GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 5, 1997, 413-419

GENERALIZATIONS OF NON-COMMUTATIVE NEUTRIX CONVOLUTION PRODUCTS OF FUNCTIONS

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ABSTRACT. The non-commutative neutrix convolution product of the functions $x^r \cos_-(\lambda x)$ and $x^s \cos_+(\mu x)$ is evaluated. Further similar non-commutative neutrix convolution products are evaluated and deduced.

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x + y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side;
see Gel'fand and Shilov [1].

Note that if f and g are locally summable functions satisfying either of the above conditions then

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x - t)\, dt = \int_{-\infty}^{\infty} f(x - t) g(t)\, dt. \tag{1}$$

It follows that if the convolution product $f * g$ exists by this definition then

$$f * g = g * f, \tag{2}$$

$$(f * g)' = f * g' = f' * g. \tag{3}$$

This definition of the convolution product is rather restrictive and so a neutrix convolution product was introduced in [2]. In order to define the

1991 *Mathematics Subject Classification.* 46F10.
Key words and phrases. Neutrix, neutrix limit, neutrix convolution product.

neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu, \end{cases}$$

for $\nu > 0$.

We now give a new neutrix convolution product which generalizes the one given in [2].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit h exists in the sense that

$$\operatorname*{N-lim}_{\nu\to\infty} \langle f_\nu * g, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix (see van der Corput [3]), having domain N' the positive reals and range N'' the complex numbers, with negligible functions finite linear sums of the functions

$$\nu^\lambda \ln^{r-1}\nu,\ \ln^r \nu,\ \nu^{r-1}e^{\mu\nu} \quad (\text{real } \lambda > 0,\ \text{complex } \mu \ne 0,\ r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g$ is defined in Gel'fand and Shilov's sense, the distribution f_ν having bounded support.

In the original definition of the neutrix convolution product, the domain of the neutrix N was the set of positive integers $N' = \{1, 2, \dots, n, \dots\}$, the range was the set of real numbers, and the negligible functions were finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \dots)$$

and all funtions which converge to zero in the usual sense as n tends to infinity. In [4], the set of negligible functions was extended to include finite linear sums of the functions $n^\lambda e^{\mu n}$ $(\mu > 0)$. In [5], the domain of the neutrix N was replaced by the set of real numbers, and the set of negligible functions was extended to include finite linear sums of the functions

$$\nu^\mu \cos\lambda\nu,\ \ \nu^\mu \sin\lambda\nu \quad (\lambda \ne 0).$$

It is easily seen that any results proved with the earlier definitions hold with this latest definition. The following theorems, proved in [2], therefore hold, the first showing that the neutrix convolution product is a generalization of the convolution product.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'.$$

Note, however, that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

We now define the locally summable functions $e_+^{\lambda x}$, $e_-^{\lambda x}$, $\cos_+(\lambda x)$, $\cos_-(\lambda x)$, $\sin_+(\lambda x)$ and $\sin_-(\lambda x)$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0, \end{cases}$$
$$\cos_+(\lambda x) = \begin{cases} \cos(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \cos_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \cos(\lambda x), & x < 0, \end{cases}$$
$$\sin_+(\lambda x) = \begin{cases} \sin(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \sin_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \sin(\lambda x), & x < 0. \end{cases}$$

It follows that

$$\cos_-(\lambda x) + \cos_+(\lambda x) = \cos(\lambda x), \quad \sin_-(\lambda x) + \sin_+(\lambda x) = \sin(\lambda x).$$

The following two theorems were proved in [4] and [5], respectively.

Theorem 3. *The neutrix convolution product $(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x})$ exists and*

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}, \tag{4}$$

where $D_\lambda = \partial/\partial\lambda$ and $D_\mu = \partial/\partial\mu$, for $\lambda \neq \mu$ and $r, s = 0, 1, 2, \ldots$, these neutrix convolution products existing as convolution products if $\lambda > \mu$ (or $\Re\lambda > \Re\mu$ for complex λ, μ) and

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\lambda x}) = B(r+1, s+1) x^{r+s+1} e_-^{\lambda x}, \tag{5}$$

where B denotes the Beta function, for all λ and $r, s = 0, 1, 2, \ldots$.

Note that for complex $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$, equation (4) can be replaced by the equation

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) = D_{\lambda_1}^r D_{\mu_1}^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} \tag{6}$$

or by the equation

$$(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\mu x}) = D_{\lambda_2}^r D_{\mu_2}^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu}. \tag{7}$$

Theorem 4. *The neutrix convolution products* $\cos_-(\lambda x) \circledast \cos_+(\mu x)$, $\cos_-(\lambda x) \circledast \sin_+(\mu x)$, $\sin_-(\lambda x) \circledast \cos_+(\mu x)$, *and* $\sin_-(\lambda x) \circledast \sin_+(\mu x)$ *exist and*

$$\cos_-(\lambda x) \circledast \cos_+(\mu x) = \frac{\lambda \sin_-(\lambda x) + \mu \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{8}$$

$$\cos_-(\lambda x) \circledast \sin_+(\mu x) = -\frac{\mu \cos_-(\lambda t) + \mu \cos_+(\mu t)}{\lambda^2 - \mu^2}, \tag{9}$$

$$\sin_-(\lambda x) \circledast \cos_+(\mu x) = -\frac{\lambda \cos_-(\lambda x) + \lambda \cos_+(\mu x)}{\lambda^2 - \mu^2}, \tag{10}$$

$$\sin_-(\lambda x) \circledast \sin_+(\mu x) = -\frac{\mu \sin_-(\lambda x) + \lambda \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{11}$$

for $\lambda \neq \pm\mu$.

We now give some generalizations of Theorems 3 and 4.

Theorem 5. *The neutrix convolution product* $[x^r e^{\lambda_1 x} \cos_-(\lambda_2 x)] \circledast [x^s e^{\mu_1 x} \cos_+(\mu_2 x)]$ *exists and*

$$\begin{aligned}
&[x^r e^{\lambda_1 x} \cos_-(\lambda_2 x)] \circledast [x^s e^{\mu_1 x} \cos_+(\mu_2 x)] = \\
&= D_{\lambda_1}^r D_{\mu_1}^s \left\{ \frac{(\lambda_1 - \mu_1)[e^{\mu_1 x} \cos_+(\mu_2 x) + e^{\lambda_1 x} \cos_-(\lambda_2 x)]}{2|\lambda - \mu|^2} + \right. \\
&\qquad + \frac{(\lambda_2 - \mu_2)[e^{\mu_1 x} \sin_+(\mu_2 x) + e^{\lambda_1 x} \sin_-(\lambda_2 x)]}{2|\lambda - \mu|^2} + \\
&\qquad + \frac{(\lambda_1 - \mu_1)[e^{\mu_1 x} \cos_+(\mu_2 x) + e^{\lambda_1 x} \cos_-(\lambda_2 x)]}{2|\lambda - \bar{\mu}|^2} - \\
&\qquad \left. - \frac{(\lambda_2 + \mu_2)[e^{\mu_1 x} \sin_+(\mu_2 x) - e^{\lambda_1 x} \sin_-(\lambda_2 x)]}{2|\lambda - \bar{\mu}|^2} \right\}
\end{aligned} \tag{12}$$

for $\lambda = \lambda_1 + i\lambda_2 \neq \mu = \mu_1 + i\mu_2$, $\lambda \neq \bar{\mu}$ *and* $r, s = 0, 1, 2, \ldots$ *and*

$$\begin{aligned}
&[x^r e^{\lambda_1 x} \cos_-(\lambda_2 x)] \circledast [x^s e^{\lambda_1 x} \cos_+(\lambda_2 x)] = \\
&\quad = \tfrac{1}{2} B(r+1, s+1) x^{r+s+1} e^{\lambda_1 x} \cos_-(\lambda_2 x) +
\end{aligned}$$

$$+ D^r_{\lambda_1} D^s_{\mu_1} \left\{ \frac{2(\lambda_1 - \mu_1)[e^{\mu_1 x}\cos_+(\lambda_2 x) + e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{(\lambda_1 - \mu_1)^2 + 4\lambda_2^2} - \right.$$
$$\left. - \frac{2\lambda_2[e^{\mu_1 x}\sin_+(\lambda_2 x) - e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{(\lambda_1 - \mu_1)^2 + 4\lambda_2^2} \right\}_{\mu = \lambda_1 - i\lambda_2}, \qquad (13)$$

for all $\lambda_1, \lambda_2 \neq 0$ *and* $r, s = 0, 1, 2, \ldots$.

In particular

$$[e^{\lambda_1 x}\cos_-(\lambda_2 x)] \circledast [e^{\lambda_1 x}\cos_+(\lambda_2 x)] =$$
$$= \frac{\lambda_2 x e^{\lambda_1 x}\cos_-(\lambda_2 x) - e^{\lambda_1 x}\sin_+(\lambda_2 x) + e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{2\lambda_2},$$

for $\lambda_2 \neq 0$.

Proof. Using equation (4) with $\lambda \neq \mu$, $\bar{\mu}$, we have

$$4[e^{\lambda_1 x}\cos_-(\lambda_2 x)] \circledast [e^{\mu_1 x}\cos_+(\mu_2 x)] =$$
$$= [e^{\lambda_1 x}(e_-^{i\lambda_2 x} + e_-^{-i\lambda_2 x})] \circledast [e^{\mu_1 x}(e_+^{i\mu_2 x} + e_+^{-i\mu_2 x})] =$$
$$= e_-^{\lambda x} \circledast e_+^{\mu x} + e_-^{\bar{\lambda} x} \circledast e_+^{\mu x} + e_-^{\lambda x} \circledast e_+^{\bar{\mu} x} + e_-^{\bar{\lambda} x} \circledast e_+^{\bar{\mu} x} =$$
$$= \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} + \frac{e_+^{\mu x} + e_-^{\bar{\lambda} x}}{\bar{\lambda} - \mu} + \frac{e_+^{\bar{\mu} x} + e_-^{\lambda x}}{\lambda - \bar{\mu}} + \frac{e_+^{\bar{\mu} x} + e_-^{\bar{\lambda} x}}{\bar{\lambda} - \bar{\mu}} =$$
$$= \frac{(\lambda_1 - \mu_1)(e_+^{\mu x} + e_+^{\bar{\mu} x} + e_-^{\lambda x} + e_-^{\bar{\lambda} x}) + i(\lambda_2 - \mu_2)(e_+^{\bar{\mu} x} - e_+^{\mu x} + e_-^{\bar{\lambda} x} - e_-^{\lambda x})}{|\lambda - \mu|^2} +$$
$$+ \frac{(\lambda_1 - \mu_1)(e_+^{\mu x} + e_+^{\bar{\mu} x} + e_-^{\lambda x} + e_-^{\bar{\lambda} x}) + i(\lambda_2 + \mu_2)(e_+^{\mu x} - e_+^{\bar{\mu} x} + e_-^{\bar{\lambda} x} - e_-^{\lambda x})}{|\lambda - \bar{\mu}|^2}$$
$$= \frac{2(\lambda_1 - \mu_1)[e^{\mu_1 x}\cos_+(\mu_2 x) + e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{|\lambda - \mu|^2} +$$
$$+ \frac{2(\lambda_2 - \mu_2)[e^{\mu_1 x}\sin_+(\mu_2 x) + e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{|\lambda - \mu|^2} +$$
$$+ \frac{2(\lambda_1 - \mu_1)[e^{\mu_1 x}\cos_+(\mu_2 x) + e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{|\lambda - \bar{\mu}|^2} -$$
$$- \frac{2(\lambda_2 + \mu_2)[e^{\mu_1 x}\sin_+(\mu_2 x) - e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{|\lambda - \bar{\mu}|^2}$$

and equation (12) follows.

Similarly, using equations (4) and (5) with $\lambda \neq \bar{\lambda}$ we have

$$4[x^r e^{\lambda_1 x}\cos_-(\lambda_2 x)] \circledast [x^s e^{\lambda_1 x}\cos_+(\lambda_2 x)] =$$
$$= [x^r e^{\lambda_1 x}(e_-^{i\lambda_2 x} + e_-^{-i\lambda_2 x})] \circledast [x^s e^{\lambda_1 x}(e_+^{i\lambda_2 x} + e_+^{-i\lambda_2 x})] =$$
$$= (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\lambda x}) + (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\bar{\lambda} x}) + (x^r e_-^{\bar{\lambda} x}) \circledast (x^s e_+^{\lambda x}) +$$

$$
\begin{aligned}
&\qquad\qquad + (x^r e_-^{\bar\lambda x}) \circledast (x^s e_+^{\bar\lambda x}) = \\
&= B(r+1,+1)x^{r+s+1}(e_-^{\lambda x} + e_-^{\bar\lambda x}) + (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\bar\lambda x}) + \\
&\qquad\qquad + (x^r e_-^{\bar\lambda x}) \circledast (x^s e_+^{\lambda x}) = \\
&= 2B(r+1,s+1)x^{r+s+1}e^{\lambda_1 x}\cos_-(\lambda_2 x) + (x^r e_-^{\lambda x}) \circledast (x^s e_+^{\bar\lambda x}) + \\
&\qquad\qquad + (x^r e_-^{\bar\lambda x}) \circledast (x^s e_+^{\lambda x}).
\end{aligned}
$$

In order to evaluate $(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\bar\lambda x}) + (x^r e_-^{\bar\lambda x}) \circledast (x^s e_+^{\lambda x})$ we put $\mu = \mu_1 - i\lambda_2$ and consider

$$
\begin{aligned}
&e_-^{\lambda x} \circledast e_+^{\mu x} + e_-^{\bar\lambda x} \circledast e_+^{\bar\mu x} = \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu} + \frac{e_+^{\bar\mu x} + e_-^{\bar\lambda x}}{\bar\lambda - \bar\mu} = \\
&= \frac{(\lambda_1 - \mu_1)(e_+^{\mu x} + e_+^{\bar\mu x} + e_-^{\lambda x} + e_-^{\bar\lambda x}) + 2i\lambda_2(e_+^{\bar\mu x} - e_+^{\mu x} + e_-^{\bar\lambda x} - e_-^{\lambda x})}{|\lambda - \mu|^2} = \\
&= \frac{2(\lambda_1 - \mu_1)[e^{\mu_1 x}\cos_+(\lambda_2 x) + e^{\lambda_1 x}\cos_-(\lambda_2 x)]}{(\lambda_1 - \mu_1)^2 + 4\lambda_2^2} - \\
&\qquad - \frac{2\lambda_2[e^{\mu_1 x}\sin_+(\lambda_2 x) - e^{\lambda_1 x}\sin_-(\lambda_2 x)]}{(\lambda_1 - \mu_1)^2 + 4\lambda_2^2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(x^r e_-^{\lambda x}) \circledast (x^s e_+^{\bar\lambda x}) + (x^r e_-^{\bar\lambda x}) \circledast (x^s e_+^{\lambda x}) = \\
&\qquad = D_{\lambda_1}^r D_{\mu_1}^s [e_-^{\lambda x} \circledast e_+^{\mu x} + e_-^{\bar\lambda x} \circledast e_+^{\bar\mu x}]_{\mu = \lambda_1 - i\lambda_2}
\end{aligned}
$$

and equation (13) follows.

Note that by replacing x by $-x$ in equation (12) gives an expression for

$$[x^r e^{\lambda_1 x}\cos_+(\lambda_2 x)] \circledast [x^s e^{\mu_2 x}\cos_-(\mu_2 x)].$$

Expressions for

$$
\begin{aligned}
&[x^r e^{\lambda_1 x}\cos_\pm(\lambda_2 x)] \circledast [x^s e^{\mu_2 x}\sin_\mp(\mu_2 x)], \\
&[x^r e^{\lambda_1 x}\sin_\pm(\lambda_2 x)] \circledast [x^s e^{\mu_2 x}\cos_\mp(\mu_2 x)], \\
&[x^r e^{\lambda_1 x}\sin_\pm(\lambda_2 x)] \circledast [x^s e^{\mu_2 x}\sin_\mp(\mu_2 x)]
\end{aligned}
$$

can be obtained similarly. □

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(Received 16.08.1995)

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