GEORGIAN MATHEMATICAL JOURNAL: Vol. 4, No. 5, 1997, 421-438

SOLUTION OF THE BASIC BOUNDARY VALUE PROBLEMS OF STATIONARY THERMOELASTIC OSCILLATIONS FOR DOMAINS BOUNDED BY SPHERICAL SURFACES

L. GIORGASHVILI

Abstract. The boundary value problems of stationary thermoelastic oscillations are investigated for the entire space with a spherical cavity, when the limit values of a displacement vector and temperature or of a stress vector and heat flow are given on the boundary. Also, consideration is given to the boundary-contact problems when a nonhomogeneous medium fills up the entire space and consists of several homogeneous parts with spherical interface surfaces. Given on an interface surface are differences of the limit values of displacement and stress vectors, also of temperature and heat flow, while given on a free boundary are the limit values of a displacement vector and temperature or of a stress vector and heat flow. Solutions of the considered problems are represented as absolutely and uniformly convergent series.

One of the main methods of solving the spatial problems of elasticity is the Fourier method based on using various representations of solutions of equilibrium equations through harmonic, biharmonic, or metaharmonic functions.

When solving problems by the said method the main difficulty consists in satisfying the boundary conditions. One of the approaches to overcoming this difficulty developed in [1] and [2] is to construct eigenfunctions of vector structure on the boundary.

In this paper, systems of homogeneous equations of stationary thermoelastic oscillations are solved in terms of four metaharmonic functions. Such a representation of solutions enables one to satisfy the boundary conditions quite easily.

1991 *Mathematics Subject Classification.* 73C02.
Key words and phrases. Thermoelasticity, Fourier method.

1072-947X/97/0900-0421$12.50/0 © 1997 Plenum Publishing Corporation

Some Auxiliary Formulas and Theorems. A system of homogeneous equations of stationary thermoelastic oscillations has the form [3], [4]

$$
\begin{aligned}
&\mu\Delta u+(\lambda+\mu)\operatorname{grad}\operatorname{div}u-\gamma\operatorname{grad}u_4+\rho\sigma^2u=0,\\
&\Delta u_4+\frac{i\sigma}{\varkappa}\,u_4+i\sigma\eta\operatorname{div}u=0,
\end{aligned}
\tag{1}
$$

where Δ is the Laplace operator, $u=(u_1,u_2,u_3)$ the elastic displacement vector, u_4 the temperature, ρ the medium density, σ the oscillation frequency; λ, μ, η, $\varkappa$, γ are the constants characterizing the physical properties of the considered elastic body and satisfying the conditions

$$
\mu>0,\quad 3\lambda+2\mu>0,\quad \gamma/\eta>0,\quad \varkappa>0,\quad \lambda+2\mu\neq\gamma\eta\varkappa.
$$

The thermoelastic stress vector is written as [4]

$$
P(\partial_x,n)U=T(\partial x,n)u-\gamma n(x)u_4, \tag{2}
$$

where $U=(u,u_4)$, $T(\partial_x,n)u$ is the stress vector of classical elasticity,

$$
T(\partial_x,n)u=2\mu\,\frac{\partial u}{\partial n}+\lambda n\operatorname{div}u+\mu[n\times\operatorname{rot}u].
$$

We introduce the notation [5]

$$
\begin{aligned}
&X_{mk}(\theta,\varphi)=e_rY_k^{(m)}(\theta,\varphi),\quad k\geq 0,\\
&Y_{mk}(\theta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(e_\theta\,\frac{\partial}{\partial\theta}+\frac{e_\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\theta,\varphi),\quad k\geq 1,\\
&Z_{mk}(\theta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\Big(\frac{e_\theta}{\sin\theta}\,\frac{\partial}{\partial\varphi}-e_\varphi\,\frac{\partial}{\partial\theta}\Big)Y_k^{(m)}(\theta,\varphi),\quad k\geq 1,
\end{aligned}
\tag{3}
$$

where $|m|\leq k$, e_r, and e_θ, e_φ are the orthogonal unit vectors:

$$
\begin{aligned}
&e_r=(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta),\\
&e_\theta=(\cos\varphi\cos\theta,\sin\varphi\cos\theta,-\sin\theta),\\
&e_\varphi=(-\sin\varphi,\cos\varphi,0),\\
&Y_k^{(m)}(\theta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\;P_k^{(m)}(\cos\theta)e^{im\varphi},
\end{aligned}
$$

where $P_k^{(m)}(\cos\theta)$ is the adjoint Legendre function of first kind, kth degree, and mth order.

On the sphere $r = const$, vectors (3) form a complete orthonormal system of vector-functions [5]. Let us show that the following formulas are valid:

$$[e_r \times X_{mk}(\theta,\varphi)] = 0, \quad [e_r \times Y_{mk}(\theta,\varphi)] = -Z_{mk}(\theta,\varphi), \\ [e_r \times Z_{mk}(\theta,\varphi)] = Y_{mk}(\theta,\varphi), \tag{4}$$

$$\operatorname{grad}\big(\Phi(r)Y_k^{(m)}(\theta,\varphi)\big) = \frac{d\Phi(r)}{dr}\,X_{mk}(\theta,\varphi) + \\ + \frac{\sqrt{k(k+1)}}{r}\,\Phi(r)Y_{mk}(\theta,\varphi), \tag{5}$$

$$\operatorname{rot}[x\Phi(r)Y_k^{(m)}(\theta,\varphi)] = \sqrt{k(k+1)}\,\Phi(r)Z_{mk}(\theta,\varphi), \tag{6}$$

where $\Phi(r)$ is a function of r, $x = (x_1, x_2, x_3)$, (r,θ,φ) are the spherical coordinates of the point x.

If in formulas (3) we set

$$[e_r \times e_r] = 0, \quad [e_r \times e_\theta] = e_\varphi, \quad [e_r \times e_\varphi] = -e_\theta,$$

then we will obtain formula (4).

We rewrite the operator grad in terms of spherical coordinates

$$\operatorname{grad} = e_r\,\frac{\partial}{\partial r} + \frac{1}{r}\Big(e_\theta\,\frac{\partial}{\partial\theta} + \frac{e_\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi}\Big),$$

and obtain

$$\operatorname{grad}[\Phi(r)Y_k^{(m)}(\theta,\varphi)] = \big(e_r Y_k^{(m)}(\theta,\varphi)\big)\,\frac{d\Phi(r)}{dr} + \\ + \frac{1}{r}\,\Phi(r)\Big(e_\theta\,\frac{\partial}{\partial\theta} + \frac{e_\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi}\Big)Y_k^{(m)}(\theta,\varphi) = \\ = \frac{d\Phi(r)}{dr}\,X_{mk}(\theta,\varphi) + \frac{\sqrt{k(k+1)}}{r}\,\Phi(r)Y_{mk}(\theta,\varphi),$$

which proves equality (5). The proof of formula (6) follows from (4), (5) and from the identity

$$\operatorname{rot}\big[x\Phi(r)Y_k^{(m)}(\theta,\varphi)\big] = -\big[x \times \operatorname{grad}\big(\Phi(r)Y_k^{(m)}(\theta,\varphi)\big)\big].$$

In what follows it will be convenient for us to represent the Fourier series of the vector-function $f(\theta,\varphi)$ by system (3) as

$$f(\theta,\varphi) = \alpha_{00}X_{00}(\theta,\varphi) + \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\big\{\alpha_{mk}X_{mk}(\theta,\varphi) + \\ + \sqrt{k(k+1)}\,[\beta_{mk}Y_{mk}(\theta,\varphi) + \gamma_{mk}Z_{mk}(\theta,\varphi)]\big\}, \tag{7}$$

where α_{mk}, β_{mk}, γ_{mk} are the Fourier coefficients:

$$
\begin{aligned}
\alpha_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} f(\theta,\varphi)\cdot \overline{X}_{mk}(\theta,\varphi)\sin\theta\, d\theta, \quad k\geq 0,\\
\beta_{mk} &= \frac{1}{\sqrt{k(k+1)}} \int_0^{2\pi} d\varphi \int_0^{\pi} f(\theta,\varphi)\cdot \overline{Y}_{mk}(\theta,\varphi)\sin\theta\, d\theta, \quad k\geq 1, \qquad (8)\\
\gamma_{mk} &= \frac{1}{\sqrt{k(k+1)}} \int_0^{2\pi} d\varphi \int_0^{\pi} f(\theta,\varphi)\cdot \overline{Z}_{mk}(\theta,\varphi)\sin\theta\, d\theta, \quad k\geq 1,
\end{aligned}
$$

$\overline{X}_{mk}$, $\overline{Y}_{mk}$, $\overline{Z}_{mk}$ are the vectors complex-conjugated to x_{mk}, y_{mk}, z_{mk}, respectively.

If in formulas (8) we take into account that on the sphere of unit radius

$$
\begin{aligned}
[Y_{mk}(\theta,\varphi)]_j &= \frac{1}{k(k+1)}\,\mathcal{D}_j Y_k^{(m)}(\theta,\varphi),\\
[Z_{mk}(\theta,\varphi)]_j &= \frac{1}{k(k+1)}\,\frac{\partial}{\partial s_j}\,Y_k^{(m)}(\theta,\varphi),
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial}{\partial s_j} &= \Big(\frac{e_\theta}{\sin\theta}\,\frac{\partial}{\partial\varphi} - e_\varphi\,\frac{\partial}{\partial\theta}\Big)_j,\\
\mathcal{D}_j &= \Big(e_\theta\,\frac{\partial}{\partial\theta} + \frac{e_\varphi}{\sin\theta}\,\frac{\partial}{\partial\varphi}\Big)_j, \quad j=1,2,3,
\end{aligned}
$$

then we will obtain

$$
\begin{aligned}
\alpha_{mk} &= \int_0^{2\pi} d\varphi \int_0^{\pi} (f\cdot e_r)\overline{Y}_k^{(m)}(\theta,\varphi)\sin\theta\, d\theta, \quad k\geq 0,\\
\beta_{mk} &= \frac{1}{\sqrt{k(k+1)}} \int_0^{2\pi} d\varphi \int_0^{\pi} \Big[2(f\cdot e_r) - \sum_{j=1}^{3}\mathcal{D}_j f_j\Big]\overline{Y}_k^{(m)}(\theta,\varphi)\sin\theta\, d\theta, \qquad (9)\\
\gamma_{mk} &= \frac{1}{\sqrt{k(k+1)}} \int_0^{2\pi} d\varphi \int_0^{\pi} \sum_{j=1}^{3}\frac{\partial f_j}{\partial s_j}\,\overline{Y}_k^{(m)}(\theta,\varphi)\sin\theta\, d\theta.
\end{aligned}
$$

Let

$$
F(\theta,\varphi) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k} a_{mk} Y_k^{(m)}(\theta,\varphi)
$$

be a Fourier series with respect to the orthonormalized system of spherical functions $Y_k^{(m)}$, where

$$a_{mk} = \int_0^{2\pi} d\varphi \int_0^{\pi} F(\theta,\varphi)\overline{Y}_k^{(m)}(\theta,\varphi)\sin\theta\, d\theta.$$

The following theorem is true [6].

Theorem 1. *If $F(y) \in C^{(l)}(S)$, then Fourier coefficients admit the estimates*

$$a_{mk} = O(k^{-l}).$$

This theorem and formula (9) imply

Theorem 2. *If $f(y) \in C^{(l)}(S)$, then the Fourier coefficients α_{mk}, β_{mk}, γ_{mk} admit the estimates*

$$\alpha_{mk} = O(k^{-l}), \quad \beta_{mk} = O(k^{-l-1}), \quad \gamma_{mk} = O(k^{-l-1}).$$

Theorem 3. *The vectors $X_{mk}(\theta,\varphi)$, $Y_{mk}(\theta,\varphi)$, and $Z_{mk}(\theta,\varphi)$ satisfy the estimates*

$$\begin{aligned} |X_{mk}(\theta,\varphi)| &\le \sqrt{\frac{2k+1}{4\pi}}\,, \\ |Y_{mk}(\theta,\varphi)| &< \sqrt{\frac{2k(k+1)}{2k+1}}\,, \\ |Z_{mk}(\theta,\varphi)| &< \sqrt{\frac{2k(k+1)}{2k+1}}\,. \end{aligned} \tag{10}$$

Proof. Using the recurrent relations of Legendre polynomials, the vector $Y_{mk}(\theta,\varphi)$ can be represented as

$$\begin{aligned} Y_{mk}(\theta,\varphi) = {} & \frac{1}{2\sqrt{k(k+1)(2k+1)}} \times \\ & \times \Big\{ -e_1\Big[\frac{k+1}{\sqrt{2k-1}}\sqrt{(k+m)(k+m-1)}\,Y_{k-1}^{(m-1)}(\theta,\varphi) + \\ & + \frac{k}{\sqrt{2k+3}}\sqrt{(k-m+1)(k-m+2)}\,Y_{k+1}^{(m-1)}(\theta,\varphi)\Big] + \\ & + e_2\Big[\frac{k+1}{\sqrt{2k-1}}\sqrt{(k-m)(k-m-1)}\,Y_{k-1}^{(m-1)}(\theta,\varphi) + \\ & + \frac{k}{\sqrt{2k+3}}\sqrt{(k+m+1)(k+m+2)}\,Y_{k+1}^{(m+1)}(\theta,\varphi)\Big] + \\ & + 2e_3\Big[\frac{k+1}{\sqrt{2k-1}}\sqrt{k^2-m^2}\,Y_{k-1}^{(m)}(\theta,\varphi) - \end{aligned}$$

$$-\frac{k}{\sqrt{2k+3}}\sqrt{(k+1)^2-m^2}\,Y_{k+1}^{(m)}(\theta,\varphi)\Big]\Big\}, \tag{11}$$

where $e_1=(1,i,0)$, $e_2=(1,-i,0)$, $e_3=(0,0,1)$.

According to [7] we have

$$|Y_k^{(m)}(\theta,\varphi)|\le\sqrt{\frac{2k+1}{4\pi}}. \tag{12}$$

By virtue of the latter inequality formulas (3) and (11) yield

$$\begin{aligned}
&|X_{mk}(\theta,\varphi)|=|e_r\cdot Y_k^{(m)}(\theta,\varphi)|=|Y_k^{(m)}(\theta,\varphi)|\le\sqrt{\frac{2k+1}{4\pi}},\\
&|Y_{mk}(\theta,\varphi)|<\sqrt{\frac{2k(k+1)}{2k+1}},\\
&|Z_{mk}(\theta,\varphi)|=|e_r\times Y_{mk}(\theta,\varphi)|=|Y_{mk}(\theta,\varphi)|<\sqrt{\frac{2k(k+1)}{2k+1}}. \qquad \square
\end{aligned}$$

Definition 4. A solution $U=(u,u_4)$ of system (1) will be called regular in the domain Ω if $U\in C^2(\Omega)\cap C^1(\overline{\Omega})$.

The following theorem is true [4].

Theorem 5. *A regular solution of equation* (1) *admits a representation of the form*

$$\begin{aligned}
u(x)&=u^{(1)}(x)+u^{(2)}(x)+u^{(3)}(x),\\
u_4(x)&=u_4^{(1)}(x)+u_4^{(2)}(x),
\end{aligned} \tag{13}$$

where

$$\begin{gathered}
(\Delta+\lambda_j^2)u^{(j)}(x)=0,\ \ j=1,2,3,\quad (\Delta+\lambda_j^2)u_4^{(j)}(x)=0,\ \ j=1,2,\\
\operatorname{rot}u^{(j)}(x)=0,\ \ j=1,2,\quad \operatorname{div}u^{(3)}(x)=0,\quad \lambda_3^2=\frac{\rho\sigma^2}{\mu},\\
\lambda_1^2+\lambda_2^2=\frac{\rho\sigma^2}{\lambda+2\mu}+\frac{i\sigma}{\varkappa}+\frac{i\sigma\gamma\eta}{\lambda+2\mu},\quad \lambda_1^2\cdot\lambda_2^2=\frac{i\sigma}{\varkappa}\cdot\frac{\rho\sigma^2}{\lambda+2\mu}.
\end{gathered} \tag{14}$$

For $\gamma\ne0$ values λ_1^2, λ_2^2 are complex numbers. Choose values λ_1 and λ_2 so that their imaginary parts are positive, i.e., $\lambda_j=\alpha_j+i\beta_j$, $\beta_j>0$, $j=1,2$.

Definition 6. A solution $U = (u, u_4)$ of system (1) will be said to satisfy the thermoelastic radiation condition at infinity if

$$\begin{aligned} &u^{(j)}(x) = o(r^{-1}), \quad \frac{\partial u^{(j)}}{\partial x_k} = O(r^{-2}), \\ &u_4^{(j)}(x) = o(r^{-1}), \quad \frac{\partial u_4^{(j)}}{\partial x_k} = O(r^{-2}), \quad j = 1, 2, \quad k = 1, 2, 3, \\ &u^{(3)}(x) = O(r^{-1}), \quad \frac{\partial u^{(3)}}{\partial r} - i\lambda_3 u^{(3)} = o(r^{-1}), \quad r = |x|. \end{aligned} \tag{15}$$

Denote by Ω_0 a ball bounded by the spherical surface S with center at the origin and radius R. A complement to the set $\overline{\Omega}_0 = \Omega_0 \cup S$ will be denoted by $\Omega_1 = E_3 \setminus \overline{\Omega}_0$.

Theorem 7. *A regular solution of equation* (1) *admits, in the domain* Ω_j*,* $j = 0, 1$*, a representation of the form*

$$\begin{aligned} u(x) &= \operatorname{grad}[\Phi_1(x) + \Phi_2(x)] + \operatorname{rot}\operatorname{rot}(x\Phi_3) + \operatorname{rot}(x\Phi_4), \\ u_4(x) &= c[(k_1^2 - \lambda_1^2)\Phi_1(x) + (k_1^2 - \lambda_2^2)\Phi_2(x)], \end{aligned} \tag{16}$$

where

$$\begin{gathered} (\Delta + \lambda_j^2)\Phi_j(x) = 0, \quad j = 1, 2, \quad (\Delta + \lambda_3^2)\Phi_j(x) = 0, \quad j = 3, 4, \\ c = (\lambda + 2\mu)/\gamma, \quad k_1^2 = \rho\sigma^2/(\lambda + 2\mu). \end{gathered} \tag{17}$$

Proof. The vectors $u^{(j)}(x)$, $j = 1, 2, 3$, satisfying equations (14) admit the following representations [5]:

$$\begin{aligned} u^{(j)}(x) &= \operatorname{grad}\Phi_j(x), \quad j = 1, 2, \\ u^{(3)}(x) &= \operatorname{rot}\operatorname{rot}(x\Phi_3) + \operatorname{rot}(x\Phi_4), \end{aligned} \tag{18}$$

where $\Phi_j(x)$, $j = 1, 2, 3, 4$, are the scalar functions satisfying equations (17).

If the values of the vector $u^{(j)}(x)$, $j = 1, 2, 3$, from (18) are substituted into (13), we will have

$$u(x) = \operatorname{grad}[\Phi_1(x) + \Phi_2(x)] + \operatorname{rot}\operatorname{rot}(x\Phi_3) + \operatorname{rot}(x\Phi_4). \tag{19}$$

System (1) implies

$$u_4(x) = \Big[\frac{(\lambda + 2\mu)\varkappa i}{\gamma\sigma}\,(\Delta + k_1^2) - \varkappa\eta\Big] \operatorname{div} u. \tag{20}$$

The substitution of the values of the vector $u(x)$ from (19) into (20) gives

$$u_4(x) = c[(k_1^2 - \lambda_1^2)\Phi_1(x) + (k_1^2 - \lambda_2^2)\Phi_2(x)],$$

which proves that representation (16) is valid. One can immediately prove that the vector (u,u_4) represented by formula (16) is a solution of system (1). $\square$

Formulation of the Problems. The following problems will be considered: find, in Ω_1, a regular vector $U=(u,u_4)$ satisfying system (1), the radiation conditions at infinity, and one of the following boundary conditions:

Problem (I)$^-$. $\big\{u(z)\big\}^-=f(z), \quad \big\{u_4(z)\big\}^-=f_4(z), \quad z\in S$;

Problem (II)$^-$. $\big\{P(\partial_z,n)U(z)\big\}^-=f(z), \quad \Big\{\dfrac{\partial u_4(z)}{\partial n(z)}\Big\}^-=f_4(z), \ \ z\in S,$

where $n(z)$ is the external normal unit vector with respect to Ω_0 at the point $z\in S$. Note that $n(x)\equiv e_r$; $f(z)=(f_1(z),f_2(z),f_3(z))$, $f_j(z)$, $j=1,2,3,4$, are the given functions.

Problem A. Find in Ω_j, $j=0,1$, a regular vector $U^{(j)}(x)=(u^{(j)}(x),u_4^{(j)}(x))$ satisfying the equation

$$\begin{aligned}&\mu_j\Delta u^{(j)}+(\lambda_j+\mu_j)\operatorname{grad}\operatorname{div}u^{(j)}-\gamma_j\operatorname{grad}u_4^{(j)}+\rho_j\sigma_j^2u^{(j)}=0,\\&\Delta u_4^{(j)}+\frac{i\sigma_j}{\varkappa_j}\,u_4^{(j)}+i\sigma_j\eta_j\operatorname{div}u^{(j)}=0, \ \ j=0,1,\end{aligned}\tag{21}$$

for $j=0,1$, the radiation conditions at infinity for $j=1$, and, on the boundary S, the contact conditions

$$\begin{aligned}&\big\{u^{(0)}(z)\big\}^+-\big\{u^{(1)}(z)\big\}^-=f^{(0)}(z),\\&\big\{u_4^{(0)}(z)\big\}^+-\big\{u_4^{(1)}(z)\big\}^-=f_4^{(0)}(z),\\&\big\{P^{(0)}(\partial_z,n)U^{(0)}(z)\big\}^+-\big\{P^{(1)}(\partial_z,n)U^{(1)}(z)\big\}^-=f^{(1)}(z),\\&\frac{\gamma_0}{\sigma_0\eta_0}\Big\{\frac{\partial u_4^{(0)}(z)}{\partial n(z)}\Big\}^+-\frac{\gamma_1}{\sigma_1\eta_1}\Big\{\frac{\partial u_4^{(1)}(z)}{\partial n(z)}\Big\}^-=f_4^{(1)}(z),\end{aligned}\tag{22}$$

where $f^{(j)}(z)=(f_1^{(j)},f_2^{(j)},f_3^{(j)})$, $j=0,1$, $f_l^{(j)}(z)$, $l=1,2,3,4$, are the given functions.

Solution of Problems (I)$^-$, (II)$^-$. A solution of these problems is sought for in form (16), where the functions $\Phi_j(x)$, $j=1,2,3,4$, are written as

$$\Phi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(\lambda_jr)Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)}, \ \ j=1,2, \tag{23}$$

$$\Phi_j(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(\lambda_3r)Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)}, \ \ j=3,4, \tag{24}$$

where $A_{mk}^{(j)}$, $j=1,2,3,4$, are the unknown constants,

$$h_k(\lambda_j r)=\sqrt{\frac{R}{r}}\,\frac{H_{k+1/2}^{(1)}(\lambda_j r)}{H_{k+1/2}^{(1)}(\lambda_j R)},\quad j=1,2,3, \tag{25}$$

$H_{k+1/2}^{(1)}(x)$ is Hankel's function of first kind.

We will impose on the function $\Phi_j(x)$, $j=3,4$, the condition

$$\int\limits_{S'}\big[\Phi_j(z)\big]^- d_z S=0,\quad j=3,4, \tag{26}$$

where S' is the spherical surface with center at the origin and radius R' $(R<R'<+\infty)$.

If the values of $\Phi_j(x)$, $j=3,4$, from (24) are substituted into (26), we will have $A_{00}^{(j)}=0$, $j=3,4$.

By putting the expression of the vector $U=(u,u_4)$ from (16) into (2) we obtain

$$\begin{aligned}P(\partial_x,n)U(x)&=2\mu\,\frac{\partial u}{\partial r}+\mu e_r\big[(2\lambda_1^2-\lambda_3^2)\Phi_1(x)+(2\lambda_2^2-\lambda_3^2)\Phi_2(x)\big]-\\&\quad-\rho\sigma^2 r\Big(e_r\,\frac{\partial}{\partial r}-\operatorname{grad}\Big)\Phi_3(x)+\\&\quad+\mu\Big[e_r\times\operatorname{grad}\Big(r\,\frac{\partial}{\partial r}+1\Big)\Phi_4(x)\Big].\end{aligned} \tag{27}$$

If the values of the function $\Phi_j(x)$, $j=1,2,3,4$, from (23), (24) are substituted into (16), (27), we will have by virtue of (5) and (6)

$$\begin{aligned}u(x)&=u_{00}(r)X_{00}(\theta,\varphi)\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\big\{u_{mk}(r)X_{mk}(\theta,\varphi)+\\&\quad+\sqrt{k(k+1)}\,\big[v_{mk}(r)Y_{mk}(\theta,\varphi)+w_{mk}(r)Z_{mk}(\theta,\varphi)\big]\big\},\\u_4(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\eta_{mk}(r)Y_k^{(m)}(\theta,\varphi),\end{aligned} \tag{28}$$

$$\begin{aligned}P(\partial_x,n)U(x)&=a_{00}(r)X_{00}(\theta,\varphi)\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\big\{a_{mk}(r)X_{mk}(\theta,\varphi)+\\&\quad+\sqrt{k(k+1)}\,\big[b_{mk}(r)Y_{mk}(\theta,\varphi)+c_{mk}(r)Z_{mk}(\theta,\varphi)\big]\big\},\\\frac{\partial u_4(x)}{\partial n(x)}&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\eta'_{mk}(r)Y_k^{(m)}(\theta,\varphi),\end{aligned} \tag{29}$$

where

$$
\begin{aligned}
u_{mk}(r) &= \sum_{j=1}^{2} \frac{d}{dr}\, h_k(\lambda_j r) A_{mk}^{(j)} + \frac{k(k+1)}{r}\, h_k(\lambda_3 r) A_{mk}^{(3)},\\
v_{mk}(r) &= \sum_{j=1}^{2} \frac{1}{r}\, h_k(\lambda_j r) A_{mk}^{(j)} + \Big(\frac{d}{dr} + \frac{1}{r}\Big) h_k(\lambda_3 r) A_{mk}^{(3)},\\
w_{mk}(r) &= h_k(\lambda_3 r) A_{mk}^{(4)},\\
a_{mk}(r) &= 2\mu \sum_{j=1}^{2} \Big[\frac{d^2}{dr^2} + \lambda_j^2 - \frac{1}{2}\,\lambda_3^2\Big] h_k(\lambda_j r) A_{mk}^{(j)} +\\
&\quad + 2\mu k(k+1)\, \frac{d}{dr}\Big[\frac{1}{r}\, h_k(\lambda_3 r)\Big] A_{mk}^{(3)},\\
b_{mk}(r) &= 2\mu \sum_{j=1}^{2} \frac{d}{dr}\Big[\frac{1}{r}\, h_k(\lambda_j r)\Big] A_{mk}^{(j)} -\\
&\quad - 2\mu\Big[\frac{1}{r}\,\frac{d}{dr} + \frac{1}{2}\,\lambda_3^2 - \frac{k(k+1)-1}{r^2}\Big] h_k(\lambda_3 r) A_{mk}^{(3)},\\
c_{mk}(r) &= \mu\Big(\frac{d}{dr} - \frac{1}{r}\Big) h_k(\lambda_j r) A_{mk}^{(j)},\\
\eta_{mk}(r) &= c \sum_{j=1}^{2} (k_1^2 - \lambda_j^2) h_k(\lambda_j r) A_{mk}^{(j)}.
\end{aligned}
\tag{30}
$$

Assume that the vector-function $f(z)$ can be expanded into series (7), while the function $f_4(z)$ can be expanded with respect to the system of spherical functions $Y_k^{(m)}(\theta,\varphi)$:

$$
f_4(z) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \delta_{mk} Y_k^{(m)}(\theta,\varphi), \tag{31}
$$

where δ_{mk} are the Fourier coefficients.

Using the boundary conditions of Problems $(\mathrm{I})^-$, $(\mathrm{II})^-$ and formulas (28), (29), (7), (31), for the constants $A_{mk}^{(j)}$, $j=1,2,3,4$, we obtain the following systems of algebraic equations:

$$
\begin{aligned}
&u_{mk}(R) = \alpha_{mk}, \quad \eta_{mk}(R) = \delta_{mk}, \quad k \geq 0,\\
&v_{mk}(R) = \beta_{mk}, \quad w_{mk}(R) = \gamma_{mk}, \quad k \geq 1,
\end{aligned}
\tag{32}
$$

for Problem $(\mathrm{I})^-$;

$$
\begin{aligned}
&a_{mk}(R) = \alpha_{mk}, \quad \eta'_{mk}(R) = \delta_{mk}, \quad k \geq 0,\\
&b_{mk}(R) = \beta_{mk}, \quad c_{mk}(R) = \gamma_{mk}, \quad k \geq 1,
\end{aligned}
\tag{33}
$$

for Problem $(\mathrm{II})^-$.

Theorem 8. *Problems* $(\mathrm{I})^-$ *and* $(\mathrm{II})^-$ *have one solution at most.*

Proof. It is enough to show that a regular solution of the homogeneous boundary value Problems $(\mathrm{I})_0^-$ and $(\mathrm{II})_0^-$, satisfying conditions (15), is identically zero. Let $U=(u,u_4)$ be a regular solution of Problem $(\mathrm{I})_0^-$ and $(\mathrm{II})_0^-$, satisfying the thermoelastic radiation condition at infinity. We write the Green formula of system (1) in the domain which is bounded by the concentric spheres S and $S(0,r)$, $r>R$ [2]:

$$\frac{2\gamma}{i\sigma\eta}\int\limits_{\Omega_r}|\operatorname{grad}u_4|^2dx=-\int\limits_{S}\Big\{\overline{u}\cdot PU-u\cdot P\overline{U}+\frac{\gamma}{i\sigma\eta}\Big(u_4\,\frac{\partial\overline{u}_4}{\partial n}+\overline{u}_4\,\frac{\partial u_4}{\partial n}\Big)\Big\}^-dS+$$
$$+\int\limits_{S(0,r)}\Big[\overline{u}\cdot PU-u\cdot P\overline{U}+\frac{\gamma}{i\sigma\eta}\Big(u_4\,\frac{\partial\overline{u}_4}{\partial n}+\overline{u}_4\,\frac{\partial u_4}{\partial n}\Big)\Big]dS. \tag{34}$$

Taking into account the boundary conditions of the homogeneous Problems $(\mathrm{I})_0^-$ and $(\mathrm{II})_0^-$ in (34), we obtain

$$\frac{2\gamma}{i\sigma\eta}\int\limits_{\Omega_r}|\operatorname{grad}u_4|^2dx=$$
$$=\int\limits_{S(0,r)}\Big[\overline{u}\cdot PU-u\cdot P\overline{U}+\frac{\gamma}{i\sigma\eta}\Big(u_4\,\frac{\partial\overline{u}_4}{\partial n}+\overline{u}_4\,\frac{\partial u_4}{\partial n}\Big)\Big]dS. \tag{35}$$

Since the imaginary parts of the constants λ_1 and λ_2 are positive, Hankel's function $H^{(1)}_{k+1/2}(\lambda_j r)$ and its complex conjugates $\overline{H}^{(1)}_{k+1/2}(\lambda_j r)$, $j=1,2$, decrease exponentially at infinity. By substituting the values u, PU, u_4, $\frac{\partial u_4}{\partial n}$, from (28), (29) into (35) and using the formulas [7]

$$\begin{aligned}&H^{(1)}_{k+1/2}(\lambda_3 r)\,\frac{d}{dr}\,H^{(2)}_{k+1/2}(\lambda_3 r)-H^{(2)}_{k+1/2}(\lambda_3 r)\,\frac{d}{dr}\,H^{(1)}_{k+1/2}(\lambda_3 r)=\frac{4}{\pi i r},\\&H^{(l)}_{k+1/2}(\lambda_3 r)=O(r^{-1/2}),\quad l=1,2,\end{aligned} \tag{36}$$

we have

$$\frac{2\gamma}{i\sigma\eta}\lim_{r\to\infty}\int\limits_{\Omega_r}|\operatorname{grad}u_4|^2dx+$$
$$+\frac{4\mu R}{\pi i}\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\frac{k(k+1)}{|H^{(1)}_{k+1/2}(\lambda_3 R)|^2}\big[\lambda_3^2|A^{(3)}_{mk}|^2+|A^{(4)}_{mk}|^2\big]=0.$$

Hence it follows that

$$\operatorname{grad}u_4(x)=0,\ \ x\in\Omega^-,\quad A^{(j)}_{mk}=0,\ \ j=3,4. \tag{37}$$

Taking into account the behavior of $u_4(x)$ at infinity and expansion (24), from equality (37) we obtain

$$u_4(x) \equiv 0, \quad \Phi_j(x) \equiv 0, \quad j=3,4, \quad x \in \Omega_1. \tag{38}$$

(16) and (38) imply

$$(k_1^2-\lambda_1^2)\Phi_1(x)+(k_1^2-\lambda_2^2)\Phi_2(x) \equiv 0, \quad x \in \Omega_1. \tag{39}$$

Applying the operator $\Delta+\lambda_j^2$, $j=1,2$, to both parts of equalities (16) and (38), we have

$$\lambda_j^2(\lambda_j^2-k_1^2)\Phi_j(x) \equiv 0, \quad j=1,2, \quad x \in \Omega_1.$$

Therefore $\Phi_j(x) \equiv 0$, $j=1,2$, $x \in \Omega_1$, and by virtue of equalities (16) and (39) we finally obtain $U=(u,u_4) \equiv 0$. $\square$

Remark. A different proof of this theorem is given in [2].

Lemma 9. *Formulas* (16), (18), (26) *establish one-to-one correspondence between the regular solution* $U=(u,u_4)$ *of equations* (1) *and the system of functions* $\{\Phi_j(x),\ j=1,2,3,4\}$.

Proof. To prove Lemma 9 it is enough to show that the triviality of the vector $U(x)$ implies the triviality of the functions $\Phi_j(x)$, $j=1,2,3,4$, and vice versa.

Let us express the functions $\Phi_j(x)$, $j=1,2,3,4$, in terms of the components of the vector (u,u_4). By formula (16) we have

$$\begin{gathered}
\Phi_j(x)=\frac{(-1)^j}{k_1^2(\lambda_1^2-\lambda_2^2)}\Big[(k_1^2-\lambda_{3-j}^2)\operatorname{div}u+\frac{\lambda_{3-j}^2}{c}\,u_4(x)\Big], \quad j=1,2, \\
r\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\,\frac{\partial}{\partial r}+\lambda_2^2\Big)\Phi_3(x)=(e_r\cdot u)+\frac{1}{k_1^2}\,\frac{\partial}{\partial r}\,\operatorname{div}u-\frac{1}{ck_1^2}\,\frac{\partial u_r}{\partial r}, \\
r\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\,\frac{\partial}{\partial r}+\lambda_2^2\Big)\Phi_4(x)=(e_r\cdot\operatorname{rot}u).
\end{gathered} \tag{40}$$

For $u(x)=0$, $u_4(x)=0$ formulas (40) imply by virtue of condition (26) that $\Phi_j(x)$, $j=1,2,3,4$. Indeed, if $u(x)=0$, $u_4(x)=0$, then it follows from (40) that $\Phi_j(x)=0$, $j=1,2$, and

$$\Big(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\,\frac{\partial}{\partial r}+\lambda_2^2\Big)\Phi_j(x)=0, \quad j=3,4. \tag{41}$$

Applying expansions (24) and equality (26) to this formula, we obtain

$$\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\frac{k(k+1)}{r^2}\,h_k(\lambda_3 r)Y_k^{(m)}(\theta,\varphi)A_{mk}^{(j)}=0, \quad j=3,4,$$

which gives us $A^{(j)}_{mk} = 0$, $k \geq 1$, and therefore $\Phi_j(x) = 0$, $j = 3, 4$.

If $\Phi_j(x) = 0$, $j = 1, 2, 3, 4$, then (16) implies $U(x) = 0$. $\square$

By Theorem 8 and Lemma 9 we conclude that systems (32) and (33) have unique solutions. If the solutions of systems (32) and (41) are put into (28) and (29), respectively, then we shall obtain formal solutions of Problems $(\mathrm{I})^-$ and $(\mathrm{II})^-$.

To substantiate the method, first we have to show that series (28) and (29) are convergent. For $k \to \infty$ the following relations are fulfilled [7]:

$$h_k(\lambda_j r) \sim \left(\frac{R}{r}\right)^{k+1}, \quad \frac{d}{dr}\, h_k(\lambda_j r) \sim -\frac{k}{r}\left(\frac{R}{r}\right)^{k+1}. \tag{42}$$

Putting the solution of system (32) (or of (33)) into formulas (30) and taking into account estimates (10) and (42), we find that series (28), (29) are majorized by the series

$$\begin{gathered} M \sum_{k=k_0}^{\infty} k^{5/2}\left(\frac{R}{r}\right)^{k+1}\big[|\alpha_{mk}| + |\delta_{mk}| + k(|\beta_{mk}| + |\gamma_{mk}|)\big], \\ M = \mathit{const} > 0. \end{gathered} \tag{43}$$

If $x \in \Omega_1$, then $R < r$ and series (43) converges. For this series to converge at the boundary, it is enough that the Fourier coefficients α_{mk}, β_{mk}, γ_{mk}, δ_{mk} admit the estimates

$$\alpha_{mk} = O(k^{-4}), \ \ \delta_{mk} = O(k^{-4}), \ \ \beta_{mk} = O(k^{-5}), \ \ \gamma_{mk} = O(k^{-5}). \tag{44}$$

Theorems 1, 2 imply that the Fourier coefficients admit estimates (44) if the vector-function $f(z) \in C^4(S)$ and the function $f_4(z) \in C^4(S)$.

Substituting the functions $\Phi_j(x)$, $j = 1, 2, 3, 4$, from (23), (24) into (18), we obtain

$$\begin{aligned} u^{(j)}(x) &= \frac{d}{dr}\, h_0(\lambda_j r) X_{00}(\theta,\varphi) \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\Big[\frac{d}{dr}\, h_k(\lambda_j r) X_{mk}(\theta,\varphi) + \\ &\quad + \frac{\sqrt{k(k+1)}}{r}\, h_k(\lambda_j r) Y_{mk}(\theta,\varphi)\Big] A^{(j)}_{mk}, \quad j = 1, 2, \\ u^{(3)}(x) &= \sum_{k=1}^{\infty}\sum_{m=-k}^{k}\bigg\{\Big[\frac{k(k+1)}{r}\, h_k(\lambda_3 r) X_{mk}(\theta,\varphi) + \\ &\quad + \sqrt{k(k+1)}\Big(\frac{d}{dr} + \frac{1}{r}\Big) h_k(\lambda_3 r) Y_{mk}(\theta,\varphi)\Big] A^{(3)}_{mk} + \\ &\quad + \sqrt{k(k+1)}\, h_k(\lambda_3 r) Z_{mk}(\theta,\varphi) A^{(4)}_{mk}\bigg\}. \end{aligned} \tag{45}$$

For $r \to \infty$ we have the relations [7]

$$H^{(1)}_{k+1/2}(\lambda_j r) = \sqrt{\frac{2}{\pi\lambda_j r}}\, e^{i(\lambda_j r - \frac{k+1}{2}\pi)}[1 + O(r^{-1}], \quad j = 1,2,3, \tag{46}$$

$$\Big|\Big(\frac{d}{dr} - i\lambda_3\Big)\frac{H^{(1)}_{k+1/2}(\lambda_3 r)}{\sqrt{r}}\Big| \leq \frac{M_1}{r^2}, \quad M_1 = const > 0, \tag{47}$$

where M_1 does not depend on k.

Applying asymptotics (46) and (47) to formulas (45) and (28), by virtue of estimates (44) and the fact that the imaginary parts of λ_j, $j = 1,2$, are positive, we conclude that the vectors $u^{(j)}(x)$, $j = 1,2,3$, and the function $u_4(x)$ satisfy condition (15) at infinity.

Thus the vector $U = (u, u_4)$ defined by formula (28), where the unknown constants $A^{(j)}_{mk}$, $j = 1,2,3,4$, are a solution of system (32) or (33), is a regular solution of Problem (I)$^-$ or (II)$^-$.

Solution of Problem A. A solution of this problem will be sought for in the form

$$\begin{aligned} u^{(j)}(x) &= \operatorname{grad}[\Phi^{(j)}_1(x) + \Phi^{(j)}_2(x)] + \operatorname{rot}\operatorname{rot}(x\Phi^{(j)}_3(x)) + \operatorname{rot}(x\Phi^{(j)}_4(x)), \\ u^{(j)}_4(x) &= c_j\big[(k^2_{1j} - \lambda^2_{1j})\Phi^{(j)}_1(x) + (k^2_{ij} - \lambda^2_{ij})\Phi^{(j)}_2(x)\big], \;\; x \in \Omega_j, \;\; j = 0,1, \end{aligned} \tag{48}$$

where

$$c_j(\lambda_j + 2\mu_j)/\gamma_j, \quad k^2_{ij} = \rho_j\sigma^2_j/(\lambda_j + 2\mu_j),$$

$$(\Delta + \lambda^2_{lj})\Phi^{(j)}_l(x) = 0, \;\; l = 1,2,$$

$$(\Delta + \lambda^2_{3j})\Phi^{(j)}_l(x) = 0, \;\; l = 3,4, \;\; j = 0,1.$$

The constants λ_{lj}, $l = 1,2,3$, $j = 0,1$, have form (14), where j corresponds to the domain Ω_j.

The functions $\Phi^{(j)}_l(x)$, $l = 1,2,3,4$, $j = 0,1$, will be sought for in the form

$$\begin{aligned} \Phi^{(0)}_l(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} g_k(\lambda_{l0} r) Y^{(m)}_k(\theta,\varphi) B^{(l)}_{mk}, \\ \Phi^{(1)}_l(x) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k} h_k(\lambda_{l1} r) Y^{(m)}_k(\theta,\varphi) A^{(l)}_{mk}, \;\; l = 1,2,3,4, \end{aligned} \tag{49}$$

where $A^{(l)}_{mk}$, $B^{(l)}_{mk}$, $l = 1, 2, 3, 4$, are the unknown constants, and $\lambda_{4j} \equiv \lambda_{3j}$,

$$g_k(\lambda_{l0} r) = \sqrt{\frac{R}{r}}\,\frac{\mathcal{I}_{k+1/2}(\lambda_{l0} r)}{\mathcal{I}_{k+1/2}(\lambda_{l0} R)}, \tag{50}$$

where $\mathcal{I}_{k+1/2}(x)$ is Bessel's function, and $h_k(\lambda_{l1} r)$ has form (25), where λ_l is replaced by λ_{l1}.

The following conditions are imposed on the functions $\Phi^{(j)}_l(x)$, $l = 3, 4$, $j = 0, 1$:

$$\int\limits_{S'} \left[\Phi^{(0)}_l(z)\right]^+ dS = 0, \quad l = 3, 4, \tag{51}$$

$$\int\limits_{S''} \left[\Phi^{(1)}_l(z)\right]^- dS = 0, \quad l = 3, 4, \tag{52}$$

where S' and S'' are the spheres with center at the origin and radii R' and R'' ($0 < R' < R < R'' < +\infty$), respectively.

If the functions $\Phi^{(j)}_l(x)$ from (49) are inserted into (51) and (52), we will obtain $A^{(l)}_{00} = 0$, $B^{(j)}_{00}$, $l = 3, 4$.

By substituting the function $\Phi^{(j)}_l(x)$, $l = 1, 2, 3, 4$, $j = 0, 1$, from formula (49) into (48) and (27) we obtain

$$\begin{aligned} u^{(j)}(x) = u^{(j)}_{00}(r) X_{00}(\theta,\varphi) \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \big\{ u^{(j)}_{mk}(r) X_{mk}(\theta,\varphi) + \\ + \sqrt{k(k+1)}\,\big[v^{(j)}_{mk}(r) Y_{mk}(\theta,\varphi) + w^{(j)}_{mk}(r) Z_{mk}(\theta,\varphi)\big]\big\}, \end{aligned} \tag{53}$$

$$u^{(j)}_4(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \eta^{(j)}_{mk}(r) Y^{(m)}_k(\theta,\varphi),$$

$$\begin{aligned} P^{(j)}(\partial_x, n) U^{(j)}(x) = a^{(j)}_{00}(r) X_{00}(\theta,\varphi) \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \big\{ a^{(j)}_{mk}(r) X_{mk}(\theta,\varphi) + \\ + \sqrt{k(k+1)}\,\big[b^{(j)}_{mk}(r) Y_{mk}(\theta,\varphi) + c^{(j)}_{mk}(r) Z_{mk}(\theta,\varphi)\big]\big\}, \quad j = 0, 1, \end{aligned} \tag{54}$$

where the expressions for $u^{(1)}_{mk}, v^{(1)}_{mk}, \dots, \eta^{(1)}_{mk}$ are given by formulas (30) if the constants $\lambda, \mu, \dots, \sigma$ there are replaced by $\lambda_1, \mu_1, \dots, \sigma_1$, while the expressions for $u^{(0)}_{mk}, \dots, \eta^{(0)}_{mk}$ are obtained from (30) if the constants $\lambda, \mu, \dots, \sigma$ there are replaced by $\lambda_0, \mu_0, \dots, \sigma_0$ and Hankel's function by Bessel's function.

Let the functions $f_4^{(j)}(z)$ and the vector-functions $f^{(j)}(z)$, $j=0,1$, be expanded into the series

$$\begin{aligned}f^{*j)}(z) &= \alpha_{00}^{(j)}X_{00}(\theta,\varphi)\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\big\{\alpha_{mk}^{(j)}X_{mk}(\theta,\varphi)+\\ &\quad+\sqrt{k(k+1)}\,\big[\beta_{mk}^{(j)}Y_{mk}(\theta,\varphi)+\gamma_{mk}^{(j)}Z_{mk}(\theta,\varphi)\big]\big\}, \qquad (55)\\ f_4^{*j)}(z) &= \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\delta_{mk}^{(j)}Y_k^{(m)}(\theta,\varphi), \quad j=0,1.\end{aligned}$$

Using the contact conditions (22) and formulas (53)–(55), we obtain the following system of algebraic equations:

$$\begin{aligned}&u_{mk}^{(0)}(R)-u_{mk}^{(1)}(R)=\alpha_{mk}^{(0)}, \quad \eta_{mk}^{(0)}(R)-\eta_{mk}^{(1)}(R)=\delta_{mk}^{(0)}, \ \ k\geq 0,\\ &v_{mk}^{(0)}(R)-v_{mk}^{(1)}(R)=\beta_{mk}^{(0)}, \quad w_{mk}^{(0)}(R)-w_{mk}^{(1)}(R)=\gamma_{mk}^{(0)}, \ \ k\geq 1,\\ &a_{mk}^{(0)}(R)-a_{mk}^{(1)}(R)=\alpha_{mk}^{(1)}, \quad \frac{\gamma}{\sigma_0\eta_0}\,\frac{d}{dR}\,\eta_{mk}^{(0)}(R)-\\ &\qquad\qquad -\frac{\gamma_1}{\sigma_1\eta_1}\,\frac{d}{dR}\,\eta_{mk}^{(1)}(R)=\delta_{mk}^{(1)}, \ \ k\geq 0, \qquad (56)\\ &b_{mk}^{(0)}(R)-b_{mk}^{(1)}(R)=\beta_{mk}^{(1)}, \quad c_{mk}^{(0)}(R)-c_{mk}^{(1)}(R)=\gamma_{mk}^{(0)}, \ \ k\geq 1.\end{aligned}$$

Theorem 10. *The homogeneous problem* $(\mathrm{A})_0$ *has only a trivial solution.*

Proof. We write Green's formulas for system (21) in the domains Ω_0 and Ω_r, where the latter is bounded by the concentric surfaces S and $S(0,r)$, $r>R$ [4]:

$$\begin{aligned}\frac{2\gamma_0}{i\sigma_0\eta_0}\int\limits_{\Omega_0}\big|\operatorname{grad}u_4^{(0)}(x)\big|^2dx &= \int\limits_S\Big\{\overline{u}^{(0)}\cdot P^{(0)}U^{(0)}-u^{(0)}\cdot P^{(0)}\overline{U}^{(0)}+\\ &\quad+\frac{\gamma_0}{i\sigma_0\eta_0}\Big(u_4^{(0)}\,\frac{\partial\overline{u}_4^{(0)}}{\partial n}+\overline{u}_4^{(0)}\,\frac{\partial u_4^{(0)}}{\partial n}\Big)\Big\}^+dS, \qquad (57)\\ \frac{2\gamma_1}{i\sigma_1\eta_1}\int\limits_{\Omega_r}\big|\operatorname{grad}u_4^{(1)}(x)\big|^2dx &= -\int\limits_S\Big\{\overline{u}^{(1)}\cdot P^{(1)}U^{(1)}-u^{(1)}\cdot P^{(1)}\overline{U}^{(1)}+\\ &\quad+\frac{\gamma_1}{i\sigma_1\eta_1}\Big(u_4^{(1)}\,\frac{\partial\overline{u}_4^{(1)}}{\partial n}+\overline{u}_4^{(1)}\,\frac{\partial u_4^{(1)}}{\partial n}\Big)\Big\}^+dS+\int\limits_{S(0,r)}\Big\{\overline{u}^{(1)}\cdot P^{(1)}U^{(1)}-\\ &\quad-u^{(1)}\cdot P^{(1)}\overline{U}^{(1)}+\frac{\gamma_1}{i\sigma_1\eta_1}\Big(u_4^{(1)}\,\frac{\partial\overline{u}_4^{(1)}}{\partial n}+\overline{u}_4^{(1)}\,\frac{\partial u_4^{(1)}}{\partial n}\Big)\Big\}^+dS. \qquad (58)\end{aligned}$$

Applying the homogeneous boundary condition of Problem $(\mathrm{A})_0$ to (57) and (59), we obtain

$$\frac{2\gamma_0}{i\sigma_0\eta_0}\int\limits_{\Omega_0}\big|\operatorname{grad}u_4^{(0)}(x)\big|^2dx+\frac{2\gamma_1}{i\sigma_1\eta_1}\int\limits_{\Omega_r}\big|\operatorname{grad}u_4^{(0)}(x)\big|^2dx=$$
$$=\int\limits_{S(0,r)}\Big\{\overline{u}^{(1)}\cdot P^{(1)}U^{(1)}-u^{(1)}\cdot P^{(1)}\overline{U}^{(1)}+$$
$$+\frac{\gamma_1}{i\sigma_1\eta_1}\Big(u_4^{(1)}\,\frac{\partial\overline{u}_4^{(1)}}{\partial n}+\overline{u}_4^{(1)}\,\frac{\partial u_4^{(1)}}{\partial n}\Big)\Big\}dS. \tag{59}$$

Substituting the expressions for $u^{(1)}$, $P^{(1)}U^{(1)}$, $u_4^{(1)}$, and $\frac{\partial u_4^{(1)}}{\partial n}$ from (53), (54) into (59), and taking into account formula (36) and the fact that the vectors X_{mk}, Y_{mk}, Z_{mk} are normalized, we have

$$\frac{2\gamma_0}{i\sigma_0\eta_0}\int\limits_{\Omega_0}\big|\operatorname{grad}u_4^{(0)}(x)\big|^2dx+\frac{2\gamma_1}{i\sigma_1\eta_1}\int\limits_{\Omega_r}\big|\operatorname{grad}u_4^{(0)}(x)\big|^2dx+$$
$$+\frac{4\mu R}{\pi i}\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\frac{k(k+1)}{|H_{k+1/2}(\lambda_3R)|^2}\big[\lambda_{31}^2|A_{mk}^{(3)}|^2+|A_{mk}^{(4)}|^2\big]=o(1). \tag{60}$$

Hence, passing to the limit as $r\to\infty$, we find

$$u_4^{(j)}(x)=const,\quad j=0,1,\quad A_{mk}^{(l)}=0,\quad l=3,4,\quad k\geq 1.$$

Since $u_4^{(j)}(x)$ is a metaharmonic function, we have $u_4^{(j)}\equiv 0$, $x\in\Omega_j$, $j=0,1$. By the equality $A_{mk}^{(l)}=0$, $l=3,4$, it follows that $\Phi_l^{(1)}(x)\equiv 0$, $x\in\Omega_1$, $l=3,4$. Using the equality $u_4^{(1)}(x)\equiv 0$, $x\in\Omega_1$, in representation (48), we obtain $\Phi_l^{(1)}\equiv 0$, $x\in\Omega_1$, $l=1,2$. Thus we have shown that $\Phi_l^{(1)}(x)\equiv 0$, $x\in\Omega_1$, $l=1,2,3,4$. By virtue of these equalities we conclude that

$$u^{(1)}(x)\equiv 0,\quad u_4^{(1)}\equiv 0,\quad x\in\Omega_1. \tag{61}$$

Using the contact conditions of Problem $(\mathrm{A})_0$ and equalities (63), we find

$$\begin{gathered}\big\{u^{(0)}(z)\big\}^+=0,\quad \big\{u_4^{(0)}(z)\big\}^+=0,\quad \big\{P^{(0)}U^{(0)}(z)\big\}^+=0,\\ \Big\{\frac{\partial u_4^{(0)}}{\partial n}\Big\}^+=0,\quad z\in S.\end{gathered} \tag{62}$$

A general representation of regular solutions of the homogeneous equation (21) in the domain Ω_0 has the form [4]

$$2U^{(0)}(x)=\int\limits_S\big\{\Gamma^{(0)}(x-y,\sigma_0)\big[R^{(0)}U^{(0)}\big]^+-$$

$$- \big[\widetilde{R}\widetilde{\Gamma}^{(0)}(y-x,\sigma_0)\big]'\big[U^{(0)}(y)\big]^{+}\big\}d_yS, \tag{63}$$

where $\Gamma^{(0)}(x-y,\sigma_0)$ is the fundamental solution of system (21), $\widetilde{\Gamma}^{(0)}(x-y,\sigma_0)$ is the matrix of fundamental solutions of the adjoint homogeneous system, $RU=(PU,\frac{\partial u_4}{\partial n})$, $\widetilde{R}U=(Tu-i\sigma\eta u_4,\frac{\partial u_4}{\partial n})$.

From (62) and (63) we finally obtain

$$u^{(0)}(x)\equiv 0,\quad u_4^{(0)}(x)\equiv 0,\quad x\in\Omega_0. \quad \square$$

By the uniqueness theorem and Lemma 9 we conclude that system (56) has a unique solution. After putting this solution into (53) , we obtain a formal solution of Problem A.

If $f^{(j)}(z)\in C^4(S)$, $f_4^{(j)}(z)\in C^4(S)$, $j=0,1$, then the constructed formal series (53) gives a regular solution of the problem posed.

References

1. A. F. Ulitko, The method of vector egenfunctions in spatial problems of the elasticity theory. (Russian) *Naukova Dumka, Kiev,* 1979.

2. L. G. Giorgashvili, Solution of the basic boundary value problems of the theory of elasticity for a ball. (Russian) *Proc. I. Vekua Inst. Appl. Math.* (*Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy*) **10**(1981), 32–36.

3. W. Novacki, Dynamical problems of thermoelasticity. (Polish) *PWN, Warszawa,* 1996; *Russian Translation: Mir, Moscow,* 1970.

4. V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. (Translated from Russian) *North-Holland series in applied mathematics and mechanics,* v. 25, *North-Holland Publishing Company, Amsterdam–New York–Oxford,* 1979; Russian original: *Nauka, Moscow,* 1976.

5. F. M. Mors and G. Feshbakh, Methods of theoretical physics, v. 2. (Russian), *Moscow,* 1958.

6 S. G. Mikhlin, Multi-dimensional singular integrals and integral equations. (Russian) *Fizmatgiz, Moscow,* 1962.

7. I. S. Gradshtein and I. M. Rizhik, Tables of integrals, sums, series, and products. (Russian) *Fizmatgiz, Moscow,* 1963.

(Received 17.03.1994; revised 10.10.1996)

Author's address:
Department of Mathematics (99)
Georgian Technical University
77, M. Kostava St., Tbilisi 380075
Georgia