UPPER ESTIMATE OF THE INTERVAL OF EXISTENCE OF SOLUTIONS OF A NONLINEAR TIMOSHENKO EQUATION

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Abstract. Solutions to the initial-boundary value problem for a nonlinear Timoshenko equation are considered. Conditions on the initial data and nonlinear term are given so that solutions to the problem under consideration do not exist for all $t > 0$. An upper estimate of the t-interval of the existence of solutions is obtained. An estimate of the growth rate of the solutions is given.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. It is assumed that the divergence theorem can be applied on Ω. Let

$$L^q(\Omega) = \left\{ u\colon \ \|u\|_{q,\Omega} = \left(\int\limits_\Omega |u(x)|^q dx \right)^{1/q} < +\infty \right\}.$$

Denote by $\overline{u}$ the complex conjugate of u.

We consider the initial-boundary value problem (IBVP) for the nonlinear Timoshenko equation

$$u_{tt} - \varphi(\|\nabla u\|^2_{2,\Omega})\Delta u + \Delta^2 u = |u|^{p-1}u, \quad t \geq 0,\ x \in \Omega, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \overline{\Omega}, \tag{2}$$

$$u_t(0,x) = u_1(x), \quad x \in \overline{\Omega}, \tag{3}$$

$$u(t,x)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \tag{4}$$

$$\Delta u(t,x)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \tag{5}$$

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where $p > 1$ is a constant, $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, u_0, $u_1 : \overline{\Omega} \to \mathbb{C}$ are given functions.

Equations of type (1) appear in a variety of models describing nonlinear vibration of beams. This kind of equations is the object of long-standing interest. We should mention the papers [1]–[6], as well as the monograph [7].

In the present paper we investigate the blowing up of solutions [8] of the initial boundary value problem for the nonlinear Timoshenko equation.

2. Main Results

Theorem 1. *Let the following conditions hold:*

1. *u is a suitably smooth classical solution of the* IBVP (1)–(5)*;*
2. *$\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $m\Phi(s) \geq s\varphi(s)$, $\forall s \geq 0$, where $\Phi(s) = \int\limits_0^s \varphi(k)\, dk$, $m \geq 1$;*
3. *$\operatorname{Re} \int\limits_\Omega u_0 \overline{u_1}\, dx > 0$;*
4. *$E(0) = \|u_1\|^2_{2,\Omega} + \Phi(\|\nabla u_0\|^2_{2,\Omega}) + \|\Delta u_0\|^2_{2,\Omega} - \frac{2}{p+1}\, \|u_0\|^{p+1}_{p+1,\Omega} \leq 0$;*
5. *$p > 2m - 1$.*

Then the maximal time interval of the existence $[0, T_{\max})$ of u can be estimated by

$$T_{\max} \leq T_0 = \int\limits_{\|u_0\|^2_{2,\Omega}}^{\infty} \left[C_1 - 4mE(0)\xi + \frac{4C}{p+3}\, \xi^{\frac{p+3}{2}} \right]^{-\frac{1}{2}} d\xi < +\infty,$$

where

$$C_1 = \left[2 \operatorname{Re} \int\limits_\Omega u_0 \overline{u_1}\, dx \right]^2 + 4mE(0) \|u_0\|^2_{2,\Omega} - \frac{4C}{p+3}\, \|u_0\|^{p+3}_{2,\Omega},$$

$$C = 2\, \frac{p+1-2m}{p+1} \left(\frac{1}{\operatorname{mes} \Omega} \right)^{\frac{p-1}{2}}$$

and if $T_{\max} = T_0$, then

$$\lim_{t \to T_0^-} \|u(t)\|^2_{2,\Omega} = +\infty$$

and

$$\lim_{t \to T_0^-} (T_0 - t) \|u(t)\|^{\frac{p-1}{2}}_{2,\Omega} \leq \frac{2}{p-1} \sqrt{\frac{p+3}{C}}.$$

Proof. We multiply both sides of equation (1) by $\overline{u}$, take the real parts of both sides, and integrate to obtain

$$\frac{1}{2}\,\frac{d^2}{dt^2}\left(\|u\|^2_{2,\Omega}\right) - \|u_t\|^2_{2,\Omega} + \varphi(\|\nabla u\|^2_{2,\Omega})\|\nabla u\|^2_{2,\Omega} + \|\Delta u\|^2_{2,\Omega} = \|u\|^{p+1}_{p+1,\Omega}. \quad (6)$$

On the other hand, if we multiply both sides of equation (1) by $\overline{u_t}$, take the real parts of both sides, and integrate, we get the energy identity

$$\|u_t\|^2_{2,\Omega} + \Phi(\|\nabla u\|^2_{2,\Omega}) + \|\Delta u\|^2_{2,\Omega} - \frac{2}{p+1}\|u\|^{p+1}_{p+1,\Omega} = E(0). \quad (7)$$

Thus, from (6), (7) and condition 2 of the theorem we conclude that

$$\begin{aligned}\frac{1}{2}\,\frac{d^2}{dt^2}\left(\|u\|^2_{2,\Omega}\right) &\geq \|u_t\|^2_{2,\Omega} - \|\Delta u\|^2_{2,\Omega} + \|u\|^{p+1}_{p+1,\Omega} + m\|u_t\|^2_{2,\Omega} + \\ &+ m\|\Delta u\|^2_{2,\Omega} - \frac{2m}{p+1}\|u\|^{p+1}_{p+1,\Omega} - mE(0) \geq \\ &\geq -mE(0) + \frac{p+1-2m}{p+1}\|u\|^{p+1}_{p+1,\Omega} \geq \\ &\geq -mE(0) + \frac{p+1-2m}{p+1}\left(\frac{1}{\operatorname{mes}\Omega}\right)^{\frac{p-1}{2}}\|u\|^{p+1}_{2,\Omega},\end{aligned}$$

where we have used the Hölder inequality in the last step.

Denote $X(t) = \|u(t,\cdot)\|^2_{2,\Omega}$. Then we have the differential inequality

$$\frac{d^2X}{dt^2} \geq -2mE(0) + CX^{\frac{p+1}{2}}, \quad (8)$$

and

$$X(0) = \|u_0\|^2_{2,\Omega}, \qquad \frac{dX}{dt}(0) = 2\operatorname{Re}\int_{\Omega} u_0\overline{u_1}\,dx.$$

It is easy to prove that $X'(t) > 0$ wherever $X(t)$ exists. If this conclusion is false then the set $Q = \{t\colon\ X'(t) \leq 0\}$ is nonempty. Denote $t_0 = \inf Q$. By integrating, we obtain

$$0 = X'(t_0) \geq X'(0) - 2mE(0)t_0 + C\int_0^{t_0} X^{\frac{p+1}{2}}(\tau)\,d\tau > 0,$$

which is a contradiction. We multiply both sides of (8) by $X'(t)$ and integrating twice we conclude that

$$T_{\max} \leq T_0 = \int_{\|u_0\|^2_{2,\Omega}}^{\infty} \left[C_1 - 4mE(0)\xi + \frac{4C}{p+3}\,\xi^{\frac{p+3}{2}}\right]^{-\frac{1}{2}} d\xi < +\infty.$$

If $T_{\max} = T_0$ then $\lim_{t \to T_0^-} \|u(t)\|_{2,\Omega}^2 = +\infty$ and from (8) we have

$$T_0 - t \le \int\limits_{X(t)}^{\infty} \left[C_1 - 4mE(0)\xi + \frac{4C}{p+3}\,\xi^{\frac{p+3}{2}} \right]^{-\frac{1}{2}} d\xi \le$$

$$\le \sqrt{\frac{p+3}{4C}}\,\frac{4}{p-1}\,\frac{1}{[X(t)]^{\frac{p-1}{4}}}, \quad t \in [T_0 - \varepsilon, T_0),$$

where $\varepsilon > 0$ is sufficiently small. Thus we obtain

$$\lim_{t \to T_0^-} (T_0 - t)\|u(t)\|_{2,\Omega}^{\frac{p-1}{2}} \le \frac{2}{p-1}\sqrt{\frac{p+3}{C}}. \quad \square$$

Remark 1. When $\varphi(s) = 1 + s$ we have $m = 2$ and $p > 3$.

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