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IMBEDDINGS BETWEEN WEIGHTED ORLICZ–LORENTZ SPACES

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ABSTRACT. We establish necessary and sufficient conditions for imbeddings of weighted Orlicz–Lorentz spaces.

1. INTRODUCTION

The purpose of this paper is to present transparent and verifiable necessary and sufficient conditions for imbeddings in a fairly general class of weighted spaces which include important representatives of r. i. spaces as Lorentz and Orlicz spaces.

Necessary and sufficient conditions for imbeddings of weighted spaces L^p were found by Avantaggiati [1] and Kabaila [2]; the latter author also considered measures not necessarily absolutely continuous with respect to the Lebesgue measure. The case of Orlicz spaces with certain mild conditions imposed on the growth of Young functions involved was the subject of Krbec and Pick's paper [3]. Here we shall consider the natural amalgam of Orlicz and Lorentz spaces, permitting one to arrive at integral conditions involving the weights in question. The generality of the concept enables one to give proofs actually simpler than those for Orlicz and/or Lorentz spaces. Observe that abstract conditions in terms of dual spaces can be given in Lorentz spaces (see Pick [4]).

Let us introduce the notation. Throughout the paper, Ω will be a measurable subset of the Euclidean space $\mathbb{R}^N$, ϱ and σ will stand for *weights* in Ω which are measurable, locally integrable, and a.e. positive function in Ω. A *Young function* F is an even continuous and non-negative function in $\mathbb{R}^1$, increasing on $(0,\infty)$, such that $\lim_{t\to 0_+} F(t)=0$, $\lim_{t\to\infty} F(t)=\infty$, $F(t)=0$ iff $t=0$. A Young function F is said to satisfy the *global* Δ_2*-condition* if there is $c>0$ such that $F(2t)\le cF(t)$ for all $t\in\mathbb{R}^1$.

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The Young functions F_0 and F_1 are said to be *equivalent* (we write $F_0 \sim F_1$) if there is a constant $c > 0$, such that

$$F_1(c^{-1}t) \leq F_0(t) \leq F_1(ct), \qquad t > 0.$$

If F is a Young function, then

$$\operatorname{comp} F(t) = \sup\{|ts| - F(s);\ s \in \mathbb{R}^1\}$$

is the *complementary function with respect to* F. If F is convex, then $\operatorname{comp} F(t)$ is equivalent to any Young function M such that

$$c_M^{-1} t \leq M^{-1}(t) F^{-1}(t) \leq c_M t, \qquad t > 0,$$

where c_M is a constant independent of t. In this case, the latter condition is sometimes used as an (equivalent) definition of $\operatorname{comp} F$.

Let F be a Young function and ϱ a weight in Ω. The *weighted Orlicz space* $L_{F,\varrho} = L_{F,\varrho}(\Omega)$ is the linear hull of the *weighted Orlicz class*

$$\widetilde{L}_{F,\varrho} = \widetilde{L}_{F,\varrho}(\Omega) = \left\{ f;\ \int_\Omega F(f(x))\varrho(x)\,dx < \infty \right\}.$$

The space $L_{F,\varrho}$ is equipped with the *Luxemburg functional*

$$\|f\|_{F,\varrho} = \inf\left\{ \lambda > 0;\ \int_\Omega F\left(f(x)/\lambda\right)\varrho(x)\,dx \leq 1 \right\}.$$

Let f be a measurable function in Ω and $m_\varrho(f,t)$ be the *weighted distribution function of* f, i.e.,

$$m_\varrho(f,\lambda) = \int_{\{x;\ |f(x)|>\lambda\}} \varrho(x)\,dx = \varrho(\{x;\ |f(x)| > \lambda\}),$$

and f_ϱ^* be the corresponding *weighted nonincreasing rearrangement of* f,

$$f_\varrho^*(t) = \inf\{\lambda;\ m_\varrho(f,\lambda) \leq t\}.$$

Further, let $1 \leq q, r < \infty$. Then

$$L_{q,r,\varrho} = L_{q,r,\varrho}(\Omega) = \left\{ f;\ \|f\|_{q,r,\varrho} = \left(\int_0^\infty [t^{1/q} f_\varrho^*(t)]^r \,\frac{dt}{t} \right)^{1/r} < \infty \right\}$$

is the *weighted Lorentz space*. As usual, for $r = \infty$, we put

$$L_{q,\infty,\varrho} = L_{q,\infty,\varrho}(\Omega) = \left\{ f;\ \|f\|_{q,\infty,\varrho} = \sup_{t>0} t^{1/q} f_\varrho^*(t) < \infty \right\}$$

and let us call the latter space the *weak weighted Lorentz* (*weighted Marcinkiewicz*) *space*.

Let us recall at least some of the basic references concerning the theory of Lorentz and Orlicz spaces such as the well-known monographs of Butzer and Berens [5], Nakano [6], Krasnosel'skii and Rutitskii [7], Musielak [8], and Ren and Rao [9].

Next, we define Orlicz–Lorentz spaces. Because of the nonhomogeneity of Young functions one can do this in several different ways, the L^p and $L^{p,q}$ spaces being always included as a special case. We shall follow the definition used in the recent papers, e.g., in Montgomery-Smith [10]: Let F and G be Young functions and ϱ a weight in Ω. For a function h even on $\mathbb{R}^1$ and positive on $(0,\infty)$ put

$$\widetilde{h}(t) = \begin{cases} 1/h(1/t), & t>0, \\ 1/h(-1/t), & t<0, \\ h(0), & t=0. \end{cases}$$

We define the *weighted Orlicz–Lorentz space* $L_{F,G,\varrho}$ as the set of all measurable f's on Ω for which the *Orlicz–Lorentz functional*

$$\begin{aligned} \|f\|_{F,G,\varrho} &= \|f^*_\varrho \circ \widetilde{F} \circ \widetilde{G}^{-1}\|_G = \\ &= \inf\left\{\lambda>0;\ \int_0^\infty G\left(\frac{f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(t)))}{\lambda}\right)dt \le 1\right\} \end{aligned} \tag{1.1}$$

is finite.

The *weak weighted Orlicz (Orlicz–Marcinkiewicz) space* $L_{F,\infty,\varrho}$ is the set of all measurable f's on Ω such that their *Orlicz–Marcinkiewicz functional*

$$\|f\|_{F,\infty,\varrho} = \sup_{t>0} \widetilde{F}^{-1}(t) f^*_\varrho(t) \tag{1.2}$$

is finite.

We shall write $L_{F_1,G_1,\varrho} \hookrightarrow L_{F_0,G_0,\sigma}$ if $\|f\|_{F_0,G_0,\sigma} \le \text{const}\ \|f\|_{F_1,G_1,\varrho}$ for all $f \in L_{F_1,G_1,\varrho}$.

We shall say that $G \preccurlyeq G_1$ (on Ω and with respect to ϱ) if $L_{F,G_1,\varrho} \hookrightarrow L_{F,G,\varrho}$ for every Young function F.

Remark 1.1. It is clear that for P and Q equal to power functions we get a weighted Lorentz space. Also, $L_{F,F,\varrho} = L_{F,\varrho}$.

Observe that for $P(t)=t^p$, $Q(t)=t^q$, and $\varrho \equiv 1$ the functional in (1.1) becomes

$$\|f\|_{P,Q,1} = \left(\int_0^\infty [f^*(t^{p/q})]^q\,dt\right)^{1/q}$$

which is equivalent to the usual quasinorm in $L_{p,q}$ and it is actually the expression giving a hint how to reasonably define the $L_{F,G,\varrho}$ spaces.

Remark 1.2. The quantities in (1.1) and (1.2) are not generally norms. Nevertheless, they are quasinorms in the many relevant cases we are interested in.

First, we shall show that if $F^{-1} \in \Delta_2$, then $\|\,.\,\|_{F,\infty,\varrho}$ is a quasinorm. Indeed, then $\widetilde{F}^{-1} = \widetilde{F^{-1}} \in \Delta_2$, too, and

$$\begin{aligned}\|f+g\|_{F,\infty,\varrho} &= \sup_{t>0} \widetilde{F}^{-1}(t)(f+g)^*_\varrho(t) \leq \\ &\leq c \sup_{t>0} \widetilde{F}^{-1}(t/2)\left[f^*_\varrho(t/2) + g^*_\varrho(t/2)\right] \leq \\ &\leq c\,(\|f\|_{F,\infty,\varrho} + \|g\|_{F,\infty,\varrho}).\end{aligned}$$

Conversely, if $\|\,.\,\|_{F,\infty,\varrho}$ is a quasinorm, then $F^{-1} \in \Delta_2$ at least near infinity. This can be shown as follows: Let $M, N \subset \Omega$ be disjoint and such that $\varrho(M) = \varrho(N)$. Then

$$\|\chi_M + \chi_N\|_{F,\infty,\varrho} = \|\chi_{M\cup N}\|_{F,\infty,\varrho} \leq c\left[\|\chi_M\|_{F,\infty,\varrho} + \|\chi_N\|_{F,\infty,\varrho}\right].$$

According to the formula for the Orlicz–Lorentz functional of $\|\chi_A\|_{F,\infty,\varrho}$ (Lemma 2.1), we have

$$\frac{1}{F^{-1}\left(1/2\varrho(M)\right)} \leq \frac{2c}{F^{-1}\left(1/\varrho(M)\right)}.$$

Putting $t = 1/2\varrho(M)$ we get $F^{-1}(2t) \leq 2cF^{-1}(t)$. If $\varrho(\Omega) = \infty$, this gives directly the Δ_2-condition for F^{-1}. If $\varrho(\Omega) < \infty$, then F^{-1} satisfies the Δ_2-condition for large t's.

Now let us consider $\|\,.\,\|_{F,G,\varrho}$. We shall not pursue the sufficient condition in detail as it is not the subject of this paper; nevertheless, let us point out one important case when $\|\,.\,\|_{F,G,\varrho}$ is a quasinorm: Suppose that $F^{-1} \in \Delta_2$. If $G \in \Delta_2$, then G is c-subadditive, i.e.,

$$G(t_1+t_2) \leq c[G(t_1) + G(t_2)], \qquad t_1, t_2 > 0, \tag{1.3}$$

for some $c > 0$ and there are $c_1 > 1$ and $c_2 > 0$ such that $c_1 G(t_1) \leq G(c_2 t_2)$ for all $t \in \mathbb{R}^1$. In particular, the Orlicz spaces generated by G and αG with $\alpha > 0$ are the same with equivalent Luxemburg functionals.

Recalling the standard estimate

$$(f+g)^*_\varrho(\tau) \leq f^*_\varrho(\tau/2) + g^*_\varrho(\tau/2),$$

we get

$$\begin{aligned}\|f+g\|_{F,G,\varrho} &\leq \left\| f^*_\varrho\left(\frac{\widetilde{F}\circ\widetilde{G}^{-1}(t)}{2}\right) + g^*_\varrho\left(\frac{\widetilde{F}\circ\widetilde{G}^{-1}(t)}{2}\right)\right\|_G \\ &= \|f^*_\varrho(\widetilde{2F}\circ\widetilde{G}^{-1}(t)) + g^*_\varrho(\widetilde{2F}\circ\widetilde{G}^{-1}(t))\|_G.\end{aligned} \tag{1.4}$$

Assuming that $F^{-1} \in \Delta_2$ there is $\alpha \in (0,1)$ such that

$$\widetilde{2F}(\tau) \geq \widetilde{F}(\alpha\tau), \qquad \tau > 0.$$

Hence

$$f_\varrho^*(\widetilde{2F} \circ \widetilde{G}^{-1}(t)) \leq f_\varrho^*(\widetilde{F}(\alpha\widetilde{G}^{-1}(t))), \quad t > 0. \tag{1.5}$$

Further we claim that there is $\beta > 0$ such that

$$\widetilde{G}(\alpha\tau) \geq \beta\widetilde{G}(\tau) = \widetilde{\beta^{-1}G}(\tau).$$

Indeed, the last inequality is nothing but the Δ_2-condition for G in terms of $\widetilde{G}$. Therefore, putting $\tau = \widetilde{G}^{-1}(t)$, we get

$$\widetilde{G}(\alpha\widetilde{G}^{-1}(t)) \geq \beta t.$$

Substituting this into (1.5) we have

$$f_\varrho^*(\widetilde{2F} \circ \widetilde{G}^{-1}(t)) \leq f_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(\beta t)).$$

Returning to (1.4) we see that

$$\|f+g\|_{F,G,\varrho} \leq \|f_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(\beta t) + g_\varrho^*(\widetilde{G} \circ \widetilde{G}^{-1}(\beta t)\|_G,$$

and, by virtue of (1.3), we get

$$\begin{aligned}\|f+g\|_{F,G,\varrho} &\leq \|f_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(\beta t))\|_{2cG} + \|g_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(\beta t))\|_{2cG} = \\ &= \|f_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(t))\|_{2c\beta^{-1}G} + \|g_\varrho^*(\widetilde{F} \circ \widetilde{G}^{-1}(t))\|_{2c\beta^{-1}G}.\end{aligned}$$

As $G \in \Delta_2$ the functions G and $2c\beta^{-1}G$ generate equivalent Luxemburg functionals.

In the sequel (see the proof of Lemma 2.2) we shall still need a simple condition for the equivalence of the Luxemburg functionals in the Orlicz spaces L_G and $L_{\alpha G}$ where α is an arbitrary positive constant. It is clear that the condition

$$\max(\alpha, \alpha^{-1})G(t) \leq G(\beta t), \qquad t > 0,$$

for some $\beta \geq 1$ independent of t, is sufficient. Following the above considerations, we see that $G \in \Delta_2$ is enough for this.

Various sufficient conditions for $\|\,.\,\|_{F,G,\varrho}$ to be a quasinorm can also be found when more is imposed on the functions F and G.

2. Weighted Imbeddings

In this section we prove the imbedding theorems for weighted strong and weak Lorentz–Orlicz spaces under fairly general conditions on the growth of the Young functions involved. Let us point out a rather surprising fact, namely, that from the point of view of weighted imbeddings there is no essential difference between $L_{P,G}$ and $L_{P,\infty}$ spaces (see conditions (ii), (iii), and (iv) of the concluding theorem in this section).

We start with the necessary condition.

Lemma 2.1. *Let any of the following condition be satisfied for each measurable f in Ω:*

$$\|f\|_{F_0,G_0,\sigma} \le K\|f\|_{F_1,G_1,\varrho}, \tag{2.1}$$

$$\|f\|_{F_0,\infty,\sigma} \le K\|f\|_{F_1,\infty,\varrho}, \tag{2.2}$$

$$\|f\|_{F_0,\infty,\sigma} \le K\|f\|_{F_1,G_1,\varrho}. \tag{2.3}$$

Then

$$\widetilde{F_0}^{-1}(\sigma(A)) \le K\widetilde{F_1}^{-1}(\varrho(A)) \tag{2.4}$$

for every measurable $A \subset \Omega$.

Proof. The necessary condition (2.4) follows directly after putting $f = \chi_A$ in (2.1)–(2.3) and calculating the corresponding norms.

As to $\|f\|_{F_1,G_1,\varrho}$, we have

$$\begin{aligned}
\|\chi_A\|_{F_1,G_1,\varrho} &= \inf\left\{\mu > 0;\ \int_0^{\widetilde{G_1}\widetilde{F_1}^{-1}(\varrho(A))} G_1(1/\mu)\,dt \le 1\right\} = \\
&= \inf\left\{\mu > 0;\ \widetilde{G_1}(\widetilde{F_1}^{-1}(\varrho(A)))G_1(1/\mu) \le 1\right\} = \\
&= \inf\left\{\mu > 0;\ \frac{1}{G_1(1/(\widetilde{F_1}^{-1}(\varrho(A))))} \le \frac{1}{G_1(1/\mu)}\right\} = \\
&= \widetilde{F_1}^{-1}(\varrho(A)).
\end{aligned}$$

Further,

$$\begin{aligned}
\|\chi_A\|_{F_1,\infty,\varrho} &= \sup_{t>0} \widetilde{F_1}^{-1}(t)(\chi_A)^*_\varrho(t) = \\
&= \sup_{t>0} \widetilde{F_1}^{-1}(t)\inf\left\{\lambda > 0;\ \varrho(\{\chi_A(x) > \lambda\}) \le t\right\} = \\
&= \widetilde{F_1}^{-1}(\varrho(A))
\end{aligned}$$

and we are done. □

Lemma 2.2. *Let $K > 1$ be such that*

$$\widetilde{F_0}^{-1}(\sigma(A)) \leq K\widetilde{F_1}^{-1}(\varrho(A)) \qquad \text{for every measurable } A \subset \Omega. \quad (2.5)$$

Assume that G satisfies the Δ_2-condition. Then there is $K_1 > 0$ such that

$$\|f\|_{F_0,G,\sigma} \leq K_1\|f\|_{F_1,G,\varrho} \qquad \text{for every } f \in L_{F_1,G,\varrho}.$$

Proof. According to the definition of the Lorentz–Orlicz functional and (2.5) we have

$$\|f\|_{F_1,G,\varrho}=\inf\left\{\mu>0; \int_0^\infty G\left(\frac{1}{\mu}\left[\inf\{\lambda>0; m_\varrho(f,\lambda)\leq\widetilde{F}\circ\widetilde{G}^{-1}(t)\}\right]\right)dt\leq 1\right\}$$

and

$$f^*_\sigma(\widetilde{F_1}\circ\widetilde{G}^{-1}(t)) \geq \inf\left\{\lambda > 0;\ K^{-1}\widetilde{F_0}^{-1}(m_\sigma(f,\lambda)) \leq \widetilde{G}^{-1}(t)\right\}.$$

As $G \in \Delta_2$ there is $K_0 > 1$ such that

$$K\widetilde{G}^{-1}(t) \leq \widetilde{G^{-1}}(K_0 t)$$

(cf. Remark 1.3) and therefore, after a simple change of variables, we get

$$\|f\|_{F_1,G,\varrho} \geq$$

$$\geq \inf\left\{\mu > 0; \int_0^\infty G\left(\frac{1}{\mu}\left[\inf\{\lambda > 0; m_\sigma(f,\lambda) \leq \widetilde{F_0}(K\widetilde{G}^{-1}(t))\}\right]\right)dt \leq 1\right\} \geq$$

$$\geq \inf\{\mu > 0; \int_0^\infty G\left(\frac{1}{\mu}\left[\inf\left\{\lambda > 0; m_\sigma(f,\lambda) \leq \widetilde{F_0}(\widetilde{G}^{-1}(t))\}\right]\right)\frac{dt}{K_0} \leq 1\right\}.$$

It is $K_0^{-1}G \sim G$ so that these functions generate the same Orlicz spaces. Hence

$$K_1\|f\|_{F_1,G,\varrho} \geq \|f\|_{F_0,G,\sigma}$$

with a suitable $K_1 > 0$. □

Next we shall consider weak Orlicz spaces.

Lemma 2.3. *Let*

$$\widetilde{F_0}^{-1}(\sigma(A)) \leq K\widetilde{F_1}^{-1}(\varrho(A)) \quad (2.6)$$

for some $K > 0$ and every measurable $A \subset \Omega$. Then

$$\|f\|_{F_0,\infty,\sigma} \leq K\|f\|_{F_1,\infty,\varrho}.$$

Proof. By virtue of (2.6) we get

$$\begin{aligned}\sup_{t>0}\widetilde{F_0}^{-1}(t)f^*_\sigma(t) &= \sup_{t>0}\widetilde{F_0}^{-1}(t)\inf\{\lambda>0;\ m_\sigma(f,\lambda)\le t\}\le\\ &\le \sup_{t>0}\widetilde{F_0}^{-1}(t)\inf\{\lambda>0;\ \widetilde{F_0}(K\widetilde{F_1}^{-1}(m_\varrho(f,\lambda)))\le t\}=\\ &=\sup_{t>0}t\inf\{\lambda>0;\ K\widetilde{F_1}^{-1}(m_\varrho(f,\lambda))\le t\}=\\ &=K\sup_{t>0}\widetilde{F_1}^{-1}(t)\inf\{\lambda>0;\ m_\varrho(f,\lambda)\le t\}=\\ &=K\|f\|_{F_1,\infty,\varrho}. \qquad\square\end{aligned}$$

The following lemma links Orlicz–Lorentz and weak Orlicz spaces.

Lemma 2.4. *Let F and G be arbitrary Young functions. Then*

$$L_{F,G,\varrho}\hookrightarrow L_{F,\infty,\varrho}. \tag{2.7}$$

Proof. By the definition of the Orlicz–Lorentz functional,

$$\begin{aligned}\|f\|_{F,\infty,\varrho} &= \sup_{t>0}\widetilde{F}^{-1}(t)\inf\{\lambda>0;\ m_\varrho(f,\lambda)\le t\}=\\ &=\sup_{t>0}t\inf\{\lambda>0;\ m_\varrho(f,\lambda)\le\widetilde{F}(t)\}=\\ &=\sup_{t>0}\widetilde{G}^{-1}(t)\inf\{\lambda>0;\ m_\varrho(f,\lambda)\le\widetilde{F}\circ\widetilde{G}^{-1}(t)\}.\end{aligned}$$

On the other hand, for every $K>0$,

$$\begin{aligned}\|f\|_{F,G,\varrho} &= \inf\left\{\lambda>0;\ \int_0^K G\left(\frac{1}{\lambda}f^*_\varrho(\widetilde{F}\circ\widetilde{G}^{-1}(t))\right)dt\le 1\right\}\ge\\ &\ge\inf\left\{\lambda>0;\ \int_0^K G\left(\frac{1}{\lambda}f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K)))\right)dt\le 1\right\}\ge\\ &\ge\inf\left\{\lambda>0;\ KG\left(\frac{1}{\lambda}f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K)))\right)dt\le 1\right\}=\\ &=\inf\left\{\lambda>0;\ \frac{1}{\lambda}f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K)))\le G^{-1}(1/K)\right\}=\\ &=\inf\left\{\lambda>0;\ \frac{1}{G^{-1}(1/K)}f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K)))\le\lambda\right\}=\\ &=\inf\left\{\lambda>0;\ \widetilde{G}^{-1}(K)f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K)))\le\lambda\right\}=\\ &=\widetilde{G}^{-1}(K)f^*_\varrho(\widetilde{F}(\widetilde{G}^{-1}(K))),\end{aligned}$$

which gives (2.7). $\square$

Now we are ready to formulate

Theorem 2.5. *Let $G_1 \in \Delta_2$. Then the following statements are equivalent:*

(i) $L_{F_1,G_1,\varrho} \hookrightarrow L_{F_0,G_1,\sigma}$,
(ii) $L_{F_1,G_1,\varrho} \hookrightarrow L_{F_0,G_0,\sigma}$ *provided* $G_1 \preccurlyeq G_0$,
(iii) $L_{F_1,G_1,\varrho} \hookrightarrow L_{F_0,\infty,\sigma}$,
(iv) $L_{F_1,\infty,\varrho} \hookrightarrow L_{F_0,\infty,\sigma}$,
(v) $\widetilde{F_0}^{-1}(\sigma(A)) \le K\widetilde{F_1}^{-1}(\varrho(A))$ *for some* $K > 0$ *and every measurable* $A \subset \Omega$.

Proof. The necessity of condition (v) follows from Lemma 2.1. The implication (v)⇒(i) was proved in Lemma 2.2. The definition of the ordering $G_1 \preccurlyeq G_0$ gives directly (i)⇒(ii) and Lemma 2.4 implies (ii)⇒(iii). Further, Lemma 2.3 gives (v)⇒(iv) and another application of Lemma 2.4 completes the proof by showing that (iv)⇒(iii). □

3. More about Weighted Imbeddings

The necessary and sufficient condition (v) for imbeddings (i)–(iv) from Theorem 2.5 is of quite another sort than those previously known for imbeddings of weighted Lebesgue and/or Orlicz spaces. It was proved in [3] that, under some additional assumptions, $L_{P,\varrho} \hookrightarrow L_{Q,\sigma}$ iff $\sigma\varrho^{-1} \in L_{N,\varrho}$, where N is the complementary function to QP^{-1}. A natural question is whether the case studied here permits an analogous condition. Let us observe that Theorem 2.5 solves the “nondiagonal” case; therefore one cannot expect a characterization in terms of Lebesgue and Orlicz spaces as in [1] and [3], respectively.

We shall show, however, that a nice condition equivalent to (v) of Theorem 2.5 can be found in important cases. First of all observe that (v) is equivalent to

$$\widetilde{F_0}^{-1}(\sigma(A)) \le \widetilde{F_2}^{-1}(\varrho(A)) \tag{3.1}$$

where $F_2(t) = F_1(Kt)$, and, consequently, equivalent to

$$\frac{1}{\widetilde{F_0}(\widetilde{F_2}^{-1}(\varrho(A)))} \int_A \sigma(x)\,dx \le 1 \quad \text{for every measurable } A \subset \Omega. \tag{3.2}$$

Put $H = F_2 \circ F_0^{-1}$. Then $\widetilde{H}^{-1} = \widetilde{F_0} \circ \widetilde{F_2}^{-1}$ and we can rewrite (3.2) as

$$\sup_{A\subset\Omega} \frac{1}{\widetilde{H}^{-1}(\varrho(A))} \int_A \sigma(x)\,dx < \infty. \tag{3.3}$$

We shall show that if $F_2 \circ F_0^{-1}$ is a convex Young function satisfying the Δ_2-condition, then (3.3) is nothing but a characterization of a certain weak Orlicz space. Indeed, following two lemmas hold.

Lemma 3.1. *Let H be a Young function and let $f \in L^1_{loc}$ be such that*

$$\sup \frac{1}{\widetilde{H}^{-1}(\varrho(A))} \int_A |f(x)|\varrho(x)\,dx < \infty$$

where the sup is taken over all measurable $A \subset \Omega$. Let J be a Young function satisfying

$$H^{-1}(t)J^{-1}(t) \geq c_0^{-1} t \tag{3.4}$$

for some $c_0 > 0$ and all $t \geq 0$. Then

$$\sup_{t>0} \widetilde{J}^{-1}(t) f_\varrho^*(t) < \infty.$$

Proof. We have

$$\begin{aligned}
\sup_{t>0} \widetilde{J}^{-1}(t) f_\varrho^*(t) &= \sup_{t>0} \frac{1}{J^{-1}(1/t)} f_\varrho^*(t) \leq \sup_{t>0} \frac{c_0 H^{-1}(1/t)}{1/t} f_\varrho^*(t) = \\
&= \sup_{t>0} c_0 t H^{-1}(1/t) f_\varrho^*(t) \leq \\
&\leq c_0 \sup_{t>0} \sup_{\substack{B\subset\Omega \\ \varrho(B)=t}} \frac{\varrho(B)}{\widetilde{H}^{-1}(\varrho(B))} f_\varrho^*(\varrho(B)) \leq \\
&\leq c_0 \sup_{B\subset\Omega} \frac{\varrho(B)}{\widetilde{H}^{-1}(\varrho(B))} f_\varrho^*(\varrho(B)).
\end{aligned} \tag{3.5}$$

Now we claim that for every $B \subset \Omega$ there is $A \subset \Omega$ such that $\varrho(A) = \varrho(B)$ and $|f(x)| \geq f_\varrho^*(\varrho(A))$ for all $x \in A$. Indeed, it suffices to choose

$$A = \{x \in \Omega;\ |f(x)| > \lambda\} \cup (\{x \in \Omega;\ |f(x)| = \lambda\} \cap \Omega_R)$$

where $\lambda = f_\varrho^*(\varrho(B))$ and Ω_R is a suitable ball centered at the origin. Then (3.5) implies

$$\sup_{t>0} \widetilde{J}^{-1}(t) f_\varrho^*(t) \leq c_0 \sup_{A\subset\Omega} \frac{1}{\widetilde{H}^{-1}(\varrho(A))} \int_A |f(x)|\varrho(x)\,dx. \qquad \square$$

Lemma 3.2. *Let H be a convex Young function and let J be complementary to H. Assume that*

$$\sup_{t>0} \frac{H'(t)t}{H(t)} = c_1 < \infty,$$

and $\sup\limits_{t>0} \widetilde{J}^{-1}(t) f_{\varrho}^{*}(t) = c_2 < \infty$. *Then*

$$\sup_{A \subset \Omega} \frac{1}{\widetilde{H}^{-1}(\varrho(A))} \int_A |f(x)|\varrho(x)\,dx < \infty.$$

Proof. Let $A \subset \Omega$ be measurable and let $f_{|A}$ be the restriction of f to A. Then

$$\begin{aligned}
\int_A |f(x)|\varrho(x)\,dx &= \int_0^{\varrho(A)} \left(f_{|A}\right)_{\varrho}^{*}(\lambda)\,d\lambda \le \int_0^{\varrho(A)} f_{\varrho}^{*}(\lambda)\,d\lambda \le \\
&\le c_2 \int_0^{\varrho(A)} \frac{d\lambda}{\widetilde{J}^{-1}(\lambda)}\,d\lambda = c_2 \int_0^{\varrho(A)} J^{-1}(1/\lambda)\,d\lambda = \\
&= c_1 c_2 \int_0^{\varrho(A)} J^{-1}(1/\lambda)\,\frac{d\lambda}{c_1} \le c_J c_1 c_2 \int_0^{\varrho(A)} \frac{d\lambda}{c_1 \lambda H^{-1}(1/\lambda)} \le \\
&\le c_J c_1 c_2 \int_0^{\varrho(A)} \frac{1}{\lambda H^{-1}(1/\lambda)} \cdot \frac{1}{H'(H^{-1}(1/\lambda))\lambda H^{-1}(1/\lambda)}\,d\lambda = \\
&= c_J c_1 c_2 \frac{1}{H^{-1}(1/\varrho(A))} = c_J c_0 c_1 \widetilde{H}^{-1}(\varrho(A)),
\end{aligned}$$

where the last step follows by taking the derivative of $1/H(1/t)$. □

Now we are in a position to reformulate Theorem 2.5.

Theorem 3.3. *Let F_0, F_1, G_0, and G_1 be Young functions, $G_1 \in \Delta_2$, and let $F_1 \circ F_0^{-1}$ be a convex Young function, $F_1 \circ F_0^{-1} \in \Delta_2$. Then conditions* (i)–(iv) *from Theorem* 2.5 *are equivalent to $\sigma/\varrho \in L_{J,\infty,\varrho}$ where J is complementary to $F_2 \circ F_0^{-1}$ where $F_2(t) = F_1(Kt)$.*

It is also worthwhile pointing out the Lorentz space version of the preceding theorem.

Corollary 3.4. *Let $1 \le p < q < \infty$, $1 \le r \le s \le \infty$. Then the following statements are equivalent:*

(i) $L_{q,r,\varrho} \hookrightarrow L_{p,r,\sigma}$,
(ii) $L_{q,r,\varrho} \hookrightarrow L_{p,s,\sigma}$,
(iii) $L_{q,r,\varrho} \hookrightarrow L_{p,\infty,\sigma}$,
(iv) $L_{q,\infty,\varrho} \hookrightarrow L_{p,\infty,\sigma}$,
(v) $\sigma/\varrho \in L_{q/(q-p),\infty,\varrho}$.

The proof follows immediately by calculation of the complementary function to $t \mapsto |t|^{q/p}, \in \mathbb{R}^1$.

Observe that for $r = s$ imbedding (ii) from Corollary 3.4 was shown to be equivalent to $\sigma(A)^{1/p} \le \text{const.}\, \varrho(A)^{1/q}$ for every measurable $A \subset \Omega$ by Carro and Soria [11]. They consider more general two-parameter Lorentz spaces which naturally lead to the question about an analogous concept using Orlicz norms instead.

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