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ON SOME ENTIRE MODULAR FORMS OF WEIGHTS $\frac{7}{2}$ AND $\frac{9}{2}$ FOR THE CONGRUENCE GROUP $\Gamma_0(4N)$

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Abstract. Entire modular forms of weights $\frac{7}{2}$ and $\frac{9}{2}$ for the congruence group $\Gamma_0(4N)$ are constructed, which will be useful for revealing the arithmetical sense of additional terms in formulas for the number of representations of positive integers by quadratic forms in 7 and 9 variables.

1.

In this paper N, a, k, q, r, t denote positive integers; u, s are odd positive integers; $H, c, g, h, j, m, n, \alpha, \beta, \gamma, \delta, \xi$ are integers; A, B, C, D, G are complex numbers; z, τ ($\operatorname{Im}\tau > 0$) are complex variables, and $\mathcal{H} = \{\tau \in \mathbb{C} | \operatorname{Im}\tau > 0\}$. Further, $(\frac{h}{u})$ is the generalized Jacobi symbol; $\binom{n}{t}$ is a binomial coefficient; $\varphi(k)$ is Euler's function; $e(z) = \exp 2\pi i z$; $\eta(\gamma) = 1$ if $\gamma \geq 0$ and $\eta(\gamma) = -1$ if $\gamma < 0$.

Let

$$\vartheta_{gh}(z|\tau; c, N) = \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} \times \times e\Big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\Big) e\Big(\big(m + \frac{g}{2}\big) z\Big); \tag{1.1}$$

hence

$$\frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) = (\pi i)^n \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} (2m + g)^n \times \times e\Big(\frac{1}{2N}\big(m + \frac{g}{2}\big)^2 \tau\Big) e\Big(\big(m + \frac{g}{2}\big) z\Big) \quad (n \geq 0). \tag{1.2}$$

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Put

$$\vartheta_{gh}^{(n)}(\tau;c,N)=\frac{\partial^n}{\partial z^n}\,\vartheta_{gh}(z|\tau;c,N)\big|_{z=0}\quad(n\geq 0). \tag{1.3}$$

It is known (see, for e.g., [1], formulas (11) and (12)) that

$$\vartheta_{g,h+2j}^{(n)}(\tau;c,N)=\vartheta_{gh}^{(n)}(\tau;c,N)\quad(n\geq 0), \tag{1.4}$$

$$\begin{aligned}\vartheta_{gh}^{(n)}(\tau+\beta;c,N)&=e\Big(\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\vartheta_{g,h+\beta g+\beta N}^{(n)}(\tau;c,N)\ \ (n\geq 0),\\ \vartheta_{gh}^{(n)}(\tau-\beta;c,N)&=(-1)^n e\Big(-\frac{\beta}{2N}\big(c+\frac{g}{2}\big)^2\Big)\times\\ &\qquad\times\vartheta_{-g,-h+\beta g-\beta N}^{(n)}(\tau;-c,N)\quad(n\geq 0).\end{aligned} \tag{1.5}$$

From (1.1) and (1.2), according to notation (1.3) it follows, in particular, that

$$\begin{aligned}\vartheta_{gh}^{(n)}(\tau;0,N)=(\pi i)^n\sum_{m=-\infty}^{\infty}(-1)^{hm}(2Nm+g)^n\times\\ \times e\Big(\frac{1}{2N}\big(Nm+\frac{g}{2}\big)^2\tau\Big)\ \ (n\geq 0).\end{aligned} \tag{1.6}$$

For $\xi_2\neq 0$, ξ_1,g,h,N with $\xi_1 g+\xi_2 h+\xi_1\xi_2 N\equiv 0 \pmod 2$, put

$$S_{gh}\Big(\begin{array}{c}\xi_1\\ \xi_2\end{array};c,N\Big)=\sum_{\substack{m \bmod N|\xi_2|\\ m\equiv c\ (\mathrm{mod}\ N)}}(-1)^{h(m-c)/N}e\Big(\frac{\xi_1}{2N\xi_2}\big(m+\frac{g}{2}\big)^2\Big).$$

Finally, let

$$\Gamma=\Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,\Big|\,\alpha\delta-\beta\gamma=1\Big\},$$

$$\Gamma_0(4N)=\Big\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\in\Gamma\,\Big|\,\gamma\equiv 0\pmod{4N}\Big\}.$$

For $\tau\in\mathcal{H}$ put

$$(\gamma\tau+\delta)^{s/2}=\big((\gamma\tau+\delta)^{1/2}\big)^s,\quad -\frac{\pi}{2}<\arg(\gamma\tau+\delta)^{1/2}\leq\frac{\pi}{2}.$$

Definition. Let M be a matrix of an arbitrary substitution from $\Gamma_0(4N)$, and let $v(M)$ be a multiplier system on $\Gamma_0(4N)$ and of weight $\frac{s}{2}$. We shall say that a function F defined on $\mathcal{H}$ is an entire modular form of weight $\frac{s}{2}$ and of multiplier system $v(M)$ on $\Gamma_0(4N)$, if

(1) F is regular on $\mathcal{H}$;

(2) for all matrices $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of substitutions from $\Gamma_0(4N)$ and all $\tau \in \mathcal{H}$,

$$F\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = v(M)(\gamma\tau+\delta)^{s/2}F(\tau);$$

(3) in the neighborhood of the point $\tau = i\infty$,

$$F(\tau) = \sum_{m=0}^{\infty} A_m e(m\tau);$$

(4) for all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma,\delta)=1$),

$$(\gamma\tau+\delta)^{s/2}F(\tau) = \sum_{m=0}^{\infty} A'_m e\Big(\frac{m}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big).$$

Lemma 1 ([1], p. 58, Lemma 3). *If g is even, then for $n \geq 0$ and all substitutions from $\Gamma_0(4N)$ we have*

$$\vartheta^{(n)}_{gh}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N\Big) =$$
$$= (\operatorname{sgn}\delta)^n i^{(2n+1)\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2}\Big(\frac{2\beta N\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{(2n+1)/2}\times$$
$$\times e\Big(-\frac{\alpha\gamma\delta^2h^2}{16N}\Big)e\Big(\frac{\beta\delta g^2}{4}\,\frac{\delta^{2\varphi(2N)-2}}{4N}\Big)\vartheta^{(n)}_{\alpha g,h}(\tau;0,2N).$$

Lemma 2 ([1], p. 61, Lemma 4). *If $\gamma \neq 0$, then for $n \geq 0$*

$$(\gamma\tau+\delta)^{(2n+1)/2}\vartheta^{(n)}_{gh}(\tau;0,2N) =$$
$$= e\big((2n+1)\operatorname{sgn}\gamma/8\big)\big(2N|\gamma|\big)^{-1/2}\big(-i\operatorname{sgn}\gamma\big)^n \times$$
$$\times \sum_{H \bmod 2N} \varphi_{g'gh}(0,H;2N)\Big\{\vartheta^{(n)}_{g'h'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big) +$$
$$+\sum_{t=1}^{n}\binom{n}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\cdot\vartheta^{(n-t)}_{g'h'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H,2N\Big)\Big\},$$

where

$$g' = \delta g - \gamma h - 2\gamma\delta N, \quad h' = -\beta g + \alpha h - 2\alpha\beta N, \tag{1.7}$$
$$\varphi_{g'gh}(0,H;2N) = e\Big(\frac{\alpha\beta}{4N}\big(H+\frac{g'}{2}\big)^2\Big)e\Big(-\frac{\beta g}{4N}\big(H+\frac{g'}{2}\big)\Big)\times$$
$$\times S_{g-\alpha g',h-\beta g'}\Big(\begin{matrix}\delta\\-\gamma\end{matrix};-\alpha H,2N\Big),$$

$$A_{tk}\big|_{z=0}=\begin{cases}(2k)!(-2N\gamma\pi i(\gamma\tau+\delta))^k & \textit{if } t=2k\\ 0 & \textit{if } t\neq 2k\end{cases}\tag{1.8}$$

$$(t=1,2,\dots,n;\quad k=1,2,\dots,t).$$

From (1.8) it follows that

$$\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}=\begin{cases}\frac{A_{t,t/2}}{(t/2)!}\big|_{z=0} & \text{if } 2|t,\\ 0 & \text{if } 2\nmid t.\end{cases}\tag{1.9}$$

2.

Lemma 3. *For given N let*

$$\begin{gathered}\Omega(\tau;g_l,h_l,c_l,N_l)=\\ =\Omega(\tau;g_1,g_2,g_3;h_1,h_2,h_3;c_1,c_2,c_3;N_1,N_2,N_3)=\\ =\Big\{\frac{1}{N_1}\,\vartheta''_{g_1h_1}(\tau;c_1,2N_1)\vartheta_{g_2h_2}(\tau;c_2,2N_2)+\\ -\frac{1}{N_2}\,\vartheta''_{g_2h_2}(\tau;c_2,2N_2)\vartheta_{g_1h_1}(\tau;c_1,2N_1)\Big\}\vartheta_{g_3h_3}(\tau;c_3,2N_3),\end{gathered}\tag{2.1}$$

where

$$2|g_k,\quad N_k|N\quad(k=1,2,3),\quad 4|N\sum_{k=1}^{3}\frac{h_k}{N_k}.\tag{2.2}$$

Then for all substitutions from Γ in the neighborhood of each rational point $\tau=-\frac{\delta}{\gamma}$ ($\gamma\neq 0$, $(\gamma,\delta)=1$), we have

$$(\gamma\tau+\delta)^{7/2}\Omega(\tau;g_l,h_l,0,N_l)=\sum_{n=0}^{\infty}G_n e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big).\tag{2.3}$$

Proof. 1. From Lemma 2 for $n=2$ and $n=0$ (respectively with g_1, h_1, N_1, g_1', h_1', H_1 and g_2, h_2, N_2, g_2', h_2', H_2 instead of g,h,N,g',h',H), according to (1.8)–(1.9), it follows that

$$\begin{gathered}\frac{1}{N_1}(\gamma\tau+\delta)^3\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)=\\ =-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\times\\ \times\sum_{\substack{H_1\bmod 2N_1\\ H_2\bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times\end{gathered}$$

$$\times\Big\{\frac{1}{N_1}\,\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+ \\ -4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\times \\ \times\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}. \tag{2.4}$$

Replacing $N_1, g_1, h_1, H_1, g_1', h_1'$ by $N_2, g_2, h_2, H_2, g_2', h_2'$ in (2.4), and vice versa, we obtain

$$\frac{1}{N_2}(\gamma\tau+\delta)^3\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)= \\ =-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\times \\ \times\sum_{\substack{H_1\bmod 2N_1\\ H_2\bmod 2N_2}}\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times \\ \times\Big\{\frac{1}{N_2}\,\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)+ \\ -4\gamma\pi i(\gamma\tau+\delta)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\times \\ \times\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.5}$$

Subtracting (2.5) from (2.4), we obtain

$$(\gamma\tau+\delta)^3\Big\{\frac{1}{N_1}\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)+ \\ -\frac{1}{N_2}\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta_{g_1h_1}(\tau;0,2N_1)\Big\}= \\ =-e\Big(\frac{3}{4}\operatorname{sgn}\gamma\Big)(4N_1N_2\gamma^2)^{-1/2}\times \\ \times\sum_{\substack{H_1\bmod 2N_1\\ H_2\bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\times \\ \times\Big\{\frac{1}{N_1}\,\vartheta''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+ \\ -\frac{1}{N_2}\,\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}. \tag{2.6}$$

From Lemma 2 for $n=0$, we have

$$(\gamma\tau+\delta)^{1/2}\vartheta_{g_3h_3}(\tau;0,2N_3)=e\big(\operatorname{sgn}\gamma/8\big)(2N_3|\gamma|)^{-1/2}\times$$

$$\times \sum_{H_3 \bmod 2N_3} \varphi_{g_3' g_3 h_3}(0, H_3; 2N_3) \vartheta_{g_3' h_3'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_3, 2N_3\Big). \qquad (2.7)$$

Multiplying (2.6) by (2.7), according to (2.1), we obtain

$$(\gamma\tau+\delta)^{7/2}\Omega(\tau; g_l, h_l, 0, N_l) =$$
$$= -e(7 \operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2} \times$$
$$\times \sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2\\ H_3 \bmod 2N_3}} \prod_{k=1}^{3} \varphi_{g_k' g_k h_k}(0, H_k; 2N_k) \Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g_l', h_l', H_l, N_l\Big). \qquad (2.8)$$

2. In (2.8) let γ be even. Then, by (1.7), g_k' $(k = 1, 2, 3)$ are also even. Now in (1.2) and (1.3) instead of m let us introduce new letters of summations m_k defined by the equalities $m - H_k = 2N_k m_k$ $(k = r, t, 3)$, respectively, and for brevity we put

$$T_k = \Big(H_k + 2N_k m_k + \frac{g_k'}{2}\Big)^2 \quad (k = r, t, 3).$$

Then, by (2.1), in (2.8) for $r = 1$, $t = 2$ and $r = 2$, $t = 1$, we get

$$\vartheta''_{g_r' h_r'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_r, 2N_r\Big) \prod_{k=t,3} \vartheta_{g_k' h_k'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; H_k, 2N_k\Big) =$$
$$= (\pi i)^2 \sum_{m_r=-\infty}^{\infty} (-1)^{h_r' m_r} \big(2(H_r + 2N_r m_r) + g_r'\big)^2 e\Big(\frac{N/N_r}{4N}\, T_r\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_t=-\infty}^{\infty} (-1)^{h_t' m_t} e\Big(\frac{N/N_t}{4N}\, T_t\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{m_3=-\infty}^{\infty} (-1)^{h_3' m_3} e\Big(\frac{N/N_3}{4N}\, T_3\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$
$$= \sum_{n_r=0}^{\infty} B_{n_r} e\Big(\frac{n_r}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_t=0}^{\infty} B_{n_t} e\Big(\frac{n_t}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \times$$
$$\times \sum_{n_3=0}^{\infty} B_{n_3} e\Big(\frac{n_3}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \sum_{n=0}^{\infty} C_n^{(r,t)} e\Big(\frac{n}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big), \qquad (2.9)$$

since by (2.2) $n_k = \frac{N}{N_k} T_k$ are non-negative integers. Thus, for even γ, (2.3) follows from (2.1),(2.8), and (2.9).

In (2.8) let now γ be odd. If h_1, h_2, h_3 are even, then by (1.7), g_1', g_2', g_3' are also even, and we obtain the same result. But if h_r is odd, then by

(1.7), h'_r and g'_r will also be odd, and in (1.2) we shall have

$$\bigl(m+g'_r/2\bigr)^2=\bigl(m+(g'_r-1)/2\bigr)^2+\bigl(m+(g'_r-1)/2\bigr)+1/4.$$

Hence

$$\vartheta''_{g'_rh'_r}\Bigl(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\Bigr)=(\pi i)^2e\Bigl(\frac{h'_r}{16N_r}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\times$$
$$\times\sum_{m_r=-\infty}^{\infty}(-1)^{h'_rm_r}\bigl(2(H_r+2N_rm_r)+g'_r\bigr)^2e\Bigl(\frac{N/N_r}{4N}\,W_r\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\quad(r=1,2).$$

Here by (1.4) we can imply that $h'_r=1$, since by (1.7), h'_r and h_r have the same parity, and

$$W_r=\bigl(H_r+2N_rm_r+(g'_r-1)/2\bigr)^2+ \\ +\bigl(H_r+2N_rm_r+(g'_r-1)/2\bigr)\quad(r=1,2). \tag{2.10}$$

Analogously,

$$\vartheta_{g'_th'_t}\Bigl(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_t,2N_t\Bigr)=e\Bigl(\frac{h'_t}{16N_t}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\times$$
$$\times\sum_{m_t=-\infty}^{\infty}(-1)^{h'_tm_t}e\Bigl(\frac{N/N_t}{4N}\,W_t\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\quad(t=2,1).$$

The same formula is valid for $t=3$ with the same remarks concerning h'_3 as for h'_t, $t=2,1$. Further, for $k=t,3$ we have

$$W_k=\begin{cases}T_k & \text{if } 2|g'_k,\\ (H_k+2N_km_k+(g'_k-1)/2)^2+ & \\ (H_k+2N_km_k+(g'_k-1)/2) & \text{if } 2\nmid g'_k.\end{cases} \tag{2.11}$$

Thus, if among h'_1,h'_2,h'_3 at least one is odd, then, as in (2.9), for $r=1$, $t=2$ and $r=2$, $t=1$ we have

$$\vartheta''_{g'_rh'_r}\Bigl(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_r,2N_r\Bigr)\prod_{k=t,3}\vartheta_{g'_kh'_k}\Bigl(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_k,2N_k\Bigr)=$$
$$=e\Bigl(\frac{1}{4N}\Bigl(\frac14\,N\sum_{k=1}^{3}\frac{h'_k}{N_k}\Bigr)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\sum_{n_r=0}^{\infty}B'_{n_r}e\Bigl(\frac{n_r}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\times$$
$$\times\sum_{n_t=0}^{\infty}B'_{n_t}e\Bigl(\frac{n_t}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\sum_{n_3=0}^{\infty}B'_{n_3}e\Bigl(\frac{n_3}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)=$$
$$=e\Bigl(\frac{1}{4N}\Bigl(\frac14\,N\sum_{k=1}^{3}\frac{h'_k}{N_k}\Bigr)\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)\sum_{n'=0}^{\infty}D^{(r,t)}_{n'}e\Bigl(\frac{n'}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Bigr)=$$

$$= \sum_{n=0}^{\infty} D_n^{(r,t)} e\Big(\frac{n}{4N}\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big), \tag{2.12}$$

since, by (2.2), (2.10), and (2.11), $n_k = \frac{N}{N_k} W_k$ and $n = \frac{1}{4} N \sum\limits_{k=1}^{3} \frac{h_k'}{N_k} + n'$ are non-negative integers. Thus, for odd γ, (2.3) follows from (2.1), (2.8), and (2.12). □

Theorem 1. *For given N the function $\Omega(\tau; g_l, h_l, 0, N_l)$ is an entire modular form of weight $7/2$ and of the multiplier system*

$$v(M) = i^{3\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} i^{3(|\delta|-1)^2/4} \Big(\frac{\beta\Delta \operatorname{sgn}\delta}{|\delta|}\Big) \tag{2.13}$$

($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is a matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 7 variables) on $\Gamma_0(4N)$ if the following conditions hold:*

(1) $2|g_k,\ \ N_k|N\ \ (k = 1, 2, 3),$ (2.14)

(2) $4|N \sum\limits_{k=1}^{3} \frac{h_k^2}{N_k}\quad 4| \sum\limits_{k=1}^{s} \frac{g_k^2}{4N_k},$ (2.15)

3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\Big(\frac{N_1 N_2 N_3}{|\delta|}\Big) \Omega(\tau; \alpha g_l, h_l, 0, N_l) = \Big(\frac{\Delta}{|\delta|}\Big) \Omega(\tau; g_l, h_l, 0, N_l). \tag{2.16}$$

Proof. 1. It is well known that theta-series (1.1)–(1.2) are regular on $\mathcal{H}$; hence the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (1) of the Definition.

2. It is easily verified that (2.15) implies

$$4|N\delta^2 \sum_{k=1}^{3} h_k^2/N_k, \quad 4| \sum_{k=1}^{3} \delta^{2\varphi(2N_k)-2} g_k^2/4N_k, \tag{2.17}$$

since $2 \nmid \delta$, because $\alpha\delta \equiv 1 \pmod{4N}$.

From (2.14) it follows that

$$\Gamma_0(4N) \subset \Gamma_0(4N_k) \quad (k = 1, 2, 3). \tag{2.18}$$

By Lemma 1 for $n = 2$ and $n = 0$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; 0, 2N_r\Big) = i^{5\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(1-|\delta|)/2} \times$$
$$\times \Big(\frac{2\beta N_r \operatorname{sgn}\delta}{|\delta|}\Big) (\gamma\tau+\delta)^{5/2} \vartheta''_{\alpha g_r, h_r}(\tau; 0, 2N_r), \tag{2.19}$$

$$\vartheta_{g_t h_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_t\Big)=i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{(1-|\delta|)/2}\times$$
$$\times\Big(\frac{2\beta N_t\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{1/2}\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t),\tag{2.20}$$
$$\vartheta_{g_3 h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{(1-|\delta|)/2}\times$$
$$\times\Big(\frac{2\beta N_3\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{1/2}\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3).\tag{2.21}$$

Thus for all substitutions from $\Gamma_0(4N)$, if $r=1$, $t=2$ and $r=2$, $t=1$, we have

$$\vartheta''_{g_r h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\prod_{k=t,3}\vartheta_{g_k h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big)=$$
$$=i^{7\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_rN_tN_3}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\times$$
$$\times\vartheta''_{\alpha g_r,h_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\prod_{k=t,3}\vartheta_{\alpha g_k,h_k}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_k\Big).\tag{2.22}$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2}\Big(\frac{2}{|\delta|}\Big)=i^{3(|\delta|-1)^2/4}.\tag{2.23}$$

Hence, by (2.1), (2.22), (2.23), (2.13), and (2.16) we get

$$\Omega\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l,h_l,0,N_l\Big)=i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{3(|\delta|-1)^2/4}\times$$
$$\times\Big(\frac{\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\Omega(\tau;\alpha g_l,h_l,0,N_l)=$$
$$=v(M)(\gamma\tau+\delta)^{7/2}\Omega(\tau;g_l,h_l,0,N_l).$$

Thus, the function $\Omega(\tau;g_l,h_l,0,N_l)$ satisfies condition (2) of the Definition.

3. From (1.6) it follows for $r=1$, $t=2$ and $r=2$, $t=1$ that

$$\vartheta''_{g_r h_r}(\tau;0,2N_r)\vartheta_{g_t h_t}(\tau;0,2N_t)\vartheta_{g_3 h_3}(\tau;0,2N_3)=$$
$$=(\pi i)^2\sum_{m_r,m_t,m_3=-\infty}^{\infty}(-1)^{h_rm_r+h_tm_t+h_3m_3}(4N_rm_r+g_r)^2\times$$
$$\times e\Big(\sum_{k=r,t,3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big)=\sum_{n=0}^{\infty}B_n^{(r,t)}e(n\tau),\tag{2.24}$$

where

$$n = \sum_{k=1}^{3} \frac{1}{4N_k}(2N_k m_k + g_k/2)^2 =$$

$$= \sum_{k=1}^{3}(N_k m_k^2 + m_k g_k/2) + \frac{1}{4}\sum_{k=1}^{3} g_k^2/4N_k,$$

by (2.14) and (2.15), is a non-negative integer. Thus, by (2.1) and (2.24), the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (3) of the Definition.

4. By Lemma 3, the function $\Omega(\tau; g_l, h_l, 0, N_l)$ satisfies condition (4) of the Definition. □

3.

Lemma 4. *For given N let*

$$\Psi_1(\tau; g_l, h_l, c_l, N_l) = \Psi_1(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) =$$
$$= \Big\{ \frac{1}{N_1}\, \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1)\vartheta_{g_2 h_2}(\tau; c_2, 2N_2) +$$
$$- \frac{1}{N_2}\, \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2)\vartheta_{g_1 h_1}(\tau; c_1, 2N_1)\Big\}\vartheta'_{g_3 h_3}(\tau; c_3, 2N_3) \qquad (3.1)$$

and

$$\Psi_2(\tau; g_l, h_l, c_l, N_l) = \Psi_2(\tau; g_1, g_2, g_3; h_1, h_2, h_3; c_1, c_2, c_3; N_1, N_2, N_3) =$$
$$= \Big\{ \frac{1}{N_1}\, \vartheta'''_{g_1 h_1}(\tau; c_1, 2N_1)\vartheta_{g_2 h_2}(\tau; c_2, 2N_2) +$$
$$- \frac{3}{N_2}\, \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2)\vartheta'_{g_1 h_1}(\tau; c_1, 2N_1)\Big\}\vartheta_{g_3 h_3}(\tau; c_3, 2N_3), \qquad (3.2)$$

where

$$2|g_k, \quad N_k|N \ \ (k = 1, 2, 3), \quad 4|N \sum_{k=1}^{3} \frac{h_k}{N_k}. \qquad (3.3)$$

Then for all substitutions from Γ, in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$), we have

$$(\gamma\tau + \delta)^{9/2}\Psi_j(\tau; g_l, h_l, 0, N_l) = \sum_{n=0}^{\infty} G_n^{(j)} e\Big(\frac{n}{4N}\, \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Big) \ (j = 1, 2). \qquad (3.4)$$

Proof. 1. Formula (2.6) is obtained in Lemma 3. From Lemma 2, for $n = 1$, we have

$$(\gamma\tau+\delta)^{3/2}\vartheta'_{g_3h_3}(\tau;0,2N_3) = -e(3\operatorname{sgn}\gamma/8)(2N_3|\gamma|)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times\sum_{H_3 \bmod 2N_3}\varphi_{g'_3g_3h_3}(0,H_3;2N_3)\vartheta'_{g'_3h'_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big). \qquad (3.5)$$

Multiplying (2.6) by (3.5), according to (3.1), we obtain

$$(\gamma\tau+\delta)^{9/2}\Psi_1(\tau;g_l,h_l,0,N_l) = \\ = e(9\operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2\\ H_3 \bmod 2N_3}}\prod_{k=1}^{3}\varphi_{g'_kg_kh_k}(0,H_k;2N_k)\Psi_1\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g'_l,h'_l,H_l,N_l\Big). \qquad (3.6)$$

Reasoning further just as in Lemma 3.2, but taking everywhere $\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)$ instead of $\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)$, from (3.1) and (3.6) we obtain (3.4) if $j = 1$.

2. From Lemma 2 for $n = 3$, $n = 0$, and respectively for $n = 2$, $n = 1$, it follows that

$$\frac{1}{N_1}(\gamma\tau+\delta)^4\vartheta'''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2) = \\ = \frac{1}{N_1}e(\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}(i\operatorname{sgn}\gamma)\times \\ \times\sum_{H_1 \bmod 2N_1}\varphi_{g'_1g_1h_1}(0,H_1;2N_1)\Big\{\vartheta'''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big) + \\ +\sum_{t=1}^{3}\binom{3}{t}\sum_{k=1}^{t}\frac{A_{tk}}{k!}\Big|_{z=0}\vartheta^{(3-t)}_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times \\ \times\sum_{H_2 \bmod 2N_2}\varphi_{g'_2g_2h_2}(0,H_2;2N_2)\vartheta_{g'_2h'_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big) = \\ = e(\operatorname{sgn}\gamma)(i\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}\sum_{\substack{H_1 \bmod 2N_1\\ H_2 \bmod 2N_2}}\varphi_{g'_1g_1h_1}(0,H_1;2N_1)\times \\ \times\varphi_{g'_2g_2h_2}(0,H_2;2N_2)\Big\{\frac{1}{N_1}\vartheta'''_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g'_2h'_2}(\tau;0,2N_2) + \\ -12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g'_1h'_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g'_2h'_2}(\tau;0,2N_2)\Big\} \qquad (3.7)$$

and

$$\frac{3}{N_2}(\gamma\tau+\delta)^4\vartheta''_{g_2h_2}(\tau;0,2N_2)\vartheta'_{g_1h_1}(\tau;0,2N_1)=$$
$$=e(\operatorname{sgn}\gamma)(i\operatorname{sgn}\gamma)(4N_1N_2\gamma^2)^{-1/2}\sum_{\substack{H_1\bmod 2N_1\\H_2\bmod 2N_2}}\varphi_{g_1'g_1h_1}(0,H_1;2N_1)\times$$
$$\times\varphi_{g_2'g_2h_2}(0,H_2;2N_2)\Big\{\frac{3}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+$$
$$-12\gamma\pi i(\gamma\tau+\delta)\vartheta'_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\Big\}\vartheta'_{g_1'h_1'}(\tau;0,2N_1). \qquad (3.8)$$

Subtracting (3.8) from (3.7) and multiplying the obtained result by (2.7), according to (3.2), we get

$$(\gamma\tau+\delta)^{9/2}\Psi_2(\tau;g_l,h_l,0,N_l)=$$
$$=e(9\operatorname{sgn}\gamma/8)(8N_1N_2N_3|\gamma|^3)^{-1/2}(i\operatorname{sgn}\gamma)\times$$
$$\times\sum_{\substack{H_1\bmod 2N_1\\H_2\bmod 2N_2\\H_3\bmod 2N_3}}\prod_{k=1}^{3}\varphi_{g_k'g_kh_k}(0,H_k;2N_k)\Psi_2\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};g_l',h_l',H_l,N_l\Big). \qquad (3.9)$$

In (3.9) let γ be even. Then, reasoning as in Lemma 3.2, by (3.2), from (3.9) we obtain

$$\Big\{\frac{1}{N_1}\vartheta'''_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\vartheta_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)+$$
$$-\frac{3}{N_2}\vartheta''_{g_2'h_2'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_2,2N_2\Big)\vartheta'_{g_1'h_1'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_1,2N_1\Big)\Big\}\times$$
$$\times\vartheta_{g_3'h_3'}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};H_3,2N_3\Big)=\Big\{\frac{(\pi i)^3}{N_1}\times$$
$$\times\sum_{m_1=-\infty}^{\infty}(-1)^{h_1'm_1}\big(2(H_1+2N_1m_1)+g_1'\big)^3e\Big(\frac{N/N_1}{4N}\,T_1\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times$$
$$\times\sum_{m_2=-\infty}^{\infty}(-1)^{h_2'm_2}e\Big(\frac{N/N_2}{4N}\,T_2\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)+$$
$$-\frac{3(\pi i)^3}{N_1}\sum_{m_2=-\infty}^{\infty}(-1)^{h_2'm_2}\big(2(H_2+2N_2m_2)+g_2'\big)^2e\Big(\frac{N/N_2}{4N}\,T_2\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\times$$
$$\times\sum_{m_1=-\infty}^{\infty}(-1)^{h_1'm_1}\big(2(H_1+2N_1m_1)+g_1'\big)e\Big(\frac{N/N_1}{4N}\,T_1\,\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\Big\}\times$$

$$\times \sum_{m_3=-\infty}^{\infty} (-1)^{h_3' m_3} e\Big(\frac{N/N_3}{4N}\, T_3\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) =$$

$$= \Big\{ \sum_{n_1=0}^{\infty} B_{n_1} e\Big(\frac{n_1}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_2=0}^{\infty} B_{n_2} e\Big(\frac{n_2}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) +$$

$$- \sum_{n_2=0}^{\infty} B'_{n_2} e\Big(\frac{n_2}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) \sum_{n_1=0}^{\infty} B'_{n_1} e\Big(\frac{n_1}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big)\Big\} \times$$

$$\times \sum_{n_3=0}^{\infty} B_{n_3} e\Big(\frac{n_3}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big) = \sum_{n=0}^{\infty} C_n e\Big(\frac{n}{4N}\, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\Big). \tag{3.10}$$

Here T_k and n_k have the same meaning as in Lemma 3.2.

Thus, for even γ, (3.4) follows from (3.2), (3.9) and (3.10) if $j = 2$.

Further, reasoning as in Lemma 3.2, we obtain (3.4) for odd γ if $j = 2$. □

Theorem 2. *For given N the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ are entire modular forms of weight $9/2$ and of the multiplier system*

$$v(M) = i^{\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} i^{(|\delta|-1)^2/4} \Big(\frac{\beta\Delta \operatorname{sgn}\delta}{|\delta|}\Big) \tag{3.11}$$

($M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is the matrix of the substitution from $\Gamma_0(4N)$ and Δ is the determinant of an arbitrary positive quadratic form with integer coefficients in 9 variables*) *on $\Gamma_0(4N)$ if the following conditions hold:*

(1) $\quad 2|g_k, \quad N_k|N \quad (k = 1, 2, 3),$ (3.12)

(2) $\quad 4|N \sum_{k=1}^{3} \frac{h_k^2}{N_k} \quad 4| \sum_{k=1}^{s} \frac{g_k^2}{4N_k},$ (3.13)

(3) *for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$*

$$\operatorname{sgn}\delta \Big(\frac{N_1 N_2 N_3}{|\delta|}\Big) \Psi_j(\tau; \alpha g_l, h_l, 0, N_l) =$$

$$= \Big(\frac{-\Delta}{|\delta|}\Big) \Psi_j(\tau; g_l, h_l, 0, N_l) \quad (j = 1, 2). \tag{3.14}$$

Proof. 1. As in the case of Theorem 1, the functions $\Psi_1(\tau; g_l, h_l, 0, N_l)$ and $\Psi_2(\tau; g_l, h_l, 0, N_l)$ satisfy condition (1) and, by Lemma 4, also condition (4) of the Definition.

2. By Lemma 1 for $n = 3$ and $n = 1$, according to (2.17) and (2.18), for all substitutions from $\Gamma_0(4N)$, we have

$$\vartheta'''_{g_1 h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; 0, 2N_1\Big) = \operatorname{sgn}\delta\, i^{7\eta(\gamma)(\operatorname{sgn}\delta - 1)/2} i^{(1-|\delta|)/2} \times$$

$$\times\Big(\frac{2\beta N_1\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{7/2}\vartheta'''_{\alpha g_1,h_1}(\tau;0,2N_1), \tag{3.15}$$

$$\vartheta'_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)=\operatorname{sgn}\delta\, i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{(1-|\delta|)/2}\times$$

$$\times\Big(\frac{2\beta N_1\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{3/2}\vartheta'_{\alpha g_1,h_1}(\tau;0,2N_1), \tag{3.16}$$

$$\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=\operatorname{sgn}\delta\, i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{(1-|\delta|)/2}\times$$

$$\times\Big(\frac{2\beta N_3\operatorname{sgn}\delta}{|\delta|}\Big)(\gamma\tau+\delta)^{3/2}\vartheta'_{\alpha g_3,h_3}(\tau;0,2N_3). \tag{3.17}$$

Thus, for all substitutions from $\Gamma_0(4N)$ we respectively get:
(1) by (2.19), (2.20), and (3.17)

$$\vartheta''_{g_rh_r}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_r\Big)\vartheta_{g_th_t}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_t\Big)\vartheta'_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta''_{\alpha g_r,h_r}(\tau;0,2N_r)\vartheta_{\alpha g_t,h_t}(\tau;0,2N_t)\vartheta'_{\alpha g_3,h_3}(\tau;0,2N_3) \tag{3.18}$$

for $r=1$, $t=2$ and $r=2$, $t=1$;
(2) by (3.15), (2.20) for $t=2$ and (2.21)

$$\vartheta'''_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)\vartheta_{g_2h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\Big)\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta'''_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta_{\alpha g_2,h_2}(\tau;0,2N_2)\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3); \tag{3.19}$$

(3) by (2.19) for $r=2$, (3.16), and (2.21)

$$\vartheta''_{g_2h_2}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_2\Big)\vartheta'_{g_1h_1}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_1\Big)\vartheta_{g_3h_3}\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta};0,2N_3\Big)=$$

$$=\operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2}i^{3(1-|\delta|)/2}\Big(\frac{2\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)\times$$

$$\times(\gamma\tau+\delta)^{9/2}\vartheta''_{\alpha g_2,h_2}(\tau;0,2N_2)\vartheta'_{\alpha g_1,h_1}(\tau;0,2N_1)\vartheta_{\alpha g_3,h_3}(\tau;0,2N_3). \tag{3.20}$$

It is not difficult to verify that

$$i^{3(1-|\delta|)/2}\Big(\frac{-2}{|\delta|}\Big)=i^{(|\delta|-1)^2/4}. \tag{3.21}$$

Hence, by (3.1), (3.18), (3.21), (3.11), (3.14) (if $j = 1$) and (3.2), (3.19)–(3.21), (3.11), (3.14) (if $j = 2$) we get

$$\Psi_j\Big(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}; g_l, h_l, 0, N_l\Big) = \operatorname{sgn}\delta\, i^{\eta(\gamma)(\operatorname{sgn}\delta-1)/2} i^{(|\delta|-1)^2/4} \times$$
$$\times\Big(\frac{-\beta\operatorname{sgn}\delta}{|\delta|}\Big)\Big(\frac{N_1N_2N_3}{|\delta|}\Big)(\gamma\tau+\delta)^{9/2}\Psi_j(\tau; \alpha g_l, h_l, 0, N_l) =$$
$$= v(M)(\gamma\tau+\delta)^{9/2}\Psi_j(\tau; g_l, h_l, 0, N_l) \ \ (j = 1, 2).$$

Thus the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 1, 2)$ satisfy condition (2) of the Definition.

From (1.6) it follows that

$$\vartheta''_{g_rh_r}(\tau; 0, 2N_r)\vartheta_{g_th_t}(\tau; 0, 2N_t)\vartheta'_{g_3h_3}(\tau; 0, 2N_3) =$$
$$= (\pi i)^3 \sum_{m_r,m_t,m_3=-\infty}^{\infty} (-1)^{\sum_{k=r,t,3} h_km_k}(4N_rm_r+g_r)^2(4N_3m_3+g_3) \times$$
$$\times e\Big(\sum_{k=r,t,3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big) = \sum_{n=0}^{\infty} C_n^{(r,t)}e(n\tau)$$

for $r = 1$, $t = 2$ and $r = 2$, $t = 1$,

$$\vartheta'''_{g_1h_1}(\tau; 0, 2N_1)\vartheta_{g_2h_2}(\tau; 0, 2N_2)\vartheta_{g_3h_3}(\tau; 0, 2N_3) =$$
$$= (\pi i)^3 \sum_{m_1,m_2,m_3=-\infty}^{\infty} (-1)^{\sum_{k=1}^{3} h_km_k}(4N_1m_1+g_1)^3 \times$$
$$\times e\Big(\sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big) = \sum_{n=0}^{\infty} C_ne(n\tau)$$

and

$$\vartheta''_{g_2h_2}(\tau; 0, 2N_2)\vartheta'_{g_1h_1}(\tau; 0, 2N_1)\vartheta_{g_3h_3}(\tau; 0, 2N_3) =$$
$$= (\pi i)^3 \sum_{m_1,m_2,m_3=-\infty}^{\infty} (-1)^{\sum_{k=1}^{3} h_km_k}(4N_2m_2+g_2)^2(4N_1m_1+g_1) \times$$
$$\times e\Big(\sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2\tau\Big) = \sum_{n=0}^{\infty} D_ne(n\tau),$$

since in all these expansions

$$n = \sum_{k=1}^{3}\frac{1}{4N_k}(2N_km_k+g_k/2)^2,$$

as has been shown by Theorem 1.3, is a non-negative integer. Therefore, in view of (3.1) and (3.2), the functions $\Psi_j(\tau; g_l, h_l, 0, N_l)$ $(j = 1, 2)$ satisfy condition (3) of the Definition. □

References

1. G. Lomadze, On some entire modular forms of weights 5 and 6 for the congruence group $\Gamma_0(4N)$. *Georgian Math. J.* **1**(1994), No. 1, 53–76.

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