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ON DIFFERENTIAL BASES FORMED OF INTERVALS

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In memory of young mathematician
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Abstract. Translation invariant subbases of the differential basis B_2 (formed of all intervals), which differentiates the same class of all non-negative functions as B_2 does, are described. A possibility for extending the results obtained to bases of more general type is discussed.

1. Definitions and Notation

A mapping B defined on $\mathbb{R}^n$ is said to be a differential basis in $\mathbb{R}^n$ if, for every $x \in \mathbb{R}^n$, $B(x)$ is a family of open bounded sets containing the point x such that there exists a sequence $\{R_k\} \subset B(x)$, $\operatorname{diam} R_k \to 0$ $(k \to \infty)$.

For $f \in L_{loc}(\mathbb{R}^n)$ the numbers

$$\overline{D}_B\Big(\int f, x\Big) = \overline{\lim_{\operatorname{diam} R\to 0, R\in B(x)}} \frac{1}{|R|}\int_R f$$

and

$$\underline{D}_B\Big(\int f, x\Big) = \underline{\lim_{\operatorname{diam} R\to 0, R\in B(x)}} \frac{1}{|R|}\int_R f$$

are said to be respectively the upper and the lower derivative of the integral of f at the point x. If the upper and the lower derivative coincide, then their common value is called the derivative of the integral of f at the point x, and we denote it by $D_B\big(\int f, x\big)$. They say that the basis B differentiates the integral of f if $D_B\big(\int f, x\big) = f(x)$ for almost all x. The set of those functions $f \in L_{loc}(\mathbb{R}^n)$, $f \geq 0$, whose integrals are differentiable with respect to the basis B will be denoted by F_B^+. Under M_B we mean the maximal operator

$$M_B(f)(x) = \sup_{R\in B(x)} \frac{1}{|R|}\int_R |f| \quad (f \in L_{loc}(\mathbb{R}^n),\ x \in \mathbb{R}^n),$$

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corresponding to the basis B. It will be assumed here that $\overline{B} = \bigcup_{x\in\mathbb{R}^n} B(x)$.

B is said to be a subbasis of B' (writing $B \subset B'$), if $B(x) \subset B'(x)$ $(x \in \mathbb{R}^n)$. The basis B is said to be translation invariant or the TI-basis, if $B(x) = \{x + R : R \in B(O)\}$ $(x \in \mathbb{R}^n)$ (here O is the origin in $\mathbb{R}^n$). Let us have the bases B and B'. We shall say that the family $\overline{B}$ is locally regular with respect to the family $\overline{B'}$ (writing $\overline{B} \in LR(\overline{B'})$), if there exist $\delta > 0$ and $c > 0$ such that for any $R \in \overline{B}$, $\operatorname{diam} R < \delta$, there is $R' \in \overline{B}'$ such that $R \subset R'$ and $|R'| < c|R|$.

We shall agree that $I^n = [0,1]^n$ and $f \in L(I^n)$, if $f \in L(\mathbb{R}^n)$ and $\operatorname{supp} f \subset I^n$.

2. TI-Bases Formed of Intervals

Let B_2 be the basis in $\mathbb{R}^2$ for which $B_2(x)$ $(x \in \mathbb{R}^2)$ consists of all two-dimensional intervals containing the point x.

The theorem below characterizes $B \subset B_2$, TI-bases for which $F_B^+ = F_{B_2}^+$.

Theorem 1. *Let $B \subset B_2$ be a TI-basis. Then the following conditions are equivalent:*

(a) $F_B^+ = F_{B_2}^+$;

(b) *for every $f \in L(\mathbb{R}^2)$, $f \geq 0$, a.e. on $\mathbb{R}^2$*

$$\overline{D}_B\Big(\int f, x\Big) = \overline{D}_{B_2}\Big(\int f, x\Big) \ \text{ and } \ \underline{D}_B\Big(\int f, x\Big) = \underline{D}_{B_2}\Big(\int f, x\Big);$$

(c) *$\overline{B}_2$ is locally regular with respect to $\overline{B}$.*

The implication $(b) \Rightarrow (a)$ is evident. Therefore to prove Theorem 1 it suffices to show that $(a) \Rightarrow (c)$ and $(c) \Rightarrow (b)$.

The implication $(a) \Rightarrow (c)$ follows from the following assertion.

Theorem 2. *Let $B \subset B_2$ be a TI-basis. If $\overline{B}_2$ is not locally regular with respect to $\overline{B}$, then there exists a function $f \in L(I^2)$, $f \geq 0$, such that*

$$\overline{D}_{B_2}\Big(\int f, x\Big) = \infty \ \text{ a.e. on } \ I^2,$$

$$D_B\Big(\int f, x\Big) = f(x) \ \text{ a.e. on } \ I^2.$$

Before proving Theorem 2 we shall give several lemmas.

For the interval I we denote by αI $(\alpha > 0)$ the interval $H(I)$, where H is the homothety with the coefficient α whose center is the center of I.

Lemma 1. *Let $I \in \overline{B}_2$ and $h > 1$. Then $\{M_{B_2}(h\chi_I) > 1\} \subset (2h+1)I$ and $|\{M_{B_2}(h\chi_I) > 1\}| \geq h(\ln h)|I|$.*

The validity of this lemma can be shown by a direct checking.

Projections of I onto the ox^1- and ox^2-axes will be denoted by $\mathrm{pr}_1 I$ and $\mathrm{pr}_2 I$, respectively. For $I \in \overline{B}_2$ and $h > 0$ we have

$$\begin{aligned} R_1(I,h) &= (2h+1)\,\mathrm{pr}_1 I \times 3\,\mathrm{pr}_2 I,\\ R_2(I,h) &= 3\,\mathrm{pr}_1 I \times (2h+1)\,\mathrm{pr}_2 I,\\ R(I,h) &= R_1(I,h) \cup R_2(I,h). \end{aligned}$$

It is clear that $|R(I,h)| \le 18h|I|$.

Lemma 2. *Let $B \subset B_2$ be a TI-basis; $h > 1$. Suppose that for $I \in \overline{B}_2$ there is no $J \in \overline{B}$, $J \supset I$, such that $|J| \le h|I|$. Then $\{M_B(h\chi_I) > 1\} \subset R(I,h)$.*

Proof. Let $J \in \overline{B}$ and $(1/|J|)\int_J h\chi_I > 1$. From the condition of the lemma we can easily find that either

$$|pr_1 J|_1 < |\,\mathrm{pr}_1 I|_1 \ \text{ or } \ |\,\mathrm{pr}_2 J|_1 < |\,\mathrm{pr}_2 I|_1, \tag{1}$$

where $|\cdot|_1$ is the Lebesgue measure on $\mathbb{R}$.

By Lemma 1, $J \subset (2h+1)I$, which by virtue of (1) implies that either $J \subset R_1(I,h)$ or $J \subset R_2(I,h)$, so that we have $J \subset R(I,h)$. It follows from the latter inclusion that $\{M_B(h\chi_I) > 1\} \subset R(I,h)$. $\square$

It can be easily seen that the following two lemmas are valid.

Lemma 3. *Let $I, J \in \overline{B}_2$; $h > 1$. If either $|\,\mathrm{pr}_1 J|_1 \ge h|\,\mathrm{pr}_1 J|_1$ or $|\,\mathrm{pr}_2 J|_1 \ge h|\,\mathrm{pr}_2 J|_1$, then $h|J \cap I| \le |I \cap R(I,h)|$.*

Lemma 4. *Let $I, J \in \overline{B}_2$. If $J \cap I \ne \varnothing$ and $J \backslash R(I,h) \ne \varnothing$, then either $|\,\mathrm{pr}_1 J|_1 \ge |\,\mathrm{pr}_1 I|_1$, or $|\,\mathrm{pr}_2 J|_1 \ge |\,\mathrm{pr}_2 I|_1$.*

The following lemma is also valid.

Lemma 5. *Let $B \subset B_2$; $h > 1$, and let $I_1, \dots, I_k \in \overline{B}_2$ be the equal intervals. If*

$$\begin{aligned} \{M_B(h\chi_{I_m}) > 1\} &\subset R(I_m,h) \quad (m = \overline{1,k}),\\ R(I_m,h) \cap R(I_{m'},h) &= \varnothing \quad (m \ne m'), \end{aligned}$$

then

$$\Big\{M_B\Big(\sum_{m=1}^{k} h\chi_{I_m}\Big) > 1\Big\} \subset \bigcup_{m=1}^{k} R(I_m,h). \tag{2}$$

Proof. Inequality (2) is equivalent to the following inequality: if $x \notin \bigcup\limits_{m=1}^{k} R(I_m, h)$ and $I \in B(x)$, then

$$\frac{1}{|I|} \int_I \sum_{m=1}^{k} h\chi_{I_m} \le 1. \tag{3}$$

Let $x \notin \bigcup\limits_{m=1}^{k} R(I_m, h)$ and $I \in B(x)$. Then we may have three cases. Let us consider each of them separately.

(i) I intersects none of the intervals I_m $(m = \overline{1,k})$; in this case the validity of (3) is evident.

(ii) I intersects only one interval I_m $(m = \overline{1,k})$.

Let I_n be the interval for which $I \cap I_n \ne \varnothing$. $x \notin R(I_m, h)$, and therefore by the condition of the lemma,

$$\frac{1}{|I|} \int_I h\chi_{I_n} \le M_B(h\chi_{I_n})(x) \le 1,$$

from which, taking into account the equality $I \cap I_m = \varnothing$ $(m \ne n)$, we obtain (3).

(iii) I intersects more than one interval I_m $(m = \overline{1,k})$.

It follows from the equality $R(I_m, h) \cap R(I_{m'}, h) = \varnothing$ $(m \ne m')$ that

$$(2h+1)I_m \cap I_{m'} = \varnothing \tag{4}$$

Denote $P = \{m \in [1,k] : I \cap I_m \ne \varnothing\}$. By (4) we have

$$I \cap I_m \ne \varnothing, \quad I \backslash (2h+1)I_m \ne \varnothing \quad (m \in P),$$

which gives either

$$|\operatorname{pr}_1 I|_1 \ge h |\operatorname{pr}_1 I_m|_1 \ \text{ or } \ |\operatorname{pr}_2 I|_1 \ge h |\operatorname{pr}_2 I_m|_1 \ \ (m \in P).$$

Now according to Lemma 3 we can conclude that

$$h|I \cap I_m| \le |I \cap R(I_m, h)| \ \ (m \in P).$$

From the obtained inequality, taking into consideration the pairwise non-intersection of $R(I_m, h)$ $(m = \overline{1,k})$, we write

$$\int_I \sum_{m=1}^{k} h\chi_{I_m} = \sum_{m \in P} \int_I h\chi_{I_m} = \sum_{m \in P} h|I \cap I_m| \le \sum_{m \in P} |I \cap R(I_m, h)| \le |I|.$$

Thus inequality (3) and Lemma 5 are proved. $\square$

Lemma 6. *Let $B \subset B_2$, $h > 1$, and for every $m \in [1,k]$ let $\{I_{m,q}\}_{q=1}^{q_m} \subset \overline{B}_2$ be a family of equal intervals. If*

$$\begin{aligned}&\{M_B(h\chi_{I_{m,q}}) > 1\} \subset R_{m,q}\\ &(R_{m,q} = R(I_{m,q}, h); m = \overline{1,k})\ \ q = \overline{1,q_m}),\end{aligned} \tag{5}$$

$$R_{m,q} \cap R_{m',q'} = \varnothing \quad (m,q) \neq (m,q'), \tag{6}$$

$$\left.\begin{aligned}&|\operatorname{pr}_1 I_{m,1}|_1 \geq h|\operatorname{pr}_1 I_{m+1,1}|_1,\\ &|\operatorname{pr}_2 I_{m,1}|_1 \geq h|\operatorname{pr}_2 I_{m+1,1}|_1,\end{aligned}\right. \quad (m = \overline{1,k-1}) \tag{7}$$

then

$$\Big\{M_B\Big(\sum_{m=1}^{k}\sum_{q=1}^{q_m} h\chi_{I_{m,q}}\Big) > 2\Big\} \subset \bigcup_{m=1}^{k}\bigcup_{q=1}^{q_m} R_{m,q}.$$

Proof. The inclusion we have to prove is equivalent to the following inequality: if $x \notin \bigcup\limits_{m=1}^{k}\bigcup\limits_{q=1}^{q_m} R_{m,q}$ and $I \in B(x)$, then

$$\frac{1}{|I|}\int_I \sum_{m=1}^{k}\sum_{q=1}^{q_m} h\chi_{I_{m,q}} \leq 2. \tag{8}$$

Assume $x \notin \bigcup\limits_{m=1}^{k}\bigcup\limits_{q=1}^{q_m} R_{m,q}$ and $I \in B(x)$. It is clear that (8) is fulfilled for $I \cap I_{m,q} = \varnothing$ $(m = \overline{1,k},\ q = \overline{1,q_m})$. Denote otherwise $n = \min\{m \in [1,k] : \exists I_{m,q}(q = \overline{1,q_m}), I \cap I_{m,q} \neq \varnothing\}$ and consider first the case with $1 < n < k$.

By virtue of Lemma 5 (see (5) and (6)), we write

$$M_B\Big(\sum_{q=1}^{q_n} h\chi_{I_{n,q}}\Big)(x) \leq 1,$$

which implies

$$\int_I \sum_{q=1}^{q_n} h\chi_{I_{n,q}} = \sum_{q=1}^{q_n} h|I \cap I_{n,q}| \leq |I|. \tag{9}$$

For some $q \in [1,q_n]$ $I \cap I_{n,q} \neq \varnothing$, it is clear that $I \backslash R_{n,q} \neq \varnothing$. Therefore, according to Lemma 4, we have either

$$|\operatorname{pr}_1 I|_1 \geq |\operatorname{pr}_1 I_{n,q}|_1 \ \text{ or } \ |\operatorname{pr}_2 I|_1 \geq |\operatorname{pr}_2 I_{n,q}|_1.$$

Taking into account that the intervals $\{I_{m,q}\}_{q=1}^{q_m}$ $(m = \overline{1,k})$ are equal and using (7), we can write that either

$$|\operatorname{pr}_1 I|_1 \geq h|\operatorname{pr}_1 I_{m,q}|_1 \quad (n < m \leq k,\ q = \overline{1,q_m})$$

or

$$|\operatorname{pr}_2 I|_1 \geq h|\operatorname{pr}_2 I_{m,q}|_1 \quad (n < m \leq k,\ q = \overline{1, q_m}),$$

whence by Lemma 3 we get

$$h|I \cap I_{m,q}| \leq |I \cap R_{m,q}| \quad (n < m \leq k;\ q = \overline{1, q_m}). \tag{10}$$

Clearly, $I \cap I_{m,q} = \varnothing$ $(n \leq m < n;\ q = \overline{1, q_m})$, so that by (6), (9), (10) and $1 \leq m < n$ we have

$$\int_I \sum_{m=1}^{k} \sum_{q=1}^{q_m} h\chi_{I_{m,q}} = \sum_{m=1}^{k} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| = \sum_{m=1}^{n-1} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| +$$

$$+ \sum_{q=1}^{q_n} h|I \cap I_{n,q}| + \sum_{m=n+1}^{k} \sum_{q=1}^{q_m} h|I \cap I_{m,q}| =$$

$$= A_1 + A_2 + A_3 \leq 0 + |I| + \sum_{m=n+1}^{k} \sum_{q=1}^{q_m} |I \cap R_{m,q}| \leq |I| + |I| = 2|I|.$$

Inequality (8) can be proved in a simpler way for $n = 1$ or $n = k$, since in these cases we do not have the terms A_1 and A_3. □

For the basis B let us define the operator

$$M_B^*(f)(x) = \sup_{R \in B(x)} \frac{1}{|R|} \Big| \int_R f \Big| \quad (f \in L_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n).$$

Lemma 7. *Let the basis B differentiate the integrals of the functions $f_k \in L(\mathbb{R}^n)$ $(k \in \mathbb{N})$, $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$. If*

$$\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty,$$

where $\lambda_k > 0$ $(k \in \mathbb{N})$, $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $|\cdot|_e$ is an outer measure, then B differentiates the integral of the function $\sum_{k=1}^{\infty} f_k$.

Note that since $M_B^*(f) \leq M_B(f)$ $(f \in L(\mathbb{R}^n))$, the conclusion of Lemma 7 will be the more so valid when the inequality $\sum_{k=1}^{\infty} |\{M_B(f_k) > \lambda_k\}|_e < \infty$ is fulfilled.

Proof. Let us note that:

(i) $\sum_{k=1}^{\infty} \|f_k\|_1 < \infty$; therefore the set $A_0 = \{x \in \mathbb{R}^n : \sum_{k=1}^{\infty} |f_k(x)| < \infty\}$ is of measure 0;

(ii) B differentiates $\int f_k$ $(k \in \mathbb{N})$, i.e., the sets $A_k = \{x \in \mathbb{R}^n : D_B(\int f, x) = f(x)\}$ $(k \in \mathbb{N})$ have a complete measure on $\mathbb{R}^n$.

(iii) $\sum_{k=1}^{\infty} |\{M_B^*(f_k) > \lambda_k\}|_e < \infty$; therefore the set $\overline{\lim}_{k \to \infty} \{M_B^*(f_k) > \lambda_k\}$ is of measure 0.

It follows from (i)–(iii) that the set $A = \bigcup_{k=1}^{\infty} A_k \setminus \big(\overline{\lim}_{k\to\infty}\{M_B^*(f_k) > \lambda_k\} \bigcup A_0\big)$ is of complete measure. Let us show that for every $x \in A$ we have $D_B\Big(\int \sum_{k=1}^{\infty} f_k, x\Big) = \sum_{k=1}^{\infty} f_k(x)$, which will prove the lemma.

Let $x \in A$ and $\varepsilon > 0$. From the inclusion $x \in A$ we can conclude that:

1) there is $k_0 \in \mathbb{N}$ ($k_0 = k(x,\varepsilon)$) such that $x \notin \bigcap_{k=k_0}^{\infty} \{M_B^*(f_k) > \lambda_k\}$, $\sum_{k=k_0}^{\infty} \lambda_k < \varepsilon/3$ and $\sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3$;

2) there is $\delta > 0$ such that for every $R \in B(x)$, $\operatorname{diam} R < \delta$

$$\left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| < \varepsilon/3.$$

According to 1) and 2) we can write that for every $R \in B(x)$, $\operatorname{diam} R < \delta$,

$$\left| \frac{1}{|R|} \int_R \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x) \right| \le \left| \frac{1}{|R|} \int_R \sum_{k=1}^{k_0-1} f_k - \sum_{k=1}^{k_0-1} f_k(x) \right| +$$
$$+ \sum_{k=k_0}^{\infty} \frac{1}{|R|} \left| \int_R f_k \right| + \sum_{k=k_0}^{\infty} |f_k(x)| < \varepsilon/3 + \sum_{k=k_0}^{\infty} M_B^*(f_k)(x) +$$
$$+\varepsilon/3 \le 2\varepsilon/3 + \sum_{k=k_0}^{\infty} \lambda_k < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the equality $D_B\Big(\int \sum_{k=1}^{\infty} f_k, x\Big) = \sum_{k=1}^{\infty} f_k(x)$ is valid. □

Proof of Theorem 2. Let $k \ge 3$. Choose h_k such that $\frac{1}{k} \ln \frac{h_k}{k} \ge 18 \cdot 2^{2k}$.

Let $\alpha_k = \frac{1}{k} \ln \frac{h_k}{k} / 2^{2(k+2)} h_k$. Obviously, $0 < \alpha_k < 1$.

For $I \in \overline{B}_2$ and $h > 0$ denote $Q^0(I,h) = (2h+1)I$. Let $2^{-m} < |\operatorname{pr}_1 Q^0(I,h)|_1 \le 2^{-m+1}$ and $2^{-m'} < |\operatorname{pr}_2 Q^0(I,h)|_1 \le 2^{-m'+1}$, where $m, m' \in \mathbb{N}$. Denote by $Q(I,h)$ the interval concentric with $Q^0(I,h)$, and $|\operatorname{pr}_1 Q(I,h)|_1 = 2^{-m+1}$, $|\operatorname{pr}_2 Q(I,h)|_1 = 2^{-m'+1}$.

For the basis B denote by $M_B^{(r)}$ $(r > 0)$ the operator

$$M_B^{(r)}(f)(x) = \sup_{R \in B(x), \operatorname{diam} R < r} \frac{1}{|R|} \int_R |f| \ \ (f \in L(\mathbb{R}^n),\ x \in \mathbb{R}^n).$$

$\overline{B}_2 \notin LR(\overline{B})$, since there is $I \in \overline{B}_2$ such that:

1) there is no $J \in \overline{B}$, $J \supset I$ for which $|J| < 2^k h_k |I|$;

2) $\operatorname{diam} Q(I, 2^k h_k) < 1/k$.

Divide I^2 into intervals equal to $Q(I, 2^k h_k)$ and denote them by $Q_{1,q}$ $(1 \le q \le q_1)$. Take $I_{1,q}$ $(1 \le q \le q_1)$ equal to $T_q(I)$, where T_q is a shift translating $Q(I, 2^k h_k)$ to $Q_{1,q}$.

Let the families $\{I_{1,q}\}_{q=1}^{q_1} \dots \{I_{m,q}\}_{q=1}^{q_m}$ consisting of equal intervals be already constructed. Consider the sets

$$A_m^1 = \bigcup_{j=1}^{m}\bigcup_{q=1}^{q_j}\{M_{B_2}^{(1/k)}(h_k,\chi_{I_{j,q}}) > k\},$$

$$A_m^2 = \bigcup_{j=1}^{m}\bigcup_{q=1}^{q_j} R_{j,q} \quad (R_{j,q} = R(I_{j,q}, 2^k h_k)).$$

If $|A_m^1| > 1 - \dfrac{1}{k}$, then we stop the construction. If $|A_m^1| \le 1 - \dfrac{1}{k}$, then we shall construct the family $\{I_{m+1,q}\}_{q=1}^{q_{m+1}}$ as follows:

Consider the set $A_m = I^2 = I^2 \setminus (A_m^1 \cup A_m^2)$ which, obviously, can be represented as $G_1 \cup G_2$, where G_1 is open and G_2 consists of a finite number of smooth closed lines. It is clear that there is $\delta \in (0, 1/k)$ such that if we divide I^2 into equal intervals $\{J_j\}$ with diameters less than δ, then

$$\Big| A_m \cap \bigcup_{J_j \subset A_m} J_j \Big| \ge \Big(1 - \frac{\alpha_k}{4}\Big)|A_m|.$$

$\overline{B}_2 \notin LR(\overline{B})$, since there is $I \in \overline{B}_2$ such that:

1) there is no $J \in \overline{B}$, $J \supset I$ for which $|J| \le 2^k h_k |I|$;

2) $\operatorname{diam} Q(I, 2^k h_k) < \delta$;

3) $|\operatorname{pr}_1 I_{m,1}|_1 \ge 2^k h_k |\operatorname{pr}_1 I|_1$ and $|\operatorname{pr}_2 I_{m,1}|_1 \ge 2^k h_k |\operatorname{pr}_2 I|_1$.

Divide I^2 into intervals equal to $Q(I, 2^k h_k)$. Denote the intervals included in A_m by $Q_{m+1,q}$ $(1 \le q \le q_{m+1})$. Take $I_{m+1,q}$ $(1 \le q \le q_{m+1})$ equal to $T_q(I)$, where T_q is a shift translating $Q(I, 2^k h_k)$ to $Q_{m+1,q}$.

By our construction we obtain

$$\Big| A_m \cap \bigcup_{q=1}^{q_{m+1}} Q_{m+1,q} \Big| \ge \Big(1 - \frac{\alpha_k}{4}\Big)|A_m|. \tag{11}$$

By Lemma 1,- for $q = \overline{1, q_{m+1}}$

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} \subset \Big(2\frac{h_k}{k} - 1\Big) I_{m+1,q} \subset Q_{m+1,q}, \tag{12}$$

$$\big|\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\}\big| > \frac{h_k}{k} \ln \frac{h_k}{k} |I_{m+1,q}|. \tag{13}$$

Since $\operatorname{diam} Q_{m+1,q} < 1/k$ $(q = \overline{1, q_{m+1}})$, because of (12) we have

$$\{M_{B_2}(h_k \chi_{I_{m+1,q}}) > k\} = \{M_{B_2}^{(1/k)}(h_k \chi_{I_{m+1,q}}) > k\} \quad (q = \overline{1, q_{m+1}}). \tag{14}$$

It can be easily seen that (see (13), (14))

$$\big|\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}})>k\}\big|\geq\alpha_k|Q_{m+1,q}|\quad(q=\overline{1,q_{m+1}}),$$

which by (11) readily implies

$$\Big|\bigcup_{q=1}^{q_{m+1}}\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}})>k\}\Big|\geq\alpha_k\Big|\bigcup_{q=1}^{q_{m+1}}Q_{m+1,q}\Big|\geq$$
$$\geq\alpha_k\Big(1-\frac{\alpha_k}{4}\Big)|A_m|>\frac{\alpha_k}{2}|A_m|. \tag{15}$$

According to the construction and Lemma 2 we have for $q=\overline{1,q_{m+1}}$

$$\{M_B(h_k\chi_{I_{m+1,q}})>1/2^k\}\subset R_{m+1,q}\quad(R_{m+1,q}=R(I_{m+1,q},2^kh_k)).$$

On account of our choice of h_k, because of (13) and (14) we write

$$\big|\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}})>k\}\big|>2^k|R_{m+1,q}|\quad(q=\overline{1,q_{m+1}}),$$

whence

$$\Big|\bigcup_{q=1}^{q_{m+1}}\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}})>k\}\Big|>2^k\Big|\bigcup_{q=1}^{q_{m+1}}R_{m+1,q}\Big|. \tag{16}$$

Let us show that for sufficiently large mk the construction ceases, i.e., we shall have the inequality

$$\Big|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m}\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m+1,q}})>k\}\Big|>1-\frac{1}{k}. \tag{17}$$

Assume the contrary, i.e., $|A_m^1|\leq 1-\dfrac{1}{k}$ $(m\in\mathbb{N})$. Introduce the notation

$$\begin{aligned}A_m^3&=\bigcup_{q=1}^{q_m}\{M_{B_2}^{(1/k)}(h_k\chi_{I_{m,q}})>k\},\\A_m^4&=\bigcup_{q=1}^{q_m}R_{m,q}.\end{aligned}\qquad(m\in\mathbb{N})$$

Clearly, (15) and (16) will be valid for all $m\in\mathbb{N}$, and therefore we write

$$|A_{m+1}^3|>\frac{\alpha_k}{2}|A_m|=\frac{\alpha_k}{2}\Big(1-\sum_{j=1}^{m}|A_j^3\cup A_j^4|\Big)\geq$$
$$\geq\frac{\alpha_k}{2}\Big(1-\sum_{j=1}^{m}\Big(1+\frac{1}{2^k}\Big)|A_j^3|\Big)\geq\frac{\alpha_k}{2}\Big(1-\frac{1}{2^k}-\sum_{j=1}^{m}|A_j^3|\Big),$$

which by the equality $|A^1_m| = \sum_{j=1}^{m} |A^3_m|$ implies that $|A^1_m| > 1 - \dfrac{1}{2^{k-1}} > 1 - \dfrac{1}{k}$ for sufficiently large m. From this contradiction we conclude that (17) is valid.

Consider the function $f_k = \sum_{m=1}^{m_k} \sum_{q=1}^{q_m} h_k \chi_{I_{m,q}}$. Clearly, $f_k \in L(I^2)$ and $f_k \geq 0$. From (17) we get

$$\left|\{M^{(1/k)}_{B_2}(f_k) > k\}\right| > 1 - \frac{1}{k}. \tag{18}$$

From the construction we can easily see that $2^k h_k$ and the families $\{I_{1,q}\}_{q=1}^{q_1}, \dots, \{I_{m_k,q}\}_{q=1}^{q_{m_k}}$ satisfy all the conditions of Lemma 6. Therefore by Lemma 6 we write

$$\begin{aligned} &\{M_B(f_k) > 1/2^{k-1}\} = \{M_B(2^k f_k) > 2\} = \\ &= \Big\{M_B\Big(\sum_{m=1}^{m_k}\sum_{q=1}^{q_m} 2^k h_k \chi_{I_{m,q}}\Big) > 2\Big\} \subset \bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} R_{m,q}. \end{aligned} \tag{19}$$

Obviously, (16) is fulfilled for all $m \in [1, m_k - 1]$, so that taking the construction into account, we have

$$2^k\left|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} R_{m,q}\right| < \left|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m}\{M^{(1/k)}_{B_2}(h_k\chi_{I_{m,q}}) > k\}\right| \leq 1. \tag{20}$$

Due to (19) and (20) we write

$$\left|\{M_B(f_k) > 1/2^{k-1}\}\right| < \frac{1}{2^k}. \tag{21}$$

It can be easily seen that

$$\frac{h_k}{k}\ln\frac{h_k}{k}\left|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} I_{m,q}\right| < \left|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m}\{M^{(1/k)}_{B_2}(h_k\chi_{I_{m,q}}) > k\}\right| \leq 1,$$

Hence, due to our choice of h_k, we obtain

$$\|f_k\|_1 = h_k\left|\bigcup_{m=1}^{m_k}\bigcup_{q=1}^{q_m} I_{m,q}\right| < 1\Big/\frac{1}{k}\ln\frac{h_k}{k} < \frac{1}{2^k}. \tag{22}$$

Consider the function $f = \sum_{k=3}^{\infty} f_k$. Clearly, $f \in L(I^2)$ (see (22)) and $f \geq 0$.

It is obvious that if $x \in \overline{\lim\limits_{k\to\infty}}\{M^{(1/k)}_{B_2}(f_k) > k\}$, then $\overline{D}_{B_2}\Big(\int \sum_{k=3}^{\infty} f_k, x\Big) = \infty$. Because of (18) $\overline{\lim\limits_{k\to\infty}}\{M^{(1/k)}_{B_2}(f_k) > k\}$ has a complete measure on I^2. Therefore the latter equality holds almost everywhere on I^2.

It is clear that B differentiates $\int f_k$ ($k \in \mathbb{N}$). Moreover, owing to (21) we get $\sum_{k=3}^{\infty} \big|\{M_B(f_k) > 1/2^{k-1}\}\big| < \sum_{k=3}^{\infty} \frac{1}{2^k} < \infty$, which by Lemma 7 implies that B differentiates $\int f$. $\square$

After making some technical changes in the proof of Theorem 2 we can obtain the following generalization.

Theorem 3. *Let $B \subset B_2$ be a TI-basis. If $\overline{B}_2$ is not locally regular with respect to $\overline{B}$ then for any function $f \in L\backslash L\ln^+ L(I^2)$, $f \geq 0$, there is a Lebesgue measure-preserving and invertible mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$, $\{x : \omega(x) \neq x\} \subset I^2$ such that*

$$\overline{D}_{B_2}\Big(\int f \circ \omega, x\Big) = \infty \ \text{ a.e. on } \ I^2,$$
$$D_B\Big(\int f \circ \omega, x\Big) = (f \circ \omega)(x) \ \text{ a.e. on } \ I^2.$$

To prove the implication $(c) \Rightarrow (b)$ let us show the validity of

Lemma 8. *Let $B \subset B_2$ be a TI-basis. If $\overline{B}_2$ is locally regular with respect to $\overline{B}$, then for every $f \in L(\mathbb{R}^2)$*

$$\big|\{M_{B_2}^{(r)}(f) > \lambda\}\big| \leq 25\Big|\Big\{M_B^{(cr)}(f) > \frac{\lambda}{4c}\Big\}\Big| \ \ (\lambda > 0, 0 < r < \delta),$$

where δ and c are the constants from the definition of local regularity of $\overline{B}_2$ with respect to $\overline{B}$.

Proof. Note first that if I and I' ($I \subset I'$) are the one-dimensional intervals and I^* is either the left or the right half of the interval I, then for the point $y \in I'$ the set $E = \{z \in \mathbb{R} : (I' + z - y) \supset I^*\}$ possesses the following properties:

1) E is an interval;
2) $E \cap I' \neq \varnothing$;
3) $|E|_1 = |I'|_1 - \dfrac{1}{2}|I|_1 \geq \dfrac{1}{2}|I'|_1$.

It follows from these properties that

4) $5E \supset I'$

is likewise valid.

Let $f \in L(\mathbb{R}^2)$. Denote by P the set of all two-dimensional intervals $E \in \overline{B}_2$ such that $E \subset \Big\{M_B^{(cr)}(f) > \dfrac{\lambda}{4c}\Big\}$ and show the inclusion

$$\{M_{B_2}^{(r)}(f) > \lambda\} \subset \bigcup_{E \in P} 5E \ \ (\lambda > 0, \ 0 < r < \delta). \tag{23}$$

Let $x \in \{M^{(r)}_{B_2}(f) > \lambda\}$ $(\lambda > 0,\ 0 < r < \delta)$. Then there is $I \in B_2(x)$ such that $\operatorname{diam} I < r$ and

$$\frac{1}{|I|}\int_I |f| > \lambda. \tag{24}$$

Since $\overline{B}_2 \in LR(\overline{B})$, there is an interval I' from $\overline{B}$ such that $I \subset I'$ and $|I'| \le c|I|$. Then it is obvious that $\operatorname{diam} I' \le c \operatorname{diam} I < cr$. Let $y \in I'$ be the point for which $I' \in B(y)$. Divide the interval I into four equal intervals. Because of (24) the integral mean will be greater than λ at least on one of these intervals. Denote this interval by I^*. Thus

$$\frac{1}{|I^*|}\int_{I^*} |f| > \lambda. \tag{25}$$

Consider the set $E = \{z \in \mathbb{R}^2 : (I' + z - y) \supset I^*\}$. By virtue of the above discussion we can easily note that

1) $E \in \overline{B}_2$;
2) $E \cap I' \ne \varnothing$;
3) $|E| \ge \frac{1}{4}|I'|$;
4) $5E \supset I'$.

Since $x \in I \subset I'$, then to prove (23) it suffices to show that $E \in P$. Let $z \in E$. Then $(I' + z - y) \supset I^*$. Since B is the TI-basis and $I' \in B(y)$, we have $(I' + z - y) \in B(z)$. Moreover, owing to (25), we have

$$\frac{1}{|I' + z - y|}\int_{I'+z-y} |f| \ge \frac{1}{|I'|}\int_{I^*} |f| \ge \frac{1}{c|I|}\int_{I^*} |f| = \frac{1}{c4|I^*|}\int_{I^*} |f| > \frac{\lambda}{4c},$$

from which by the inequality $\operatorname{diam}(I' + z - y) < cr$, we conclude that $z \in \Big\{M^{(cr)}_B(f) > \dfrac{\lambda}{4c}\Big\}$. Hence $E \in P$. Thus inclusion (23) is proved.

Now, using (23) and Lemma 1 from [2], we will have

$$\big|\{M^{(r)}_{B_2}(f) > \lambda\}\big| \le \Big|\bigcup_{E\in P} 5E\Big| \le 25\Big|\bigcup_{E\in P} E\Big| \le 25\Big|\Big\{M^{(cr)}_B(f) > \frac{\lambda}{4c}\Big\}\Big|$$

for $\lambda > 0$, $0 < r < \delta$. □

The basis B is said to be regular if there is a constant $c < \infty$ such that for every $R \in \overline{B}$ there exists a cubic interval Q with the following properties: $Q \supset R$, $|Q| \le c|R|$.

According to the well-known Lebesgue theorem on differentiability of integrals, the regular basis B differentiates L, i.e., $D_B\big(\int f, x\big) = f(x)$ a.e. on $\mathbb{R}^n$ for every $f \in L(\mathbb{R}^n)$.

We shall say that the basis B possesses property (E) if for every $f \in L(\mathbb{R}^n)$, $\underline{D}_B\big(\int f, x\big) \le f(x) \le \overline{D}_B\big(\int f, x\big)$ a.e. on $\mathbb{R}^n$.

Obviously, the basis B, containing a regular subbasis, possesses the property (E).

It directly follows from the relation $\overline{B}_2 \in LR(\overline{B})$ that B contains a regular subbasis, and hence possesses property (E). Now, to prove the implication $(b) \Rightarrow (c)$, it suffices to consider the following assertion.

Lemma 9. *Let B be a subbasis of B_2 with property (E). Suppose that there exist positive constants c_1, c_2, c_3 and δ such that for every $f \in L(\mathbb{R}^2)$,*

$$\big|\{M_{B_2}^{(r)}(f) > \lambda\}\big| \le c_1 \Big|\Big\{M_B^{(c_2 r)}(f) > \frac{\lambda}{c_3}\Big\}\Big| \ \ (\lambda > 0,\ 0 < r < \delta).$$

Then for every $f \in L(\mathbb{R}^2)$, $f \ge 0$,

$$\underline{D}_B\Big(\int f, x\Big) = \underline{D}_{B_2}\Big(\int f, x\Big) \ \text{ and } \ \overline{D}_B\Big(\int f, x\Big) = \overline{D}_{B_2}\Big(\int f, x\Big),$$

a.e. on $\mathbb{R}^2$.

The proof of Lemma 9 is based on the well-known Besicovitch theorem on possible values of upper and lower derivative numbers (see [1], Ch.IV, §3) and is carried out analogously to the proof of Theorem 1 from [2].

3. On a Possibility of Extending Theorem 1 to More General Bases

It should be noted that beyond the scope of TI-bases Theorem 1 becomes invalid. Moreover, even for bases which are similar enough to TI-bases the local regularity of $\overline{B}_2$ with respect to $\overline{B}$ is insufficient for the equality $F_B^+ = F_{B_2}^+$ to be fulfilled.

A basis B will be called a TI^*-basis if for every $x \in \mathbb{R}^n$, $R \in B(x)$ and $y \in \mathbb{R}^n$ there is a translation T such that $T(R) \in B(y)$.

The following theorem is valid.

Theorem 4. *There is a TI^*-basis $B \subset B_2$ with the properties:*
1) *$\overline{B}_2$ is locally regular with respect to $\overline{B}$, and $B(O) = B_2(O)$;*
2) *there is a function $f \in L(I^2)$, $f \ge 0$ such that*

$$\overline{D}_{B_2}\Big(\int f, x\Big) = \infty \ \text{ a.e. on } I^2,$$
$$D_B\Big(\int f, x\Big) = f(x) \ \text{ a.e. on } I^2.$$

Proof. Consider the sequences $\alpha_k \uparrow \infty$, $\alpha_k > 0$ and $h_k \uparrow \infty$, $h_k > 10$ with the properties

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty, \ \ \sum_{k=1}^{\infty} \frac{\ln h_k}{h_k} = \infty. \tag{26}$$

Let $q_k = 2^{m_k}$, where $m_k \in \mathbb{N}$ $(k \in \mathbb{N})$ and $m_k \uparrow \infty$. For every k, we divide I^2 into q_k^2 equal square interval and denote them by $I_{k,q}^2$. The length of square intervals sides will be denoted by Δ_k. For every $I_{k,q}^2$ let us consider the square interval $I_{k,q}$ concentric with $I_{k,q}^2$ having the side length $\delta_k = \Delta_k/(2h_k+1)$. We shall assume $q_k \uparrow \infty$ so that $\delta_k > \Delta_{k+1}$ $(k \in \mathbb{N})$.

Let $f_k = \sup\{\alpha_k h_k \chi_{I_{k,q}} : q = \overline{1, q_k^2}\}$ $(k \in \mathbb{N})$ and $f = \sum_{k=1}^{\infty} f_k$. Clearly, $f \geq 0$. It can be easily checked that

$$\|f_k\|_1 = \alpha_k h_k \Big|\bigcup_{q=1}^{q_k^2} I_{k,q}\Big| = \alpha_k h_k \sum_{q=1}^{q_k^2} \frac{1}{(2h_k+1)^2} |I_{k,q}^2| < \frac{\alpha_k}{h_k},$$

which according to (26) implies

$$\|f\|_1 \leq \sum_{k=1}^{\infty} \|f_k\|_1 < \sum_{k=1}^{\infty} \frac{\alpha_k}{h_k} < \infty,$$

i.e., $f \in L(I^2)$.

By Lemma 1 for $k \in \mathbb{N}$, $q = \overline{1, q_k^2}$ we have

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset (2h_k+1) I_{k,q} = I_{k,q}^2, \tag{27}$$

and

$$\big|\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\big| \geq h_k \ln h_k |I_{k,q}|.$$

Thus we write

$$\Big|\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\Big| = \sum_{q=1}^{q_k^2} \big|\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\big| \geq$$
$$\geq \sum_{q=1}^{q_k^2} h_k \ln h_k |I_{k,q}| = \sum_{q=1}^{q_k^2} h_k \ln h_k \frac{|I_{k,q}^2|}{(2h_k+1)^2} > \frac{\ln h_k}{9h_k} \tag{28}$$

for $k \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let us consider those intervals $I_{k+1,q}^2$ $(q = \overline{1, q_{k+1}^2})$ which are contained in $\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}$. Denote their union by O_k. It can be easily seen that if $\{q_k\}$ tends rapidly enough to ∞, then

$$|O_k| \geq \frac{1}{2} \Big|\bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\}\Big| \quad (k \in \mathbb{N}). \tag{29}$$

Using the property of dyadic intervals, we can see that $\{O_k\}$ is the sequence of independent sets, i.e., for every $n \geq 2$ and for arbitrary pairwise different natural numbers $k_1, k_2 \ldots, k_n$,

$$\Big|\bigcap_{m=1}^{n} O_{k_m}\Big| = \prod_{m=1}^{n} |O_{k_m}|.$$

Owing to (26), (28), and (29), $\sum_{k=1}^{\infty} |O_k| > \sum_{k=1}^{\infty} \frac{1}{2}\frac{\ln h_k}{9h_k} = \infty$. Using now Borel–Kantelly's lemma, we get

$$\overline{\lim_{k\to\infty}} O_k, \text{ has a complete measure in } I^2. \tag{30}$$

From (27) we have for $k \in \mathbb{N}$, $q = \overline{1, q_k^2}$

$$\{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} = \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\},$$

which for $k \in \mathbb{N}$ implies

$$O_k \subset \bigcup_{q=1}^{q_k^2} \{M_{B_2}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} =$$

$$= \bigcup_{q=1}^{q_k^2} \{M_{B_2}^{(\Delta_k)}(\alpha_k h_k \chi_{I_{k,q}}) > \alpha_k\} \subset \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}.$$

Because of (30), from the latter inclusion we obtain

$$\overline{\lim_{k\to\infty}} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\} \text{ has a complete measure in } I^2. \tag{31}$$

Since $\alpha_k \uparrow \infty$, $\Delta_k \downarrow 0$ $(k \to \infty)$, we can easily see that if $x \in \overline{\lim\limits_{k\to\infty}} \{M_{B_2}^{(\Delta_k)}(f_k) > \alpha_k\}$, then $\overline{D}_{B_2}\big(\int f, x\big) = \infty$. By virtue of (31) we conclude that the latter equality holds a.e. on I^2.

Assume k to be fixed. For every $I_{k,q}$ $(q = \overline{1, q_k^2})$ let us consider the intervals $J_{k,q}^1$ $(\mathrm{pr}_1\, J_{k,q}^1 = 3\, \mathrm{pr}_1\, I_{k,q},\ \mathrm{pr}_2\, J_{k,q}^1 = (0,1))$ and $J_{k,q}^2$ $(\mathrm{pr}_1\, J_{k,q}^2 = (0,1),\ \mathrm{pr}_2\, J_{k,q}^2 = 3\, \mathrm{pr}_2\, I_{k,q})$. Denote $G_k^i = \bigcup\limits_{q=1}^{q_k^2} J_{k,q}^i$ $(i = \overline{1,2})$, $G_k = G_k^1 \cup G_k^2$. It can be easily verified (see (26)) that $\sum_{k=1}^{\infty} |G_k| < \infty$ which implies that $\overline{\lim\limits_{k\to\infty}} G_k$ has zero measure.

Determine now an unknown basis B. If $x \in \Big(\overline{\lim\limits_{k\to\infty}} G_k\Big) \cup \mathbb{Q}^2 \cup (\mathbb{R}^2 \backslash I^2)$ ($\mathbb{Q}$ is a set of rational numbers), then we put $B(x) = B_2(x)$. For $x \in I^2 \backslash \Big(\overline{\lim\limits_{k\to\infty}} G_k \cup \mathbb{Q}^2\Big)$ choose $B(x)$ such that if d_I is the length of the lesser side of the interval $I \ni x$, and $d_I \notin \bigcup_{k\geq k(x)} [\delta_k, \Delta_k/2]$ (for $x \notin \overline{\lim_{k\to\infty}} G_k$, $k(x)$ is the number with the property $x \notin G_k$ for $k \geq k(x)$), then $I \in B(x)$, while if $d_I \in [\delta_k, \Delta_k/2]$ for some $k \geq k(x)$, then $I \in B(x)$ iff either $I \cap G_k^1 = \varnothing$ or $I \cap G_k^2 = \varnothing$.

It can be easily verified that for every $I \in \overline{B}_2$ and $x \in \mathbb{R}^2$ there is a translation T, $T(I) \in B(x)$; this implies that B is the TI^*-basis. The local regularity of $\overline{B}_2$ with respect to $\overline{B}$ is also easily checked. By proving the inclusion $f \in F_B^+$ the proof of the theorem will be completed.

Denote $A_k = \{x \in \mathbb{R}^n : D_B(\int f_k, x) = f_k(x)\}$ ($k \in \mathbb{N}$). Clearly, A_k ($k \in \mathbb{N}$) has a complete measure on $\mathbb{R}^2$. Denote $A = \Big(I^2 \cap \bigcap_{k=1}^{\infty} A_k\Big) \backslash \Big(\overline{\lim_{k\to\infty}} G_k \cup \mathbb{Q}^2\Big)$. Since $f(x) = 0$ for $x \notin I^2$, B differentiates $\int f$ on $\mathbb{R}^2 \backslash I^2$; A is the set of a complete measure on I^2. Therefore to prove the inclusion $f \in F_B^+$, it suffices to show the differentiability of $\int f$ with respect to B on the set A.

Let $x \in A$ and $\varepsilon > 0$. Find $\delta > 0$ for which $\Big|(1/|I|)\int_I f - f(x)\Big| < \varepsilon$ when $I \in B(x)$, $\operatorname{diam} I < \delta$. Consider $k'(x) \geq k(x)$ for which

$$\sum_{k\geq k'(x)} 16\|f_k\|_1 < \varepsilon/2. \tag{32}$$

It is clear that $\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{k'(x)-1} f_k(x)$, and B differentiates $\int \sum_{k=1}^{k'(x)-1} f_k$ at the point x. This implies that there is $\delta \in (0, \Delta_{k'(x)}/2)$ such that

$$\Big|\frac{1}{|I|}\int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x)\Big| < \varepsilon/2, \ \text{ for } \ I \in B(x), \ \operatorname{diam} I < \delta. \tag{33}$$

Consider an arbitrary interval $I \in B(x)$, $\operatorname{diam} I < \delta$. For $k_I \in \mathbb{N}$ let $\Delta_{K_I+1}/2 \leq d_I \leq \Delta_{k_I}/2$. Clearly, $k_I \geq k'(x)$. By virtue of the inequality $\Delta_{k+1} < \delta_k$ ($k \in \mathbb{N}$) and the condition $x \notin G_k$ ($k \geq k(x)$), we have $I \cap \operatorname{supp} f_k = \varnothing$ for $k'(x) \leq k \leq k_I - 1$, which gives

$$\int_I f_k = 0 \ \text{ for } \ k'(x) \leq k \leq k_I - 1. \tag{34}$$

Analogously, $\int_I f_{k_I} = 0$ if $d_I < \delta_{k_I}$, while if $\delta_{k_I} \leq d_I \leq \Delta_{k_I}/2$, then $\int_I f_{k_I} = 0$ because $I \cap \operatorname{supp} f_k = \varnothing$ by construction of the basis B. Thus

$$\int_I f_{k_I} = 0. \tag{35}$$

Let is show that

$$|I \cap \operatorname{supp} f_k| \leq 16|I|\, |\operatorname{supp} f_k| \ \text{ for } \ k \geq k_I + 1. \tag{36}$$

To this end it is sufficient to consider the following easily verifiable facts:

1) $d_I \geq \Delta_k/2$ for $k \geq k_I + 1$;

2) let P_k be a set of vertices of the squares $I^2_{k,q}$ $(q = \overline{1, q_k^2})$. If the vertices of the interval $J \subset I^2$ belong to the set P_k, then $|J \cap \operatorname{supp} f_k| = |I||\operatorname{supp} f_k|$;

3) for every interval $J \subset I^2$, $d_J \geq \Delta_k/2$ there is an interval $J' \supset J$, $|J'| \leq 16|J|$ with vertices at the points of the set P_k.

By (36) we can write that for $k \geq k_I + 1$,

$$\int_I f_k = \alpha_k h_k |I \cap \operatorname{supp} f_k| \leq 16\alpha_k h_k |I|\, |\operatorname{supp} f_k| = 16|I|\, \|f_k\|_1,$$

which, owing to (32) and the inequality $k_I \geq k'(x)$, implies

$$\frac{1}{|I|}\int_I \sum_{k \geq k_I+1} f_k = \frac{1}{|I|}\sum_{k\geq k_I+1}\int_I f_k \leq \frac{1}{|I|}\sum_{k\geq k_I+1} 16|I|\, \|f_k\|_1 < \varepsilon/2. \quad (37)$$

By virtue of relations (33)–(35) and (37) (it should also be noted that $f_k(x) = 0$ for $k \geq k'(x)$), we have

$$\left|\frac{1}{|I|}\int_I \sum_{k=1}^{\infty} f_k - \sum_{k=1}^{\infty} f_k(x)\right| \leq \left|\frac{1}{|I|}\int_I \sum_{k=1}^{k'(x)-1} f_k - \sum_{k=1}^{k'(x)-1} f_k(x)\right| +$$

$$+\frac{1}{|I|}\int_I \sum_{k=k'(x)}^{k_I} f_k + \frac{1}{|I|}\int_I \sum_{k\geq k_I+1} f_k + \sum_{k\geq k'(x)} f_k(x) < \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} + 0 = \varepsilon.$$

Due to the arbitrariness of $x \in A$ and $\varepsilon > 0$, we conclude that $D_B\left(\int f, x\right) = f(x)$ for every $x \in A$. □

4. Remarks

(1) Let us consider one application of the obtained results. Let B_Z be the basis in $\mathbb{R}^2$ for which $B_Z(x)$ $(x \in \mathbb{R}^2)$ consists of all intervals $I \ni x$, $D_I^2 \leq d_I \leq D_I \leq 1$, where D_I and d_I are the lengths, respectively, of the greater and of the lesser side of I. This basis was introduced by Zygmund and it was he who initiated (see [1], Ch. VI, §4) the study of the differential properties of B_Z.

Morion showed (see [1], Appendix IV) that for the integral classes B_Z behaves like B_2, i.e., B_Z does not differentiate a wider integral class than $L\ln^+ L$.

A question arises: does there exist in general a function whose integral does not differentiate B_2 and differentiates B_Z? From the above proven theorems the answer is positive.

B_Z is the TI-basis. The fact that $\overline{B}_2 \notin LR(\overline{B}_z)$ is easily verified. By Theorem 1 this implies that the strict inclusion $F^+_{B_2} \subset F^+_{B_Z}$ holds. Moreover, Theorem 3 implies that by perturbing the values ω of any function $f \in L\backslash L\ln^+ L(I^2)$, $f \geq 0$, one can get the function $f \circ \omega$ with the following properties: $\overline{D}_{B_2}\big(\int f \circ \omega, x\big) = \infty$ a.e. on I^2, and B_Z differentiates $\int f \circ \omega$.

Thus, for the integral classes the bases B_2 and B_Z behave similarly, while for individual nonnegative functions the basis B_Z behaves better than the basis B_2.

(2) The bases B and B' are said to be positive equivalent ($B \overset{+}{\Leftrightarrow} B'$) if for every $f \in L(\mathbb{R}^n)$, $f \geq 0$, $\overline{D}_B\big(\int f, x\big) = \overline{D}_{B'}\big(\int f, x\big)$ and $\underline{D}_B\big(\int f, x\big) = \underline{D}_{B'}\big(\int f, x\big)$ a.e. on $\mathbb{R}^n$ (i.e., condition (b) in Theorem 1 means that the bases B and B' are positive equivalent).

B is said to be a Busemann–Feller type basis (BF-basis) if for any $R \in \overline{B}$ and $x \in R$ we have $R \in B(x)$.

We shall say that B exactly differentiates $\varphi(L)$ (writing $B \in D(\varphi(L))$ if B differentiates $\varphi(L)$ and does not differentiate a wider integral class than $\varphi(L)$.

For integral classes the behavior of $B \subset B_2$, BF, TI-bases was studied by Stokoloc in [3], where he introduced the property (S) and proved that if B possesses the property (S), then $B \in D(L\ln^+ L)$, and if B does not possess this property, then $B \in D(L)$.

One can easily see that ignoring the BF property, this result remains valid. Thus, for the integral classes $B \subset B_2$, the TI-basis behaves like B_2 or B_1 (B_1 is the basis formed of square intervals). The B_Z-basis illustrates that an analogous fact does not hold for nonnegative individual functions and, generally speaking, if we combine Stokoloc's assertion and Theorem 1, then we shall have:

1) if $\overline{B}_2 \in LR(\overline{B})$, then $B \overset{+}{\Leftrightarrow} B_2$;

2) if $\overline{B}_2 \notin LR(\overline{B})$ and B possesses the property (S), then $B \in D(L\ln^+ L)$ and $F^+_{B_2} \subset F^+_B$ (strictly);

3) if B does not possess the property (S), then $B \in D(L)$.

(3) Let $B \subset B_2$ be a TI-basis. Consider the intervals $I \in \overline{B}$ of the type $I = (0, x^1) \times (0, x^2)$. Denote the set of points (x^1, x^2) by A_B. The set A_B indicates how rich the family $\overline{B}$ is.

One can easily prove the following criterion of local regularity of $\overline{B}_2$ with respect to $\overline{B}$: $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\exists m, k_0 \in \mathbb{N} : A_B \cap ([1/m^k, 1/m^{k-1}) \times [1/m^{k'}, 1/m^{k'-1})) \neq \varnothing$ for $k, k' > k_0)$.

(4) The basis B is said to be invariant (HI-basis) with respect to homotheties if for every $x \in \mathbb{R}^n$ and every homothety H centered at x we have $B(x) = \{H(R) : R \in B(x)\}$.

If $\overline{B}$ is locally regular with respect to $\overline{B}'$ and, moreover, if δ is equal to ∞, then $\overline{B}$ is called regular with respect to $\overline{B}'$.

For the basis $B \subset B_2$ define the sets: $R_{1,B} = \{r > 1 : \exists I \in \overline{B}, |\operatorname{pr}_1 I|_1 = r|\operatorname{pr}_2 I|_1\}$, $R_{2,B} = \{r > 1 : \exists I \in \overline{B}, |\operatorname{pr}_2 I|_1 = r|\operatorname{pr}_1 I|_1\}$.

For the basis $B \subset B_2$ which is simultaneously the TI- and HI basis one can easily verify that $(\overline{B}_2 \in LR(\overline{B})) \Leftrightarrow (\overline{B}_2$ is regular with respect to $B) \equiv\leftrightarrow (\exists m \in \mathbb{N} : R_{i,B} \cap [m^k, m^{m+1}) \neq \varnothing,\ k \in \mathbb{N},\ i = \overline{1,2})$.

(5) Let B_2 $(B_2 = B_2(\mathbb{R}^n))$ be the basis in $\mathbb{R}^n$ for which $B_2(x)$ $(x \in \mathbb{R}^n)$ consists of all n-dimensional intervals containing the point x.

Theorems 1–4 are also valid for $B \subset B_2(\mathbb{R}^n)$ $(n \geq 3)$, TI-bases. Proofs for the n-dimensional case are similar to those for the two-dimensional case.

(6) Let $\delta_k^1 \downarrow 0, \dots, \delta_k^n \downarrow 0$ $(k \to \infty)$. Denote $\Delta_{k,m}^i = [(m-1)\delta_k^i, m\delta_k^i)$ $(i = \overline{1,n}; k, m \in \mathbb{N})$. Let B be the basis in $\mathbb{R}^n$ for which $B(x) = \{\prod_{i=1}^n \Delta_{k^i,m^i}^i : \prod_{i=1}^n \Delta_{k^i,m^i}^i \ni x; k^i, m^i \in \mathbb{N})$. Such bases are sometimes called nets. We call them N-bases.

Let B be the N-basis in $\mathbb{R}^n$ $(n \geq 2)$. Denote by B_T the least TI-basis containing B. For B the following analogue of Theorem 1 is valid: the following conditions are equivalent: (i) $F_B^+ = F_{B_2}^+$; (ii) $B \overset{+}{\Leftrightarrow} B_2$; (iii) $\overline{B}_2$ is locally regular with respect to $\overline{B}_T$.

Obviously, (ii)⇒(i), the implication (i)⇒(iii) follows directly from Theorem 1. Thus it remains only to show that (iii)⇒(i). To this end, note that using Lemma 1 of [2] one can easily obtain the inequality

$$|\{M_{B_T}^{(r)}(f) > \lambda\}| \leq 3^n \left|\left\{M_B^{(r)}(f) > \frac{\lambda}{2^n}\right\}\right| \quad (f \in L(\mathbb{R}^n); \lambda, r > 0).$$

From this and Lemma 5 we get the upper bound of the distribution function of $M_{B_2}^{(r)}(f)$ by means of the distribution function of $M_B^{(cr)}(f)$. It easily follows from the relation $\overline{B}_2 \in LR(\overline{B})$ that B contains a regular subbasis. Hence B possesses the property (E). Next, we obtain the implication (iii)⇒(ii) from Lemma 6.

Note that for $\delta_k^i = 1/2^k$ $(i = \overline{1,n}; k \in \mathbb{N})$ the relation $B \overset{+}{\Leftrightarrow} B_2$ was proved earlier in [2].

(7) The basis B constructed in Theorem 4 is not a BF-basis, which is not a casual fact. In particular, the following assertion is valid: let $B \subset B_2$ $(B_2 = B_2(\mathbb{R}^n))$ be a BF-basis and let $\overline{B}_2 \in LR(\overline{B})$. Then $B \overset{+}{\Leftrightarrow} B_2$.

To prove the above assertion we have to consider the following facts which easily follow from the Busemann-Feller property of the basis B and from the relation $\overline{B}_2 \in LR(\overline{B}$: (i) B contains a regular subbasis; (ii) for every $f \in L(\mathbb{R}^n)$,

$$|\{M_{B_2}^{(r)}(f) > \lambda\}| \leq \left|\left\{M_B^{(cr)}(f) > \frac{\lambda}{c}\right\}\right| \quad (\lambda > 0,\ 0 < r < \delta),$$

where c and δ are the constants from the definition of local regularity of $\overline{B}_2$ with respect to $\overline{B}$.

From (i) we have that B possesses the property (E). Now, taking into account (ii) and using Lemma 6, we can conclude that the relation $B \overset{+}{\Leftrightarrow} B_2$ is valid.

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