big)(z)\Big)^q d\nu. \tag{2.6}$$

Our aim is to prove that the operator T has a strong type (p,q). It is obvious that the operator T has a strong type (∞,∞). Now let us show that T is the operator of a weak type $(1,\frac{q}{p})$, i.e.,

$$\begin{aligned}\nu\Gamma(\lambda) = \nu\big\{(j,k)\in\mathbb{Z}^2 :\ T(g)(j,k) > \lambda\big\} \le \\ c\lambda^{-\frac{q}{p}}\bigg(\int\limits_X |g(z)|^p\sigma(z)d\mu\bigg)^{\frac{q}{p}}\end{aligned} \tag{2.7}$$

for any $\lambda>0$ and $g\in L^1(X,\sigma d\mu)$.

We fix some $k_0\in\mathbb{Z}$ and first show that

$$\begin{aligned}\lambda^{\frac{q}{p}}\nu\Gamma_{k_0}(\lambda) = \nu\big\{(j,k)\in\mathbb{Z}\times\mathbb{Z}_{k_0} :\ T(g)(j,k) > \lambda\big\} \le \\ \le \bigg(\int\limits_X |g(z)|^p\sigma(z)d\mu\bigg)^{\frac{1}{p}},\end{aligned} \tag{2.8}$$

where $\mathbb{Z}_{k_0} = \{k_0, k_0+1,\dots\}$.

Now fix $\lambda>0$ and consider the system of balls $(B_j^k)_{(j,k)\in\Gamma_{k_0}(\lambda)}$. Choose from the latter system a subsystem of nonintersecting balls in the following manner: take all balls of "rank" k_0, i.e., all balls $B_j^{k_0}$, $j=1,2,\dots$. Pass to "rank" k_0+1. If some $B_i^{k_0+1}$ intersects with none of $B_j^{k_0}$, then include it in the subsystem and otherwise discard. Next compare the balls of rank k_0+2 with the ones already chosen in the above-described manner and so on. We obtain thus the sequence of nonintersecting balls $\{B_i\}_i$. According to (2.4) if $B_j^k\in\{B_i\}_i$, then $NB_j^k\subset NB_{i_0}$ for some $i_0\ge 1$ and therefore

$$\bigcup_{i=1}^{\infty} NB_i = \bigcup_{(j,k)\in\Gamma_{k_0}(\lambda)} NB_j^k.$$

Hence we obtain

$$\begin{aligned}\lambda^{\frac{q}{p}}\nu\Gamma_{k_0}(\lambda) &= \lambda^{\frac{q}{p}} \sum_{(j,k)\in\Gamma_{k_0}(\lambda)} \Big(\frac{\sigma(NB_j^k)}{(\mu B_j^k)^{1-\gamma}}\Big)^q vE_j^k \le \\ &\le c\lambda^{\frac{q}{p}} \sum_{i=1}^{\infty} \sum_{NB_j^k\subset NB_i} \Big(\frac{\sigma(NB_j^k)}{(\mu(NB_j^k))^{1-\gamma}}\Big)^q vE_j^k \le\end{aligned}$$

$$\leq c\lambda^{\frac{q}{p}}\sum_{i=1}^{\infty}\sum_{NB_j^k\subset NB_i}\int\limits_{E_j^k}\big(M_\gamma(\chi_{_{NB_i}}\sigma)(z)\big)^q v(z)d\mu\leq$$

$$\leq c\lambda^{\frac{q}{p}}\sum_{i=1}^{\infty}\int\limits_{NB_i}\big(M_\gamma(\chi_{_{NB_i}}\sigma)(z)\big)^q v(z)d\mu\leq c\lambda^{\frac{q}{p}}\sum_{i=1}^{\infty}\big(\sigma(NB_i)\big)^{\frac{q}{p}}\leq$$

$$\leq c\sum_{i=1}^{\infty}\big(\sigma(NB_i)\big)^{\frac{q}{p}}\bigg(\frac{1}{\sigma(NB_i)}\int\limits_{B_i}|g(z)|\sigma(z)d\mu\bigg)^{\frac{q}{p}}=$$

$$=c\sum_{i=1}^{\infty}\bigg(\int\limits_{B_i}|g(z)|\sigma(z)d\mu\bigg)^{\frac{q}{p}}\leq c\bigg(\sum_{i=1}^{\infty}\int\limits_{B_i}|g(z)|\sigma(z)d\mu\bigg)^{\frac{q}{p}}\leq$$

$$\leq c\bigg(\int\limits_{X}|g(z)|\sigma(z)d\mu\bigg)^{\frac{q}{p}}.$$

Thus we have proved (2.8) where the constant does not depend on k_0. Making now k_0 tend to $-\infty$, we obtain (2.7). We have shown that the operator T has a weak type $(1,\frac{q}{p})$.

Since the operator T has a weak type $(1,\frac{q}{p})$ and a strong type (∞,∞), by the Marcinkiewicz interpolation theorem we conclude that T has a strong type (p,q). Then (3.6) implies

$$\int\limits_X\big(M_\gamma(f)(x)\big)^q v(x)d\mu\leq c\int\limits_{\mathbb{Z}^2}\Big(T\Big(\frac{f}{\sigma}\Big)(x)\Big)^q d\nu\leq$$

$$\leq c_1\bigg(\int\limits_X\Big(\frac{f}{\sigma}\Big)^p(x)\sigma(x)d\mu\bigg)^{\frac{q}{p}}=c_1\bigg(\int\limits_X f^p(x)w(x)d\mu\bigg)^{\frac{q}{p}}.$$

Let now f be an arbitrary function from $L^p(X,wd\mu)$. By virtue of the foregoing arguments, for an arbitrary ball B_0 we will have

$$\bigg(\int\limits_X\big(M_\gamma(\chi_{B_0}f)\big)^q v(x)d\mu\bigg)^{\frac{1}{q}}\leq$$

$$\leq c\bigg(\int\limits_{B_0}f^p(x)w(x)d\mu\bigg)^{\frac{1}{p}}\leq c\bigg(\int\limits_X f^p(x)w(x)d\mu\bigg)^{\frac{1}{p}}.$$

Making $\operatorname{rad}B_0$ tend to infinity, by the Fatou lemma we obtain (2.1). $\square$

Next we will give two theorems which are proved in [7]. They concern two-weight estimates of integral transforms with a positive kernel, in particular, analogs of fractional integrals defined in spaces with a quasi-metric and measure.

Consider the integral operators

$$\mathcal{K}(f)(x)=\int_X k(x,y)f(y)dy$$
$$\mathcal{K}^*(f)(x)=\int_X k(y,x)f(y)dy.$$

Theorem A. *Let* $1<p<q<\infty$*,* $k:X\times X\to R^1$ *be an arbitrary positive measurable kernel;* v *and* μ *be arbitrary finite measures on* X *so that* $\mu\{x\}=0$ *for any* $x\in X$*. If the condition*

$$c_0=\sup_{\substack{x\in X\\ r>0}}\big(vB(x,2N_0r)\big)^{\frac{1}{q}}\bigg(\int\limits_{X\setminus B(x,r)}k^{p'}(x,y)w^{1-p'}(y)d\mu\bigg)^{\frac{1}{p'}}<\infty,$$

where $N_0=a_1(1+2a_0)$ *and the constants* a_0 *and* a_1 *are from the definition of a quasi-metric, is fulfilled, then there exists the constant* $c>0$ *such that the inequality*

$$v\big\{x\in X:\ \mathcal{K}(f)(x)>\lambda\big\}\le c\lambda^{-q}\bigg(\int_X|f(x)|^p d\mu\bigg)^{\frac{q}{p}}$$

holds for any μ*-measurable nonnegative function* $f:X\to R^1$ *and arbitrary* $\lambda>0$*.*

Definition 2.1. A positive measurable kernel $k:X\times X\to R^1$ will be said to satisfy condition (V) $(k\in V)$ if there exists a constant $c>0$ such that

$$k(x,y)<ck(x',y)$$

for arbitrary x, y, and x' from X which satisfy the condition $d(x,x')<Nd(x,y)$, where $N=2N_0$.

Theorem B. *Let* $1<p<q<\infty$*,* μ *be an arbitrary locally finite measure,* w *be a weight, and* $k\in V$*. Then the following conditiona are equivalent:*

(i) *there exists a constant* $c_1>0$ *such that the inequality*

$$w^{1-p'}\big\{x\in X:\ \mathcal{K}(f)(x)>\lambda\big\}\le c_1\lambda^{-q}\bigg(\int_X|f(x)|^p v^{\frac{1}{1-q}}(x)d\mu\bigg)^{\frac{p'}{q'}}$$

holds for arbitrary $\lambda>0$ *and nonnegative* $f\in L^p(X,wd\mu)$*;*

(ii) *there exists a constant $c_2 > 0$ such that*

$$\bigg(\int\limits_X \big(\mathcal{K}^*(\chi_B w^{1-p'})(x)\big)^q v(x)d\mu\bigg)^{\frac{1}{q}} \le c_2\bigg(\int\limits_B w^{1-p'}(y)d\mu\bigg)^{\frac{1}{p}}$$

for an arbitrary ball $B \subset X$;

(iii)

$$\sup_{\substack{x\in X\\ r>0}} \big(w^{1-p'}B(x,2N_0r)\big)^{\frac{1}{p'}}\bigg(\int\limits_{X\setminus B(x,r)} k^q(x,y)v(y)d\mu\bigg)^{\frac{1}{q}} < \infty.$$

3. Proof of the Main Theorem

Using the results of the preceding sections we will prove the main theorem of this paper.

Proof. Our aim is to show that the implication (1.1) $\Leftrightarrow$ (1.2) is valid. First we will prove the implication (1.2) $\Rightarrow$ (1.1). Consider an operator given on $L^{q'}(X, v^{\frac{1}{1-q}})$ in the form

$$T_\gamma(f)(x) = \int\limits_X \frac{|f(y)|}{(\mu B(x,d(x,y)))^{1-\gamma}}\,d\mu.$$

The latter operator is an analog of the Riesz potential for homogeneous type spaces.

Using Theorem A, from condition (1.2) we conclude that the weak type inequality

$$w^{1-p'}\big\{x\in X:\ T_\gamma(f)(x) > \lambda\big\} \le c_4\lambda^{-p'}\bigg(\int\limits_X |f(x)|^{q'} v^{\frac{1}{1-q}}(x)d\mu\bigg)^{\frac{p'}{q'}} \tag{3.1}$$

with the constant not depending on $\lambda > 0$ and f is valid.

Further, by virtue of Theorem B the latter inequality implies that there exists a constant $c_2 > 0$ such that for any ball $B \subset X$ we have

$$\bigg(\int\limits_X \big(T^*_\gamma(\chi_B w^{1-p'})(x)\big)^q v(x)d\mu\bigg)^{\frac{1}{q}} \le c_2\bigg(\int\limits_B w^{1-p'}(x)d\mu\bigg)^{\frac{1}{p'}}, \tag{3.2}$$

where

$$T^*_\gamma(\varphi)(x) = \int\limits_X \frac{|\varphi(y)|}{(\mu B(y,d(x,y)))^{1-\gamma}}\,d\mu.$$

On the other hand, there exist constants $c_3 > 0$ and $c_4 > 0$ such that

$$c_3 \mu B\big(y, d(x,y)\big) \le \mu B\big(x, d(x,y)\big) \le c_4 \mu B\big(y, d(x,y)\big). \tag{3.3}$$

The latter follows from the fact that $B(x, d(x,y)) \subset a_1(a_0+1)d(x,y)$. Indeed, let $d(x,z) \le d(x,y)$. Then

$$d(y,z) \le a_1\big(d(y,x) + d(x,z)\big) \le a_1(a_0+1)d(x,y).$$

By virtue of the doubling property for measure we obtain

$$\mu B\big(y, a_1(a_0+1)d(x,y)\big) \le c_5 \mu B\big(y, d(x,y)\big).$$

Hence we conclude that (3.3) is valid. Next, from (3.3) and (3.2) we derive

$$\bigg(\int\limits_X \big(T_\gamma(\chi_B w^{1-p'})(x)\big)^q v(x) d\mu\bigg)^{\frac{1}{q}} \le c_6 \bigg(\int\limits_B w^{1-p'}(x) d\mu\bigg)^{\frac{1}{p}}. \tag{3.4}$$

Now we will show that for an arbitrary nonnegative measurable function φ we have

$$M_\gamma(\varphi)(x) \le c_7 T_\gamma(\varphi)(x), \tag{3.5}$$

where the constant c_7 does not depend on φ and x.

First we will show that for any x there exists a ball $B_x = B(x, r_x)$ such that

$$M_\gamma(\varphi)(x) \le \frac{c_8}{(\mu B_x)^{1-\gamma}} \int\limits_B \varphi(z) d\mu, \tag{3.6}$$

where the positive constant c_8 does not depend on φ and B_x. Indeed, there exists a ball $B(y,r)$ such that $x \in B(y,r)$ and

$$M_\gamma(\varphi)(x) \le \frac{2}{(\mu B(y,r))^{1-\gamma}} \int\limits_{B(y,r)} \varphi(z) d\mu. \tag{3.7}$$

Assuming now that $z \in B(y,r)$, we obtain

$$d(x,z) \le a_1\big(d(x,y) + d(y,z)\big) \le a_1(a_0+1)r.$$

Therefore $B(y,r) \subset B(x, a_1(1+a_0)r)$. On the other hand, we have $B(x, a_1(1+a_0)r) \subset B(y, a_1(1+a_1(1+a_0))r)$ since

$$d(y,z) \le a_1\big(d(y,x) + d(x,z)\big) \le a_1\big(r + a_1(1+a_0 r)\big) = a_1\big(1 + a_1(1+a_0)\big)r$$

for any $z \in B(x, a_1(1+a_0)r)$.

Now applying the doubling condition, from (3.7) we find that

$$M_\gamma(\varphi)(x) \le \frac{2}{(\mu B(y,r))^{1-\gamma}} \int\limits_{B(x,a_1(1+a_0)r)} \varphi(z)d\mu \le$$

$$\le \frac{c_8}{(\mu B(x,a_1(1+a_0)r))^{1-\gamma}} \int\limits_{B(x,a_1(1+a_0)r)} \varphi(z)d\mu.$$

Replacing r_x by the number $a_1(1+a_0)r$, we obtain (3.6).

Now we will prove (3.5). We have

$$T_\gamma(\varphi)(x) \ge \int\limits_{B(x,r_x)} \frac{\varphi(y)}{(\mu B(x,d(x,y)))^{1-\gamma}}\, d\mu \ge$$

$$\ge \frac{1}{(\mu B(x,r_x))^{1-\gamma}} \int\limits_{B(x,r_x)} \varphi(z)d\mu \ge \frac{1}{c_8} M_\gamma(\varphi)(x).$$

From (3.4) and (3.5) we obtain

$$\bigg(\int\limits_X \big(M_\gamma(\chi_{_B} w^{1-p'})(x)\big)^q v(x)d\mu\bigg)^{\frac{1}{q}} \le c_9 \bigg(\int\limits_B w^{1-p'}(x)d\mu\bigg)^{\frac{1}{p}}.$$

By Theorem 2.1 we conclude that inequality (1.1) is valid.

Thus we have proved the implication (1.2) $\Rightarrow$ (1.1).

Let us show the validity of the inverse implication (1.1) $\Rightarrow$ (1.2). Fix an arbitrary ball $B(x,r)$ and assume

$$f(y) = \chi_{_{NB(x,r)}} w^{1-p'}(y),$$

where N is an arbitrary number greater than unity. Obviously, by virtue of the doubling condition we have

$$M_\gamma(f)(y) \ge \frac{1}{\mu B(x,Nd(x,y))} \int\limits_{B(x,Nd(x,y))\cap NB(x,r)} w^{1-p'}(z)dz \ge$$

$$\ge \frac{c_{10}}{\mu B(x,d(x,y))} \int\limits_{NB(x,r)} w^{1-p'}(z)d\mu$$

for any $y \in X\backslash B(x,r)$.

Therefore (1.1) implies

$$\Big(\int\limits_{NB(x,r)} w^{1-p'}(z)d\mu\Big)\Big(\int\limits_{X\setminus B(x,r)} \frac{v(y)}{(\mu B(x,d(x,y))^{(1-\gamma)q}}\,d\mu\Big)^{\frac{1}{q}} \le$$

$$\le c\Big(\int\limits_{NB(x,r)} w^{1-p'}(z)d\mu\Big)^{\frac{1}{p}}.$$

From the latter inequality we obtain the validity of (1.2). □

Definition 3.1. Measure ν satisfies the reverse doubling condition ($\nu \in (RD)$) if there exist numbers δ and ε from $(0,1)$ such that

$$\nu B(x,r_1) \le \varepsilon \nu B(x,r_2)$$

for $\mu B(x,r_1) \le \delta \mu B(x,r_2)$, $0 < r_1 < r_2$.

In the particular case where measure $w^{1-p'}$ satisfies the reverse doubling condition, (1.2) in the main theorem can be replaced by a simpler condition.

Theorem 3.1. *Let* $1 < p < q < \infty$, $0 < \gamma < 1$ *and measure* $w^{1-p'}$ *satisfy the reverse doubling condition. Then* (1.1) *holds iff*

$$\sup_{\substack{x\in X\\ r>0}} \big(\mu B(x,r)\big)^{\gamma-1}\big(w^{1-p'}B(x,r)\big)^{\frac{1}{p'}}\big(vB(x,r)\big)^{\frac{1}{q}} < \infty. \tag{3.8}$$

Proof. The implication (1.1) ⇒ (3.8) is obtained immediately by substituting the function $f(y) = \chi_B w^{1-p'}(y)$ into (1.1).

By virtue of the main theorem to prove the implication (3.8) ⇒ (1.1) it is sufficient to show that if $w^{1-p'} \in (RD)$, then (3.8) ⇒ (1.2). Let $x \in X$ and $r > 0$ be fixed. Choose numbers r_k ($k = 0,1,\dots$) as follows:

$$r_0 = 2N_0 r \quad \text{and} \quad r_k = \inf\big\{r :\ \mu B(x,r_{k-1}) < \delta\mu B(x,r)\big\}.$$

Obviously,

$$\mu B(x,r_{k-1}) < \delta\mu B(x,r_k) < c\mu B(x,r_{k-1}).$$

Again applying the condition (RD), we obtain

$$w^{1-p'}B(x,r_0) \le \varepsilon^k w^{1-p'}B(x,r_k) \quad k = 1,2,\dots .$$

The latter inequalities imply

$$\big(w^{1-p'}B(x,2N_0 r)\big)^{\frac{1}{p'}}\Big(\int\limits_{X\setminus B(x,r)} v(y)\big(\mu B(x,d(x,y))\big)^{(\gamma-1)q}d\mu(y)\Big)^{\frac{1}{q}} \le$$

$$\leq \big(w^{1-p'}B(x,r_0)\big)^{\frac{1}{p'}} \sum_{k=1}^{\infty} \Bigg(\int\limits_{B(x,r_k)\setminus B(x,r_{k-1})} v(y)\big(\mu B(x,d(x,y))\big)^{(\gamma-1)q} d\mu(y)\Bigg)^{\frac{1}{q}} +$$

$$+\big(w^{1-p'}B(x,r_0)\big)^{\frac{1}{p'}} \Bigg(\int\limits_{B(x,r_0)\setminus B(x,r)} v(y)\big(\mu B(x,d(x,y))\big)^{(\gamma-1)q} d\mu(y)\Bigg)^{\frac{1}{q}} \leq$$

$$\leq \sum_{k=1}^{\infty} \varepsilon^{\frac{k}{p'}} \big(w^{1-p'}B(x,r_k)\big)^{\frac{1}{p'}} \big(vB(x,r_k)\big)^{\frac{1}{q}} \big(\mu B(x,r_{k-1})\big)^{\gamma-1} +$$

$$+\big(w^{1-p'}B(x,r_0)\big)^{\frac{1}{p'}} \big(vB(x,r_0)\big)^{\frac{1}{q}} \big(\mu B(x,r_0)\big)^{\gamma-1} \leq$$

$$\leq c \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{p'}} \big(\mu B(x,r_k)\big)^{\gamma-1} \big(w^{1-p'}B(x,r_k)\big)^{\frac{1}{p'}} \big(vB(x,r_k)\big)^{\frac{1}{q}} \leq$$

$$\leq c \sum_{k=0}^{\infty} \varepsilon^{\frac{k}{p'}} < \infty. \qquad \square$$

Remark. A similar result was obtained in [2] under the assumption that the measure $w^{1-p'}$ satisfies the doubling condition. Theorem 3.1 contains a stronger result, since condition (RD) is weaker than the doubling condition. (For instance, the function $e^{|x|} \in (RD)$ but it does not satisfy the doubling condition.) Along with this, the proof in [2] essentially differs from the above.

The above proofs also remains valid for fractional maximal functions:

$$M_\gamma(fd\sigma)(x) = \sup_{B\ni x} \frac{1}{(\mu B)^{1-\gamma}} \int\limits_B |f(y)| d\sigma(y), \tag{3.9}$$

where $0 \leq \gamma < 1$ and σ is a Borel measure while the supremum is taken over all balls $B \subset X$ with a positive measure containing x.

Theorem 3.2. *Let X be an arbitrary homogeneous type space, $1 < p < q < \infty$, $0 < \gamma < 1$, ω and σ be positive Borel measures. For a constant $c > 0$ to exist such that the inequality*

$$\Bigg(\int\limits_X \big(M_\gamma(fd\sigma)(x)\big)^q d\omega(x)\Bigg)^{\frac{1}{q}} \leq c \Bigg(\int\limits_X |f(x)|^p d\omega(x)\Bigg)^{\frac{1}{p}} \tag{3.10}$$

holds, it is necessary and sufficient that the condition

$$\sup_{\substack{x\in X\\ r>0\\ \mu B(x,r)\neq 0}} \big(\sigma B(2N_0 r)\big)^{\frac{1}{p'}} \times$$

$$\times\Bigg(\int\limits_{X\setminus B(x,r)} \big(\mu B(x,d(x,y))\big)^{(\gamma-1)q} d\omega(y)\Bigg)^{\frac{1}{q}} < \infty \tag{3.11}$$

be fulfilled.

Theorem 3.3. *Let X be an arbitrary homogeneous type space, $1 < p \leq q < \infty$, $0 \leq \gamma < 1$, and σ and ω be positive Borel measures. Then the following two conditions are equivalent:*

(i) *there exists a constant $c_1 > 0$ such that*

$$\Bigg(\int\limits_X \big(M_\gamma(f d\sigma)(x)\big)^q d\omega(x)\Bigg)^{\frac{1}{q}} \leq c_1\Bigg(\int\limits_X |f(x)|^p d\sigma(x)\Bigg)^{\frac{1}{p}}, \tag{3.12}$$

for any $f \in L^p(X, d\sigma)$;

(ii) *there exists a constant $c_2 > 0$ such that*

$$\Bigg(\int\limits_B \big(M_\gamma(\chi_B d\sigma)(x)\big)^q d\omega(x)\Bigg)^{\frac{1}{q}} \leq c_2(\sigma B)^{\frac{1}{p}} \tag{3.13}$$

for any $B \subset X$.

Theorem 3.4. *Let $1 < p < q < \infty$, $0 < \gamma < 1$, and the measure σ satisfy the reverse doubling condition ($\sigma \in (RD)$). Then* (3.10) *holds iff*

$$\sup_{\substack{B\\ \mu B\neq 0}} (\mu B)^{\gamma-1}(\sigma B)^{\frac{1}{p'}}(\omega B)^{\frac{1}{q}} < \infty.$$

Theorems 3.2 and 3.3 in fact were previously proved in [2] under the assumption that the space X possesses group structure, while Theorem 3.4 was proved for homogeneous type spaces under a stronger restriction imposed on the measure σ.

Remark. The above-formulated theorems remain valid also in the case where $X = R^1$ is an arbitrary positive Borel measure. This follows from the fact that we proved Theorem A for such measures and from the general conclusion that the well-known technique is applicable for maximal functions with respect to a Borel measure on the real axis (see, for instance, [9]).

4. A Strong Type Two-Weight Problem for Fractional Maximal Functions in a Half-Space

This section deals with "Hörmander type" fractional maximal functions. Let $\widehat{X} = X \times [0, \infty)$ and for $(x,t) \in \widehat{X}$

$$\widetilde{M}_\gamma(f d\sigma)(x,t) = \sup (\mu B)^{\gamma-1} \int\limits_B |f(y)| d\sigma(y), \tag{4.1}$$

where $0 \le \gamma < 1$, σ is a Borel measure in X and the supremum is taken over all balls $B \subset X$ containing x and having a radius greater than $\frac{t}{2}$.

We have

Theorem 4.1. *If $1 < p < q < \infty$, $0 < \gamma < 1$, and X is an arbitrary homogeneous type space, then the inequality*

$$\Bigg(\int\limits_{\widehat{X}} \big(\widetilde{M}_\gamma(f d\sigma))(x,t)\big)^q d\omega(x,t) \Bigg)^{\frac{1}{q}} \le c \Bigg(\int\limits_X |f(x)|^p d\sigma(x) \Bigg)^{\frac{1}{p}} \tag{4.2}$$

holds for all f with c independent of f iff

$$\sup_{\substack{x \in X \\ r>0}} \big(\sigma B(x, 2N_0 r)\big)^{\frac{1}{p'}} \times$$
$$\times \Bigg(\int\limits_{\widehat{X} \setminus \widehat{B}(x,r)} \big(\mu B(x, d(x,y)+t)\big)^{(\gamma-1)q} d\omega(x,t) \Bigg)^{\frac{1}{q}} < \infty, \tag{4.3}$$

where $\widehat{B}(x,r) = B(x,r) \times [0,r)$.

Proof. The validity of the theorem follows from the main theorem using the following arguments. A similar idea is used in [18].

On the product space $X \times R$ we define the quasi-metric as

$$\widehat{d}\big((x,t),(y,s)\big) = \max\big\{d(x,y), |s-t|\big\}$$

and the measure $\widehat{\mu} = \mu \oplus \delta_0$, where δ_0 is the Dirac measure concentrated at zero.

Note that the center of a ball from $\widehat{X}$ with respect to $\widehat{d}$ lies in $\widehat{X}$ and is obtained by the intersection of the respective ball in $X \times R$ with $\widehat{X}$. Since $\widehat{B}((x,t),r) = B(x,r) \times (t-r, t+r) \cap \widehat{X}$, we have

$$\widehat{\mu}\widehat{B}\big((x,t),r\big) = \mu B(x,r) \quad \text{for} \quad r \ge t,$$
$$\widehat{\mu}\widehat{B}\big((x,t),r\big) = 0 \quad \text{for} \quad r < t.$$

Therefore $(\widehat{X}, \widehat{d}, \widehat{\mu})$ is a homogeneous type space. It can be easily verified that the maximal function $\widehat{M}$ is actually a function of the form

$$M_\gamma(f d\sigma)(x,t) = \sup \big(\mu \widehat{B}((z,t),r)\big)^{\gamma-1} \int\limits_{\widehat{B}((z,\tau),r)} |f(y)| d\widehat{\sigma},$$

where $\widehat{\sigma} = \sigma \oplus \delta_0$.

Hence to derive Theorem 4.2 from the main theorem it is sufficient to show that in the considered case condition (2.11) can be reduced to condition (4.3). For this, in turn, since $\widehat{\mu}\widehat{B}((x,\tau),r) \neq 0$ for $\tau < r$ and $\widehat{B}((x,\tau),r) = B(x,r) \times [0, r+\tau)$ it is sufficient to show that

$$\begin{gathered}\widehat{\mu}\widehat{B}\big((x,\tau), \widehat{d}((x,\tau),(y,t))\big) \sim \mu B\big(x, d(x,y)+t\big) \\ \text{for} \quad \tau < r < \widehat{d}\big((x,\tau),(y,t)\big) = \max\big\{d(x,y), |\tau - t|\big\}.\end{gathered}$$

Indeed,

$$d(x,y) + t \le d(x,y) + |\tau - t| + \tau \le d(x,y) + |\tau - t| + r \le 3\widehat{d}\big((x,\tau),(y,t)\big).$$

Since $0 < \tau < \max\{d(x,y), |\tau - t|\}$, we have $\{d(x,y), |\tau - t|\} < d(x,y) + t$, i.e.,

$$d(x,y) + t \sim \widehat{d}((x,\tau),(y,t)).$$

Therefore by virtue of the doubling condition

$$\begin{gathered}\widehat{\mu}\widehat{B}\big((x,t), \widehat{d}((x,t),(y,s))\big) = \\ = \mu B\big(x, \widehat{d}((x,\tau),(y,t))\big) \sim \mu B\big(x, d(x,y)+t\big). \qquad \square\end{gathered}$$

Theorem 4.2. *Let X be an arbitrary homogeneous type space, $1 < p \le q < \infty$, $0 \le \gamma < 1$. Then (4.2) holds for all f with c independent of f iff*

$$\bigg(\int\limits_{\widehat{\widehat{B}}} \big(M_\gamma(\chi_B d\sigma)(x,t)\big)^q d\omega(x,t)\bigg)^{\frac{1}{q}} \le c_1 (\sigma B)^{\frac{1}{p}}, \tag{4.4}$$

where $\widehat{\widehat{B}} = B \times [0, 2\operatorname{rad} B)$.

Theorem 4.3. *If $1 < p < q < \infty$, $0 < \gamma < 1$, $\sigma \in (RD)$, then (4.2) holds iff*

$$\sup (\mu B)^{\gamma - 1} (\sigma B)^{\frac{1}{p'}} \big(\omega \widehat{\widehat{B}}\big)^{\frac{1}{q}} < \infty, \tag{4.5}$$

where the supremum is taken with respect to all balls B.

It should be said that theorems of the type of Theorems 4.1 and 4.2 have previously been known only for homogeneous type spaces with a group structure [10], [11]. In the latter papers instead of condition (4.3) there figures a condition which actually implies the simultaneous fulfillment of conditions (4.3) and (4.5). As for Theorem 4.3, it is proved in [10], [11] but under a more restrictive doubling condition for the measure σ.

We should also note papers [12], [13], where some more particular results are obtained.

5. A Strong Type Two-Weight Problem for One-Sided Fractional Maximal Functions

The method developed in the preceding sections enables one to solve the following strong type two-weight problem:

$$M_\gamma^+(f)(x) = \sup_{h>0} h^{\gamma-1} \int\limits_x^{x+h} |f(t)|dt,$$

and

$$M_\gamma^-(f)(x) = \sup_{h>0} h^{\gamma-1} \int\limits_{x-h}^{x} |f(t)|dt,$$

where $x > 0$ and $0 < \gamma < 1$.

To this end it is sufficient to follow the proof of the main theorem and apply an analog of Theorem 2.1 for one-sided maximal functions (see, for instance, [17, Theorem 2]), Theorem 1.6 from [7], and also the fact that the Riemann-Liouville operator R_γ and the Weyl operator W_γ control the one-sided maximal functions

$$M_\gamma^-(f)(x) \le R_\gamma(|f|)(x)$$

and

$$M_\gamma^+(f)(x) \le W_\gamma(|f|)(x),$$

where

$$R_\gamma(f)(x) = \int\limits_0^x (x-t)^{\gamma-1} f(t)dt$$

and

$$W_\gamma(f)(x) = \int\limits_x^\infty (t-x)^{\gamma-1} f(t)dt,$$

where $0 < \gamma < 1$, $x > 0$.

We thus come to the validity of the following two statements:

Theorem 5.1. *Let $1 < p < q < \infty$, $0 < \gamma < 1$. For the inequality*

$$\bigg(\int_0^\infty \big(M_\gamma^+(f)(x)\big)^q v(x)dx\bigg)^{\frac{1}{q}} \le c\bigg(\int_0^\infty |f(x)|^p w(x)dx\bigg)^{\frac{1}{p}}$$

with the positive constant c independent of f to hold, it is necessary and sufficient that the condition

$$\sup_{\substack{a,h\\0<h<a}} \bigg(\int_{a-h}^{a+h} w^{1-p'}(y)dy\bigg)^{\frac{1}{p'}} \bigg(\int_{a+h}^\infty \frac{v(y)}{(y-a)^{(1-\gamma)q}}\,dy\bigg)^{\frac{1}{q}} < \infty$$

be fulfilled.

Theorem 5.2. *Let $1 < p < q < \infty$, $0 < \gamma < 1$. For the inequality*

$$\bigg(\int_0^\infty \big(M_\gamma^-(f)(x)\big)^q v(x)dx\bigg)^{\frac{1}{q}} \le c\bigg(\int_0^\infty |f(x)|^p w(x)dx\bigg)^{\frac{1}{p}}$$

with the positive constant c independent of f to hold, it is necessary and sufficient that the condition

$$\sup_{\substack{a,h\\0<h<a}} \bigg(\int_{a-h}^{a+h} w^{1-p'}(y)dy\bigg)^{\frac{1}{p'}} \bigg(\int_0^{a-h} \frac{v(y)}{(a-y)^{(1-\gamma)q}}\,dy\bigg)^{\frac{1}{q}} < \infty$$

be fulfilled.

The solution of a weak type two-weight problem for one-sided maximal functions was obtained as a particular case of a more general theorem in [7] (see also [8]).

6. Criteria of Two-Weighted Inequalities for Integral Transforms with Positive Kernels

In this section it will be assumed that the space X is such that there exists a number $L > 1$ such that any ball $B(x,r)$ contains at most L points x_i such that

$$d(x_i, x_m) > \frac{r}{2}\,.$$

The assumption that a measure μ given in X satisfies the doubling condition guarantees the fulfillment of the above-formulated condition (see [15], Lemma 1.1).

For such space X the following statements are valid.

Lemma B ([16], p. 623). *Let Ω be an open bounded subset of X and $c \ge 1$. Then there exists a sequence of balls $B_j = B(x_j, r_j)$ such that*

(i)

$$\Omega = \bigcup_{j=1}^{\infty} \overline{B}_j = \bigcup_{j=1}^{\infty} B_j,$$

where $\overline{B}_j = B(x_j, cr_j)$*;*

(ii) *there exists a positive number* $M = M(c, L, a_0, a_1)$ *such that*

$$\sum_{j=1}^{\infty} \chi_{\overline{B}_j}(x) \leq M;$$

(iii) $(X \backslash \Omega) \cap \overline{\overline{B}}_j \neq \varnothing$ *for each* $\overline{\overline{B}}_j = B(x_j, 3ca_1 r_j)$*;*

Theorem 6.1. *Let* $1 < p \leq q < \infty$*, let* v *and* μ *be locally finite measures in* X*, and let the kernel* k *satisfy condition* (V)*. Then the following conditions are equivalent:*

(i) *there exists* $c_1 > 0$ *such that*

$$v\{x \in X : \mathcal{K}(f)(x) > \lambda\} \leq c_1 \lambda^{-q} \bigg(\int_X f^p(x) d\mu \bigg)^{\frac{q}{p}} \tag{6.1}$$

for any $\lambda > 0$ *and measurable nonnegative* f*;*

(ii) *there exists* $c_2 > 0$ *such that*

$$\bigg(\int_B \big(\mathcal{K}^*(\chi_B dv)\big)^{p'} d\mu \bigg)^{\frac{1}{p'}} \leq c_2 (vB)^{\frac{1}{q'}}. \tag{6.2}$$

The theorem also hold in the case $p = 1$*,* $1 < q < \infty$ *provided that condition* (6.2) *is replaced by the condition*

$$\exp_B K^*\big(\chi_B dv\big)(x) \leq c_2 (vB)^{\frac{1}{q'}}$$

for each ball B*.*

Proof. Let f be a bounded nonnegative function with a compact support. For given $\lambda > 0$ we set

$$\Omega_\lambda = \big\{x \in X : K(f)(x) > \lambda\big\}.$$

Let $B_j = B(x_j, r_j)$ be the sequence of balls from Lemma B corresponding to the number $c = 2a_1$. Then there exists a constant $c_3 > 0$ such that (see, for instance, [17])

$$\mathcal{K}\big(\chi_{X \backslash cB_j} f\big)(x) \leq c_3 \lambda \tag{6.3}$$

for any $x \in B_j$, where $cB_j = B(x_j, cr_j)$. Hence $K(\chi_{X \backslash cB_j} f)(x) > c_3 \lambda$ for any $x \in B_j \cap \Omega_{2c_3 \lambda}$.

Next,

$$v\Omega_{2c_3\lambda} \le \sum_{j=1}^{\infty} v\big(B_j \cap \Omega_{2c_3\lambda}\big) = \\ = \sum_{j\in F} + \sum_{j\in G} v\big(B_j \cap \Omega_{2c_3\lambda}\big) = I_1 + I_2,$$

where

$$F = \big\{j:\ v\big(B_j \cap \Omega_{2c_3\lambda}\big) > \beta v(cB_j)\big\}$$

and

$$G = \big\{v\big(B_j \cap \Omega_{2c_3\lambda}\big) \le \beta v(cB_j)\big\},$$

while the number β will be chosen below so that $0 < \beta < 1$.

Applying the Hölder inequality and condition (6.2) we obtain

$$c_3\lambda v(cB_j) \le c_3\beta^{-1}\lambda v\big(B_j \cap \Omega_{2c_3\lambda}\big) \le \\ \le \beta^{-1} \int\limits_{B_j} \bigg(\int\limits_{cB_j} k(x,y) f(y) d\mu(y) \bigg) dv(x) \le \\ \le \beta^{-1} \int\limits_{cB_j} \bigg(\int\limits_{B_j} k(x,y) dv(x) \bigg) f(y) d\mu(y) \le \\ \le \beta^{-1} \bigg(\int\limits_{cB_j} \bigg(\int\limits_{B_j} k(x,y) dv(x) \bigg)^{p'} d\mu(y) \bigg)^{\frac{1}{p'}} \bigg(\int\limits_{B_j} f^p(y) d\mu(y) \bigg)^{\frac{1}{p}} \le \\ \le \beta^{-1} c_2 v(cB_j)^{\frac{1}{q'}} \bigg(\int\limits_{cB_j} f^p(y) d\mu \bigg)^{\frac{1}{p}}.$$

Therefore

$$\lambda^q v(cB_j) \le (c_3^{-1}\beta^{-1}c_2)^q \bigg(\int\limits_{cB_j} |f(y)|^p d\mu(y) \bigg)^{\frac{q}{p}}.$$

The summation of the latter inequality gives

$$I_1 \le \lambda^{-q} (c_3^{-1}\beta^{-1}c_2)^q \bigg(\int\limits_X |f(y)|^p d\mu \bigg)^{\frac{q}{p}}.$$

Here we have used the fact that $\frac{q}{p} > 1$ and $\sum_{j=1}^{\infty} \chi_{cB_j} \le M$. Next for $j \in G$ we obtain

$$v\big(B_j \cap \Omega_{2c_3\lambda}\big) \le \beta v(cB_j).$$

Therefore

$$I_2 \le \beta M v\Omega_\lambda.$$

Finally, we find that

$$v\Omega_{2c_3\lambda} \le c_4\lambda^{-q}\bigg(\int\limits_X |f(y)|^p d\mu\bigg)^{\frac{q}{p}} + \beta M v\Omega_\lambda.$$

Multiplying the latter inequality by λ^q and taking the exact upper bound with respect to λ, $0<\lambda<\frac{\delta}{2c_3}$, we obtain

$$\sup_{0<\lambda<s} \lambda^q v\Omega_\lambda \le c_4\bigg(\int\limits_X |f(y)|^p d\mu\bigg)^{\frac{q}{p}} + (2c_3)^q M \sup_{0<\lambda<s} \lambda^q v\Omega_\lambda.$$

If we set $\beta = \frac{1}{2M(2c_3)^q}$, the latter inequlity will imply

$$\sup_{0<\lambda<s} \lambda^q v\Omega_\lambda \le c_4\bigg(\int\limits_X |f(y)|^p d\mu\bigg)^{\frac{q}{p}} + \frac{1}{2} \sup_{0<\lambda<s} \lambda^q v\Omega_\lambda.$$

If the second term on the right-hand side is assumed to be finite, then from the latter inequlity we obtain

$$\sup_{0<\lambda<s} \lambda^q v\Omega_\lambda \le c_4\bigg(\int\limits_X |f(y)|^p d\mu\bigg)^{\frac{q}{p}}.$$

Making s tend to infinity, we concluded that (6.1) holds.

It remains to show that

$$\sup_{0<\lambda<s} \lambda^q v\Omega_\lambda < \infty \tag{6.4}$$

for arbitrary finite s.

Let $\operatorname{supp} f \subset B$, where B is a ball in X. It is obvious that

$$\lambda^q v(2a_1B) \le s^q v(2a_1B) < \infty.$$

Therefore it is suffiecient to show that

$$\sup_{\lambda>0} \lambda^q v\big(\Omega_\lambda \backslash 2a_1B\big) < \infty.$$

Let $x \in \Omega_\lambda \backslash 2a_1B$ and $B_x = B(x_B, 2d(x_B,x))$, where x_B is the center of B. For $x' \in B_x$ and $y \in B$ we have

$$2a_1 d(x_B,y) \le d(x_B,x) \le a_1\big(d(x_B,y)+d(y,x)\big).$$

Hence

$$d(x_B,y) \le a_0 d(x,y).$$

Further for $x' \in B_x$ and $y \in B$ we have

$$d(x',y) \le a_1\big(d(x',x_B)+d(x_B,y)\big) \le a_1a_0d(x,y) + 2a_1a_0d(x_B,x) \le$$

$$\le a_1 a_0 d(x,y) + 2a_1^2 a_0 \big(d(x_B,y) + d(y,x)\big) \le$$
$$\le a_1 a_0 d(x,y) + 4a_1^2 a_0^2 d(x,y) = (a_1 a_0 + 4a_1^2 a_0^2) d(x,y).$$

Now by virtue of the remark made at the beginning of the proof of Proposition 3.1 from [7] there exists a constant c' such that

$$k(x,y) \le c' k(x',y)$$

for any $x' \in B_x$ and $y \in B$. Therefore using (6.2) and the Hölder inequality we obtain

$$\lambda v B_x \le v B_x \int\limits_B k(x,y) f(y) d\mu(y) \le$$
$$\le \bigg(\int\limits_B |f(y)|^p d\mu(y)\bigg)^{\frac{1}{p}} \bigg(\int\limits_B \big(k(x,y) v B_x\big)^{p'} d\mu(y)\bigg)^{\frac{1}{p'}} \le$$
$$\le c_5 \bigg(\int\limits_B |f(y)|^p d\mu(y)\bigg)^{\frac{1}{p}} \bigg(\int\limits_B \Big(\int\limits_{B_x} k(x',y) dv(x')\Big)^{p'} d\mu(y)\bigg)^{\frac{1}{p'}} \le$$
$$\le c_6 \bigg(\int\limits_B |f(y)|^p d\mu(y)\bigg)^{\frac{1}{p}} (v B_x)^{\frac{1}{q'}}.$$

Hence we conclude that

$$\lambda (v B_x)^{\frac{1}{q}} \le c_6 \bigg(\int\limits_B |f(y)|^p d\mu(y)\bigg)^{\frac{1}{p}}.$$

Since $\Omega_\lambda \backslash 2a_1 B \subset \cup B_x$ and balls B_x have a common center, by virtue of the Fatou theorem the latter estimate implies that (6.4) is valid. Thus the implication (6.2) $\Rightarrow$ (6.1) is proved for functions f with a compact support. One can easily pass over to the general case.

The implication (6.1) $\Rightarrow$ (6.1) is proved by Theorem 1.5 from [7]. $\square$

Insignificant changes in the proof show that the theorem also holds for $p = 1$ and $1 < q < \infty$.

For the above-mentioned space (X, d, μ) we will consider an upper half-space of the product space $X \times R$. We set $\widehat{X} = X \times [0, \infty)$.

Let $k : X \times X \times [0, \infty)$ be a positive measurable kernel satisfying the condition: there exists a constant $c > 0$ such that

$$k(x,y,t) \le c k(x',y,t')$$

for x, x', y from X, $t \ge 0$, $t' \ge 0$, satisfying the condition $d(x,x') + t' \le 5N_0(d(x,y) + t)$.

Consider the integral operators

$$T(f)(x,t) = \int_X k(x,y,t)f(y)d\mu$$

and

$$T^*(gdv)(y) = \int_{\widehat{X}} k(x,y,t)g(x,t)dv(x,t), \quad y \in X,$$

where v is some locally finite measure in $\widehat{X}$.

Using the consideration of Section 2 from [7] we can deduce the validity of Theorem 1.3 from [18] in this more general case.

Note that in Theorem 6.1 the measure μ was not assumed to satisfy the doubling condition so that we used only the special geometric property of the space. The latter condition is fulfilled if the measure μ has the doubling property.

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(Received 31.01.1995)

Authors' address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Alexidze St., Tbilisi 380093,
Republic of Georgia