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MONOTONE SOLUTIONS OF A HIGHER ORDER NEUTRAL DIFFERENCE EQUATION

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Abstract. A real sequence $\{x_k\}$ is said to be $(*)$-monotone with respect to a sequence $\{p_k\}$ and a positive integer σ if $x_k > 0$ and $(-1)^n \Delta^n (x_k - p_k x_{k-\sigma}) \geq 0$ for $n \geq 0$. This paper is concerned with the existence of $(*)$-monotone solutions of a neutral difference equation. Existence criteria are derived by means of a comparison theorem and by establishing explicit existence criteria for positive and/or bounded solutions of a majorant recurrence relation.

§ 1. Introduction

A sequence $\{x_k\}$ is said to be $(*)$-monotone if it satisfies

$$x_k > 0, \;\; \Delta x_k \geq 0, \;\; \Delta^2 x_k \leq 0, \;\; \Delta^3 x_k \geq 0, \dots, (-1)^n \Delta^{n+1} x_k \geq 0.$$

A typical example is the sequence defined by $x_k = 1 - \lambda^k$, $0 < \lambda < 1$, $k = 0, 1, 2, \dots$. Given a positive integer σ and a sequence $\{p_k\}$, we can generalize the concept of a $(*)$-monotone sequence $\{x_k\}$ by requiring

$$x_k > 0, \;\; x_k - p_k x_{k-\sigma} \geq 0,$$
$$\Delta (x_k - p_k x_{k-\sigma}) \leq 0, \dots, (-1)^n \Delta^n (x_k - p_k x_{k-\sigma}) \geq 0.$$

Such a sequence will again be called $(*)$-monotone with respect to the integers n, σ and the sequence $\{p_k\}$.

This note is concerned with the $(*)$-monotone solutions of a class of nonlinear recurrence relations of the form

$$\Delta^n (y_k - p_k y_{k-\sigma}) + q_k f(y_{k-\tau}) = 0, \quad k = 0, 1, 2, \dots, \tag{1}$$

where n is a positive odd integer, σ is a positive integer, τ is a non-negative integer, $\{p_k\}$ and $\{q_k\}$ are non-negative sequences such that $\{q_k\}$ does not vanish identically for all large k, and f is a real function defined on R such

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that f is positive nondecreasing for $x > 0$. The forward difference operator is defined as usual, i.e., $\Delta x_k = x_{k+1} - x_k$.

Let $\mu = \max\{\sigma, \tau\}$. Then by a solution of (1) we mean a real sequence $\{y_k\}$ which is defined for $k \geq -\mu$ and which satisfies equation (1) for $k \geq 0$. By writing equation (1) in the form of a recurrence relation$y_{n+k} = F(y_{n+k-1}, \dots, y_k, y_{k-\sigma}, y_{k-\tau})$ it is clear that an existence and uniqueness theorem for the solutions of (1) satisfying appropriate initial conditions can easily be formulated and proved by induction. A solution $\{y_k\}$ of (1) is said to be eventually positive if $y_k > 0$ for all large k, and eventually negative if $y_k < 0$ for all large k. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Finally, it is said to be eventually $(*)$-monotone if it is $(*)$-monotone for all large k.

We will be concerned with the existence of eventually $(*)$-monotone solutions of (1). Similar problems related to both differential and difference equations have been considered in [1–4].

§ 2. Comparison Theorem

We first establish a Sturmian type criterion which has not been explored before. To be more precise, we will assume the existence of an eventually $(*)$-monotone solution of a majorant relation of the form

$$\Delta^n \left(x_k - P_k x_{k-\sigma}\right) + Q_k F(x_{k-\tau}) \leq 0, \quad k = 0, 1, 2, \dots , \tag{2}$$

and then show that (1) also has an eventually $(*)$-monotone solution. The correct assumptions will be stated later, for now, we will assume in the sequel that the sequences $\{P_k\}$ and $\{Q_k\}$, and the function F satisfy the same assumptions that have respectively been imposed on $\{p_k\}$, $\{q_k\}$ and f.

Let $\{x_k\}$ be an eventually $(*)$-monotone solution of (2) such that $x_k > 0$ for $k \geq N - \mu$ and the sequence $\{z_k\}$ defined by

$$z_k = x_k - P_k x_{k-\sigma}, \quad k \geq 0, \tag{3}$$

satisfies$z_k \geq 0$, $\Delta z_k \leq 0, \dots, \Delta^n z_k \leq 0$, for $k \geq N - \mu$. Then in view of (2),$\Delta^n z_k \leq -Q_k F(x_{k-\tau})$, $k \geq N$. By summing the above inequality n times from k to infinity, we obtain respectively

$$\Delta^{n-1} z_k \geq \sum_{j=k}^{\infty} Q_j F(x_{j-\tau}), \quad \Delta^{n-2} z_k \leq -\sum_{j=k}^{\infty} \sum_{k=i}^{\infty} Q_i F(x_{i-\tau}),$$

$$\dots$$

$$x_k - P_k x_{k-\sigma} = z_k \geq \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{(n-1)}}{(n-1)!} Q_j F(x_{j-\tau}), \ \ k \geq N,$$

where the factorial function $h^{(m)}(i)$ is defined by $h(i)h(i-1)\cdots h(i-m+1)$. As a consequence, by assuming $P_k \geq p_k$ and $Q_k \geq q_k$ for $k \geq 0$, $F(x) \geq f(x)$ for $x > 0$, as well as f is positive nondecreasing on $(0, \infty)$, we see that

$$x_k \geq p_k x_{k-\sigma} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{(n-1)}}{(n-1)!} q_j f(x_{j-\tau}), \quad k \geq N. \tag{4}$$

Let Ω be the set of all real sequences $w = \{w_k\}_{n=N-\mu}^{\infty}$. Define an operator $T : \Omega \to \Omega$ by$(Tw)_k = 1$, $N - \mu \leq k \leq N - 1$, and

$$(Tw)_k = \frac{1}{x_k}\left\{p_k w_{k-\sigma} x_{k-\sigma} + \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{(n-1)}}{(n-1)!} q_j f(w_{j-\tau} x_{j-\tau})\right\},\ k \geq N.$$

Consider the following successive approximations: $w^{(0)} \equiv 1$, $w^{(j+1)} = Tw^{(j)}$ for $j = 0, 1, 2, \ldots$. By means of (4) and induction, it is easily seen that$0 \leq w_k^{(j+1)} \leq w_k^{(j)} \leq 1$, $k \geq N$, $j \geq 0$. Thus, as $m \to \infty$, $w^{(m)}$ converges (pointwise) to some non-negative sequence w^* which satisfies $w_k^* = 1$ for $N - \mu \leq k \leq N - 1$, and furthermore, by means of the Lebesgue dominated convergence theorem, we may take limits on both sides of $w^{(j+1)} = Tw^{(j)}$ to obtain

$$w_k^* x_k - p_k w_{k-\tau}^* x_{k-\tau} = \sum_{j=k}^{\infty} \frac{(j-k+n-1)^{(n-1)}}{(n-1)!} q_j f(w_{k-\tau}^* x_{k-\tau}), ;\ k \geq N.$$

Taking differences on both sides of the above equality, we see that $\{y_k\}$ defined by$y_k = w_k^* x_k$, $k \geq N - \mu$, is an eventually non-negative solution of the recurrence relation (1). It will be a $(*)$-monotone solution if we can show that $\{y_n\}$ is eventually positive. Indeed, note that $y_k > 0$ for $N - \mu \leq k \leq N - 1$. Suppose to the contrary that $y_k > 0$ for $N - \mu \leq k < k^*$ and $y_{k^*} = 0$, then

$$0 = y_{k^*} = p_{k^*} y_{k^*-\tau} + \sum_{j=k^*}^{\infty} \frac{(j-k+n-1)^{(n-1)}}{(n-1)!} q_j f(y_{j-\tau}).$$

Since $y_{k^*-\sigma} > 0$, $(j - k^* + n - 1)^{(n-1)} > 0$ for $j \geq k^*$, we must have $p_{k^*} = 0$ and $q_j f(y_{j-\tau}) = 0$ for $j \geq k^*$. In other words, if $\tau = 0$, we may prevent this from happening by imposing the condition $p_k > 0$ for all $k \geq 0$, or, if $\tau > 0$, by imposing the condition $p_k > 0$, or, the vector $(q_k, q_{k+1}, \ldots, q_{k+\tau-1}) \neq 0$ for all $k \geq 0$.

We summarize these results.

Theorem 1. *Suppose that either* (H1) $\tau = 0$ *and* $p_k > 0$ *for* $k \geq 0$, *or* (H2) $\tau > 0$, *and,* $p_k > 0$ *or the vector* $(q_k, \ldots, q_{k+\tau-1}) \neq 0$ *for* $k \geq 0$. *Suppose that* $\{P_k\}$ *and* $\{Q_k\}$ *are two sequences such that* $P_k \geq p_k \geq 0$ *and* $Q_k \geq q_k \geq 0$ *for* $k \geq 0$. *Suppose further that* F *and* f *are real functions* *such that* $F(x) \geq f(x)$ *for* $x > 0$. *If* (2) *has a* $(*)$*-monotone solution with respect to* σ, n *and* $\{P_k\}$, *then* (1) *will have a* $(*)$*-monotone solution with respect to* σ, n *and* $\{p_k\}$.

§ 3. Existence Theorems

Given a real sequence $\{u_k\}$ with sign conditions on its difference sequences $\{\Delta^r u_k\}$ and $\{\Delta^t u_k\}$, the in-between difference sequences $\{\Delta^s u_k\}$ will also satisfy certain sign conditions. Similar results are well known in the theory of ordinary differential equations (see, for example, Kiguradze and Chanturia [7]). We will make use of these criteria to derive the existence of $(*)$-monotone solutions.

Lemma 1 (Zhou and Yan [5]). *Let* $\{u_k\}$ *be a real bounded sequence of fixed sign. Suppose* $u_k \Delta^t u_k \leq 0$ *for some odd integer* $t > 1$ *and all large* k. *Then* $(-1)^s u_k \Delta^s u_k \geq 0$ *for* $s = 1, 2, \ldots, t$ *and all large* k.

Lemma 2 (Zafer and Dahiya [6]). *Let* m *be a positive integer. Let* $\{y_k\}_{k=0}^{\infty}$ *be a real sequence such that the sequences* $\{y_k\}, \ldots, \{\Delta^{m-1} y_k\}$ *are of constant sign. Suppose further that* $y_k \Delta^m y_k \geq 0$ *for* $k \geq 0$. *Then either*

(i) $y_k \Delta^j y_k \geq 0$ *for each* $j \in \{1, 2, \ldots, m-1\}$ *and all large* k, *or*

(ii) *there is an integer* $t \in \{1, 2, \ldots, m-2\}$ *such that* $(-1)^{m-t} = 1$ *and for each* $j \in \{1, \ldots, t\}$, $y_k \Delta^j y_k > 0$ *for all large* k, *and for each* $j \in \{t+1, \ldots, m-2\}$, $(-1)^{j-t} y_k \Delta^j y_k > 0$ *for all large* k.

Lemma 3. *Suppose there is an integer* N *such that*$P_{N+j\sigma} \leq 1$, $j = 0, 1, 2, \ldots$. *Then for any eventually positive solution* $\{x_k\}$ *of* (2), *the sequence* $\{z_k\}$ *defined by* (3) *is also eventually positive.*

Proof. Suppose $x_{k-\tau} > 0$ for all large k. Then in view of (2) we see that $\Delta^n z_k \leq -Q_k F(x_{k-\tau}) \leq 0$ for all large k. Since $\{Q_k\}$ is not identically zero for all large k, thus for each $j \in \{0, 1, \ldots, n-1\}$, $\{\Delta^j z_k\}$ is of constant sign for all large k. In particular, $\{z_k\}$ is eventually positive or eventually negative. Suppose to the contrary that $\{z_k\}$ is eventually negative, then in view of Lemma 2 and the fact that n is an odd integer, we see that $\Delta z_k < 0$ for all large k. Thus there is a positive number α such that $z_k \leq -\alpha$ for all large k. Therefore $x_k \leq -\alpha + P_k x_{k-\sigma}$ for k greater than or equal to some integer, which, without loss of generality, may be taken to be $N+\sigma\ N^*$. Then we have $x_{N+\sigma} \leq -\alpha + P_{N+\sigma} x_N \leq -\alpha + x_N$, $x_{N+2\sigma} \leq -\alpha + P_{N+2\sigma} x_{N+\sigma} \leq -2\alpha + x_N, \ldots, x_{N+j\sigma} \leq -j\sigma + x_N$. But for sufficiently large j, the right-hand side is negative, while the right-hand side remains positive. A contradiction is obtained. □

We remark that the proof of the above lemma is similar to that of Lemma 3.4 in [8] but is included here for the sake of completeness.

Theorem 2. *Suppose* $\{P_k\}$ *is bounded and there is an integer* N *such that* $P_{N+j\sigma} \leq 1$ *for* $j \geq 0$. *Then any eventually positive and bounded solution* $\{x_k\}$ *of* (2) *is eventually* $(*)$*-monotone.*

Proof. In view of Lemma 3, the sequence $\{z_k\}$ defined by $z_k = x_k - P_k x_{k-\sigma}$ is eventually positive. Since $\{x_k\}$ and $\{P_k\}$ is bounded, we see further that $\{z_k\}$ is bounded. But then by means of Lemma 1, $(-1)^j \Delta^j z_k \geq 0$ for $j = 1, 2, \ldots, n$ and all large k. □

We also have a result which removes the boundedness condition in Theorem 2.

Theorem 3. *Suppose there is an integer* N *such that* $P_{N+j\sigma} \leq 1$ *for* $j \geq 0$. *Suppose further that* $\liminf_{x\to\infty} F(x) \geq d > 0$ *and that*

$$\sum_{i=0}^{\infty} Q_i = \infty. \tag{5}$$

Then an eventually positive solution $\{x_k\}$ *of* (2) *is also a* $(*)$*-monotone solution.*

Proof. Let $\{x_k\}$ be an eventually positive solution of (2). Then the sequence $\{z_k\}$ defined by (3) will satisfy $\Delta^n z_k \leq -Q_k F(x_{k-\tau}) \leq 0$ for all large k. Furthermore, since $\{Q\}$ does not vanish identically for all large k, thus either

$$\lim_{k\to\infty} \Delta^{n-1} z_k = -\infty, \tag{6}$$

or

$$\lim_{k\to\infty} \Delta^{n-1} z_k = c. \tag{7}$$

If (6) holds, then it is easy to see that $\lim\limits_{k\to\infty} z_k = -\infty$. This implies that $x_k \leq -\alpha + P_k x_{k-\sigma}$ for some positive number and all large k. As we have seen in the proof of Lemma 3, this is impossible. Therefore (7) must hold. We assert further that $c = 0$. Otherwise, it is easy to see that $\lim_{k\to\infty} z_k = -\infty$ (which is impossible) or $\lim_{k\to\infty} z_k = +\infty$. If $z_k \to +\infty$, then in view of (3) it is clear that $\lim\limits_{k\to\infty} x_k = +\infty$. By summing both sides of (2) from 0 to ∞ we obtain $c + \sum\limits_{i=0}^{\infty} Q_i F(x_{i-\sigma}) = \Delta^{n-1} z_0$, so that $d \sum\limits_{i=0}^{\infty} Q_i < \infty$, which is contrary to (5). We have thus shown that $\Delta^{n-1} z_k > 0$ for all large k and strictly decreases to zero as $k \to \infty$. By repeating the above arguments it is not difficult to see that $(-1)^j \Delta^j z_k > 0$, $j = 1, 2, \ldots, n-1$, for all large k, and also, $\lim\limits_{k\to\infty} \Delta^j z_k = 0$, $j = 1, 2, \ldots, n-1$. □

In view of Theorems 2 and 3, we need to find some explicit existence criteria for eventually positive and/or bounded solutions of recurrence relations of form (2) so that our comparison Theorem 1 can be used to produce existence criteria for ($*$)-monotone solutions of (1). For the linear equation $\Delta^n (x_k - Px_{k-\sigma}) + Q_k x_{k-\tau} = 0,\ k = 0, 1, 2, \dots,$ such existence criteria can be found in [8].

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