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SINGULAR NONLINEAR $(n-1,1)$ CONJUGATE BOUNDARY VALUE PROBLEMS

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Abstract. Solutions are obtained for the boundary value problem, $y^{(n)} + f(x,y) = 0$, $y^{(i)}(0) = y(1) = 0$, $0 \le i \le n-2$, where $f(x,y)$ is singular at $y = 0$. An application is made of a fixed point theorem for operators that are decreasing with respect to a cone.

§ 1. Introduction

In this paper, we establish the existence of solutions for the $(n-1,1)$ conjugate boundary value problem,

$$y^{(n)} + f(x,y) = 0, 0 < x < 1, \tag{1}$$

$$\begin{aligned} y^{(i)}(0) &= 0, \quad 0 \le i \le n-2, \\ y(1) &= 0, \end{aligned} \tag{2}$$

where $f(x,y)$ has a singularity at $y = 0$. Our assumptions throughout are:

(A) $f(x,y) : (0,1) \times (0,\infty) \to (0,\infty)$ is continuous,
(B) $f(x,y)$ is decreasing in y, for each fixed x,
(C) $\int\limits_0^1 f(x,y)dx < \infty$, for each fixed y,
(D) $\lim\limits_{y \to 0^+} f(x,y) = \infty$ uniformly on compact subsets of $(0,1)$, and
(E) $\lim\limits_{y \to \infty} f(x,y) = 0$ uniformly on compact subsets of $(0,1)$.

We note that, if y is a solution of (1), (2), then (A) implies $y(x) > 0$ on $(0,1)$.

Singular nonlinear two-point boundary value problems appear frequently in applications, and usually, only positive solutions are meaningful. This is especially true for the case $n = 2$, with Taliaferro [1] treating the general problem, Callegari and Nachman [2] considering existence questions

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in boundary layer theory, and Luning and Perry [3] obtaining constructive results for generalized Emden–Fowler problems. Results have also been obtained for singular boundary value problems arising in reaction-diffusion theory and in non-Newtonian fluid theory [4]. A number of papers have been devoted to singular boundary value problems in which topological transversality methods were applied; see, for example, [5]–[10].

The results and methods of this work are outgrowths of papers on second-order singular boundary value problems by Gatica, Hernandez, and Waltman [11] and Gatica, Oliker, and Waltman [12] which in turn received extensive embellishment by Eloe and Henderson [13] and Henderson and Yin [14], [15]. In attempting to improve some of these generalizations, the recent paper by Wang [16] did contain some flaws; however, that paper was corrected in a subsequent work by Agarwal and Wong [17].

We obtain solutions of (1), (2) by arguments involving positivity properties, an iteration, and a fixed point theorem due to [12] for mappings that are decreasing with respect to a cone in a Banach space. We remark that, for $n = 2$, positive solutions of (1), (2) are concave. This concavity was exploited in [12], and later in the generalizations [14]–[18], in defining an appropriate subset of a cone on which a positive operator was defined to which the fixed point theorem was applied. The crucial property in defining this subset in [12] made use of an inequality that provides lower bounds on positive concave functions as a function of their maximum. Namely, this inequality may be stated as:

If $y \in C^{(2)}[0,1]$ *is such that* $y(x) \geq 0, 0 \leq x \leq 1$, *and* $y''(x) \leq 0$, $0 \leq x \leq 1$, *then*

$$y(x) \geq \frac{1}{4} \max_{0 \leq s \leq 1} |y(s)|, \quad \frac{1}{4} \leq x \leq \frac{3}{4}. \tag{3}$$

Although (3) can be developed using concavity, it can also be obtained directly with the classical maximum principle. This observation was exploited by Eloe and Henderson [18], and a generalization of (3) was given for positive functions satisfying the boundary conditions (2).

In Section 2, we provide preliminary definitions and some properties of cones in a Banach space. We also state the fixed point theorem from [12] for mappings that are decreasing with respect to a cone. In that section, we state the generalization of (3) as it extends to solutions of (1), (2). An analogous inequality is also stated for a related Green's function.

In Section 3, we apply the generalization of (3) in defining a subset of a cone on which we define an operator which is decreasing with respect to the cone. A sequence of perturbations of f is constructed, with each term of the sequence lacking the singularity of f. In terms of this sequence, we define a sequence of decreasing operators to which the fixed point theorem yields a sequence of iterates. This sequence of iterates is shown to converge to a positive solution of (1), (2).

§ 2. Some Preliminaries and a Fixed Point Theorem

In this section, we first give definitions and some properties of cones in a Banach space [19]. After that, we state a fixed point theorem due to [12] for operators that are decreasing with respect to a cone. We then state a theorem from [18] generalizing (3) followed by an analogous inequality for a Green's function.

Let $\mathcal{B}$ be a Banach space, and K a closed, nonempty subset of $\mathcal{B}$. K is a *cone* provided (i) $\alpha u+\beta v \in K$, for all u, $v \in K$ and all α, $\beta \geq 0$, and (ii) u, $-u \in K$ imply $u=0$. Given a cone K, a partial order, $\leq$, is induced on $\mathcal{B}$ by $x \leq y$, for x, $y \in \mathcal{B}$ iff $y-x \in K$. (For clarity, we may sometimes write $x \leq y(wrtK)$.) If x, $y \in \mathcal{B}$ with $x \leq y$, let $\langle x,y\rangle$ denote the *closed order interval between* x *and* y given by, $\langle x,y\rangle=\{z \in \mathcal{B} \mid x \leq z \leq y\}$. A cone K is *normal* in B provided there exists $\delta>0$ such that $\|e_1+e_2\| \geq \delta$, for all e_1, $e_2 \in K$, with $\|e_1\|=\|e_2\|=1$.

Remark 1. If K is a normal cone in $\mathcal{B}$, then closed order intervals are norm bounded.

The following fixed point theorem can be found in [12].

Theorem 1. *Let $\mathcal{B}$ be a Banach space, K a normal cone in $\mathcal{B}$, $E \subseteq K$ such that, if x, $y \in E$ with $x \leq y$, then $\langle x,y\rangle \subseteq E$, and let $T: E \to K$ be a continuous mapping that is decreasing with respect to K, and which is compact on any closed order interval contained in E. Suppose there exists $x_0 \in E$ such that $T^2x_0=T(Tx_0)$ is defined, and furthermore, Tx_0, T^2x_0 are orders comparable to x_0. If, either*

(I) *$Tx_0 \leq x_0$ and $T^2x_0 \leq x_0$, or $x_0 \leq Tx_0$ and $x_0 \leq T^2x_0$, or*
(II) *The complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is defined, and there exists $y_0 \in E$ such that $Ty_0 \in E$ and $y_0 \leq T^n x_0$, for all $n \geq 0$,*

then T has a fixed point in E.

In extending (3), Eloe and Henderson [13] first established the following.

Theorem 2. *Let $n \geq 2$ and $h \in C^{(n)}[a,b]$ be such that $h^{(n)}(x) \leq 0$, $a \leq x \leq b$, and*

$$h^{(i)}(a) \geq 0, \quad 0 \leq i \leq n-2, \tag{4}$$

$$h(b) \geq 0. \tag{5}$$

Then $h(x) \geq 0$, $a \leq x \leq b$. Moreover, if $h^{(n)}(x)<0$ on any compact subinterval of $[a,b]$, or if either (4) or (5) is strict inequality, then $h(x)>0$, $a<x<b$.

We now state the extension of (3) which will play a fundamental role in our future arguments.

Theorem 3. *Let $y \in C^{(n)}[0,1]$ be such that $y^{(n)}(x) < 0$, $0 < x < 1$, and $y^{(i)}(0) = y(1) = 0$, $0 \le i \le n-2$. Then $y(x) > 0$ on $(0,1)$, and there exists a unique $x_0 \in (0,1)$ such that $|y|_\infty = \sup_{0 \le x \le 1} |y(x)| = y(x_0)$. Moreover, $y(x)$ is increasing on $[0, x_0]$, $y(x)$ is concave on $[x_0, 1]$, and*

$$y(x) \ge \frac{|y|_\infty}{4^{n-1}}, \quad \frac{1}{4} \le x \le \frac{3}{4}. \tag{6}$$

For the final result to be stated in this section, let $G(x,s)$ denote the Green's function for the boundary value problem,

$$-y^{(n)} = 0, \quad 0 \le x \le 1, \tag{7}$$

satisfying (2). It is well known [20] that

$$G(x,s) > 0 \quad \text{on} \quad (0,1) \times (0,1), \tag{8}$$

and

$$\frac{\partial^{n-1}}{\partial x^{n-1}} G(0,s) > 0 > \frac{\partial}{\partial x} G(1,s), \quad 0 < s < 1. \tag{9}$$

Also, for the remainder of the paper for $0 < s < 1$, let $\tau(s) \in [0,1]$ be defined by

$$G(\tau(s), s) = \sup_{0 \le x \le 1} G(x,s). \tag{10}$$

The following analogue of (6) for $G(x,s)$ was also obtained in [18].

Theorem 4. *Let $G(x,s)$ denote the Green's function for (7), (2). Then, for $0 < s < 1$,*

$$G(x,s) \ge \frac{1}{4^{n-1}} G(\tau(s), s), \quad \frac{1}{4} \le x \le \frac{3}{4}. \tag{11}$$

§ 3. Solutions of (1), (2)

In this section, we apply Theorem 1 to a sequence of operators that are decreasing with respect to a cone. The obtained fixed points provided a sequence of iterates which converges to a solution of (1), (2). Positivity of solutions and Theorems 2–4 are fundamental in this construction.

To that end, let the Banach space $\mathcal{B} = C[0,1]$, with norm $\|y\| = |y|_\infty$, and let

$$K = \{y \in \mathcal{B} \mid y(x) \ge 0 \quad \text{on} \quad [0,1]\}.$$

K is a normal cone in $\mathcal{B}$.

To obtain a solution of (1), (2), we seek a fixed point of the integral operator,

$$T\varphi(x) = \int_0^1 G(x,s) f(s,\varphi(s))ds,$$

where $G(x,s)$ is the Green's function for (7), (2). Due to the singularity of f given by (D), T is not defined on all of the cone K.

Next, define $g : [0,1] \to [0,1]$ by

$$g(x) = \begin{cases} (2x)^{n-1}, & 0 \le x \le \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \le x \le 1, \end{cases}$$

and for each $\theta > 0$, define $g_\theta(x) = \theta g(x)$. Then for the remainder of this work, assume the condition:

(F) For each $\theta > 0$, $0 < \int_0^1 f(x, g_\theta(x))dx < \infty$.

We remark, for each $\theta > 0$, that $g_\theta \in K$, $g_\theta(x) > 0$ on $(0,1)$, and $g_\theta^{(i)}(0) = g(1) = 0$, $0 \le i \le n-2$.

Our first result of this section is a consequence of Theorem 3 and its proof in [18].

Theorem 5. *Let $y \in C^{(n)}[0,1]$ be such that $y^{(n)}(x) < 0$ on $(0,1)$ and $y^{(i)}(0) = y(1) = 0$, $0 \le i \le n-2$. Then, there exists a $\theta > 0$ such that $g_\theta(x) \le y(x)$ on $[0,1]$.*

Proof. Let y be as stated above and let $x_0 \in (0,1)$ be the unique point from Theorem 3 such that $y(x_0) = |y|_\infty$. Define the piecewise polynomial

$$p(x) = \begin{cases} \frac{|y|_\infty}{x_0^{n-1}} x^{n-1}, & 0 \le x \le x_0, \\ \frac{|y|_\infty}{x_0 - 1}(x-1), & x_0 \le x \le 1. \end{cases}$$

The proof of Theorem 3 in [18] yields that $y(x) \ge p(x)$ on $[0,1]$. If we choose $\theta = p(\frac{1}{2})$, then

$$p(x) \ge p(\tfrac{1}{2}) g(x) = g_\theta(x) \quad \text{on} \quad [0,1],$$

and so, $y(x) \ge g_\theta(x)$ on $[0,1]$. $\square$

In view of Theorem 5, let $D \subseteq K$ be defined by

$$D = \{\varphi \in \mathcal{B} \mid \text{ there exists } \theta(\varphi) > 0 \text{ such that } g_\theta(x) \le \varphi(x) \text{ on } [0,1]\},$$

(i.e., $D = \{\varphi \in \mathcal{B} \mid$ there exists $\theta(\varphi) > 0$ such that $g_\theta \le \varphi(wrtK)\}$). Then, define $T : D \to K$ by

$$T\varphi(x) = \int_0^1 G(x,s) f(s,\varphi(s))ds, \quad 0 \le x \le 1, \quad \varphi \in D.$$

Note that, from conditions (A)–(F) and properties of $G(x,s)$ in (8)–(9), if $\varphi \in D$, then $(T\varphi)^{(n)} < 0$ on $(0,1)$, and $T\varphi$ satisfies the boundary conditions (2). Application of Theorem 5 yields that $T\varphi \in D$ so that $T : D \to D$. Moreover, if φ is a solution of (1), (2), then by Theorem 5 again, $\varphi \in D$. As a consequence, $\varphi \in D$ is a solution of (1), (2) if, and only if, $T\varphi = \varphi$.

Our next result establishes *a priori* bounds on solutions of (1), (2) which belong to D.

Theorem 6. *Assume that conditions* (A)–(F) *are satisfied. Then, there exists an* $R > 0$ *such that* $\|\varphi\| = |\varphi|_\infty \leq R$, *for all solutions,* φ, *of* (1), (2) *that belong to* D.

Proof. Assume to the contrary that the conclusion is false. This implies there exists a sequence, $\{\varphi_\ell\} \subset D$, of solutions of (1), (2) such that $\lim_{\ell\to\infty} |\varphi_\ell| = \infty$. Without loss of generality, we may assume that, for each $\ell \geq 1$,

$$|\varphi_\ell|_\infty \leq |\varphi_{\ell+1}|_\infty. \tag{12}$$

For each $\ell \geq 1$, let $x_\ell \in (0,1)$ be the unique point from Theorem 3 such that

$$0 < \varphi_\ell(x_\ell) = |\varphi_\ell|_\infty,$$

and also

$$\varphi_\ell(x) \geq \frac{1}{4^{n-1}}\varphi_\ell(x_\ell), \quad \frac{1}{4} \leq x \leq \frac{3}{4}.$$

By the monotonicity in (12), $\varphi_\ell(x_\ell) \geq \varphi_1(x_1)$, for all ℓ, and so

$$\varphi_\ell(x) \geq \frac{1}{4^{n-1}}\varphi_1(x_1), \quad \frac{1}{4} \leq x \leq \frac{3}{4} \quad \text{and} \quad \ell \geq 1. \tag{13}$$

Let $\theta = \frac{1}{4^{n-1}}\varphi_1(x_1)$. Then

$$g_\theta(x) \leq \frac{1}{4^{n-1}}\varphi_1(x_1) \leq \varphi_\ell(x), \quad \frac{1}{4} \leq x \leq \frac{3}{4} \ \text{ and } \ \ell \geq 1.$$

Next, if we apply Theorem 2 to $\varphi_\ell(x) - g_\theta(x)$ on $[0, \frac{1}{4}]$, for each $\ell \geq 1$, then $\varphi_\ell(x) \geq g_\theta(x)$ on $[0, \frac{1}{4}]$. Also, Theorem 3 implies that $\varphi_\ell(x)$ increases on $[0, x_\ell]$ and is concave on $[x_\ell, 1]$, together implying $\varphi_\ell(x) \geq g_\theta(x)$ on $[\frac{3}{4}, 1]$. We conclude

$$g_\theta(x) \leq \varphi_\ell(x), \quad 0 \leq x \leq 1 \ \text{ and } \ \ell \geq 1.$$

Now, set

$$0 < M = \sup\{G(x,s) \mid (x,s) \in [0,1] \times [0,1]\}.$$

Then, assumptions (B) and (F) yield, for $0 \le x \le 1$ and all $\ell \ge 1$,

$$\varphi_\ell(x) = T\varphi_\ell(x) = \int_0^1 G(x,s) f(s, \varphi_\ell(s)) ds =$$
$$\le M \int_0^1 f(s, g_\theta(s)) ds = N,$$

for some $0 < N < \infty$. In particular,

$$|\varphi_\ell|_\infty \le N, \quad \text{for all} \quad \ell \ge 1,$$

which contradicts $\lim_{\ell\to\infty} |\varphi_\ell|_\infty = \infty$. The proof is complete. $\square$

Remark 2. With R as in Theorem 6, $\varphi \le R(wrtK)$, for all solutions $\varphi \in D$ of (1), (2).

Our next step in obtaining solutions of (1), (2) is to construct a sequence of nonsingular perturbations of f. For each $\ell \ge 1$, define $\psi_\ell : [0,1] \to [0,\infty)$ by

$$\psi_\ell(x) = \int_0^1 G(x,s) f(s,\ell) ds.$$

By conditions (A)–(E), for $\ell \ge 1$,

$$0 < \psi_{\ell+1}(x) \le \psi_\ell(x) \ \text{ on } \ (0,1),$$

and

$$\lim_{\ell\to\infty} \psi_\ell(x) = 0 \ \text{ uniformly on } \ [0,1]. \tag{14}$$

Now define a sequence of functions $f_\ell : (0,1) \times [0,\infty) \to (0,\infty)$, $\ell \ge 1$, by

$$f_\ell(x,y) = f(x, \max\{y, \psi_\ell(x)\}).$$

Then, for each $\ell \ge 1$, f_ℓ is continuous and satisfies (B). Furthermore, for $\ell \ge 1$,

$$\begin{aligned} &f_\ell(x,y) \le f(x,y) \text{ on } (0,1) \times (0,\infty), \ \text{ and} \\ &f_\ell(x,y) \le f(x, \psi_\ell(x)) \ \text{ on } \ (0,1) \times (0,\infty). \end{aligned} \tag{15}$$

Theorem 7. *Assume that conditions* (A)–(F) *are satisfied. Then the boundary value problem* (1), (2) *has a solution* $y \in D$.

Proof. We begin by defining a sequence of operators $T_\ell : K \to K$, $\ell \geq 1$, by

$$T_\ell \varphi(x) = \int_0^1 G(x,s) f_\ell(s, \varphi(s)) ds.$$

Note that, for $\ell \geq 1$ and $\varphi \in K$, $(T_\ell \varphi)^{(n)}(x) < 0$ on $(0,1)$, $T_\ell \varphi$ satisfies the boundary conditions (2), and $T_\ell \varphi(x) > 0$ on $(0,1)$; in particular, $T_\ell \varphi \in D$. Since each f_ℓ satisfies (B), it follows that, if φ_1, $\varphi_2 \in K$ with $\varphi_1 \leq \varphi_2 (wrtK)$, then for $\ell \geq 1$, $T_\ell \varphi_2 \leq T_\ell \varphi_1 (wrtK)$; that is, each T_ℓ is decreasiing with respect to K. It is also clear that $0 \leq T_\ell(0)$ and $0 \leq T_\ell^2(0)(wrtK)$, for each ℓ.

Hence when we apply Theorem 1, for each ℓ, there exists a $\varphi_\ell \in K$ such that $T_\ell \varphi_\ell = \varphi_\ell$. The above note implies, for $\ell \geq 1$, that $\varphi_\ell^{(n)}(x) < 0$ on $(0,1)$, φ_ℓ satisfies (2), and $\varphi_\ell(x) > 0$ on $(0,1)$. In addition, inequality (15), coupled with the positivity of $G(x,s)$, yields $T_\ell \varphi \leq T\psi_\ell (wrtK)$, for each $\varphi \in K$ and $\ell \geq 1$. Thus,

$$\varphi_\ell = T_\ell \varphi_\ell \leq T\psi_\ell (wrtK), \ell \geq 1. \tag{16}$$

By essentially the same argument as in Theorem 6, in conjunction with inequality (16), it can be shown that there exists an $R > 0$ such that, for each $\ell \geq 1$,

$$\varphi_\ell \leq R(wrtK). \tag{17}$$

Our next claim is that there exists a $\kappa > 0$ such that $\kappa \leq |\varphi_\ell|_\infty$, for all ℓ. We assume this claim to be false. Then, by passing to a subsequence and relabeling, we assume with no loss of generality that $\lim_{\ell \to \infty} |\varphi_\ell|_\infty = 0$. This implies

$$\lim_{\ell \to \infty} \varphi_\ell(x) = 0 \quad \text{uniformly on} \quad [0,1]. \tag{18}$$

Next set

$$0 < m = \inf \left\{ G(x,s) \mid (x,s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[\frac{1}{4}, \frac{3}{4}\right] \right\}.$$

By condition (D), there exists a $\delta > 0$ such that, for $\frac{1}{4} \leq x \leq \frac{3}{4}$ and $0 < y < \delta$,

$$f(x,y) > \frac{2}{m}.$$

The limit (18) implies there exists an $\ell_0 \geq 1$ such that, for $\ell \geq \ell_0$,

$$0 < \varphi_\ell(x) < \frac{\delta}{2} \quad \text{for} \quad 0 \leq x \leq 1.$$

Also, from (14), there exists an $\ell_1 \geq \ell_0$ such that, for $\ell \geq \ell_1$,

$$0 < \psi_\ell(x) < \frac{\delta}{2} \quad \text{for} \quad \frac{1}{4} \leq x \leq \frac{3}{4}.$$

Thus, for $\ell \geq \ell_1$ and $\frac{1}{4} \leq x \leq \frac{3}{4}$,

$$\varphi_\ell(x) = \int_0^1 G(x,s) f_\ell(s, \varphi_\ell(s)) ds \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G(x,s) f_\ell(s, \varphi_\ell(s)) ds \geq$$

$$\geq m \int_{\frac{1}{4}}^{\frac{3}{4}} f(s, \max\{\varphi_\ell(s), \psi_\ell(s)\}) ds \geq m \int_{\frac{1}{4}}^{\frac{3}{4}} f(s, \frac{\delta}{2}) ds \geq 1.$$

But this contradicts the uniform limit (18). Hence, our claim is verified. That is, there exists a $\kappa > 0$ such that

$$\kappa \leq |\varphi_\ell|_\infty \leq R \quad \text{for all} \quad \ell.$$

Applying Theorem 3,

$$\varphi_\ell(x) \geq \frac{1}{4^{n-1}} |\varphi_\ell|_\infty \geq \frac{\kappa}{4^{n-1}}, \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \quad \ell \geq 1.$$

One can mimic part of the proof of Theorem 6 to show, if $\theta = \frac{\kappa}{4^{n-1}}$, then

$$g_\theta(x) \leq \varphi_\ell(x) \quad \text{on} \quad [0,1] \ \text{for} \ \ell \geq 1.$$

By (17), we now have

$$g_\theta \leq \varphi_\ell \leq R(wrtK) \quad \text{for} \quad \ell \geq 1;$$

that is, the sequence $\{\varphi_\ell\}$ belongs to the closed order interval $\langle g_\theta, R\rangle \subset D$. When restricted to this closed order interval, T is a compact mapping, and so, there is a subsequence of $\{T\varphi_\ell\}$ which converges to some $\varphi^* \in K$. We relabel the subsequence as the original sequence so that $\lim_{\ell\to\infty} \|T\varphi_\ell - \varphi^*\| = 0$.

The final part of the proof is to establish that $\lim_{\ell\to\infty} \|T\varphi_\ell - \varphi_\ell\| = 0$. To this end, let $\theta = \frac{\kappa}{4^{n-1}}$ be as above, and set

$$0 < M = \sup\{G(x,s) \mid (x,s) \in [0,1] \times [0,1]\}.$$

Let $\epsilon > 0$ be given. By the integrabilty condition (F), there exists $0 < \delta < 1$ such that

$$2M \left[\int_0^\delta f(s, g_\theta(s)) ds + \int_{1-\delta}^1 f(s, g_\theta(s)) ds \right] < \epsilon.$$

Further, by (14), there exists an ℓ_0 such that, for $\ell \geq \ell_0$,

$$\psi_\ell(x) \leq g_\theta(x) \quad \text{on} \quad [\delta, 1-\delta],$$

so that

$$\psi_\ell(x) \leq g_\theta(x) \leq \varphi_\ell(x) \quad \text{on} \quad [\delta, 1-\delta].$$

Observe also that, for $\delta \leq s \leq 1-\delta$ and $\ell \geq \ell_0$,

$$f_\ell(s, \varphi_\ell(s)) = f(s, \varphi_\ell(s)).$$

Hence, for $\ell \geq \ell_0$ and $0 \leq x \leq 1$,

$$\begin{aligned} T\varphi_\ell(x) - \varphi_\ell(x) &= T\varphi_\ell(x) - T_\ell\varphi_\ell(x) = \\ &= \int_0^\delta G(x,s)[f(s,\varphi_\ell(s)) - f_\ell(s,\varphi_\ell(s))]ds + \\ &+ \int_{1-\delta}^1 G(x,s)[f(s,\varphi_\ell(s)) - f_\ell(s,\varphi_\ell(s))]ds. \end{aligned}$$

So, for $\ell \geq \ell_0$ and $0 \leq x \leq 1$,

$$\begin{aligned} |T\varphi_\ell(x) - \varphi_\ell(x)| &\leq M\Big[\int_0^\delta [f(s,\varphi_\ell(s)) + f(s, \max\{\varphi_\ell(s), \psi_\ell(s)\})]ds + \\ &+ \int_{1-\delta}^1 [f(s,\varphi_\ell(s)) + f(s, \max\{\varphi_\ell(s), \psi_\ell(s)\})]ds\Big] \leq \\ &\leq 2M\Big[\int_0^\delta f(s,\varphi_\ell(s))ds + \int_{1-\delta}^1 f(s,\varphi_\ell(s))ds\Big] \leq \\ &\leq 2M\Big[\int_0^\delta f(s,g_\theta(s))ds + \int_{1-\delta}^1 f(s,g_\theta(s))ds\Big] < \epsilon. \end{aligned}$$

In particular,

$$\lim_{\ell\to\infty} \|T\varphi_\ell - \varphi_\ell\| = 0.$$

In turn, we have $\lim_{\ell\to\infty} \|\varphi_\ell - \varphi^*\| = 0$, and thus

$$\varphi^* \in \langle g_\theta, R\rangle \subset D,$$

and

$$\varphi^* = \lim_{\ell\to\infty} T\varphi_\ell = T(\lim_{\ell\to\infty} \varphi_\ell) = T\varphi^*,$$

which is sufficient for the conclusion of the theorem. □

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