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ON THE REPRESENTATION OF NUMBERS BY POSITIVE DIAGONAL QUADRATIC FORMS WITH FIVE VARIABLES OF LEVEL 16

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ABSTRACT. A general formula is derived for the number of representations $r(n;f)$ of a natural number n by diagonal quadratic forms f with five variables of level 16. For f belonging to one-class series, $r(n;f)$ coincides with the sum of a singular series, while in the case of a many-class series an additional term is required, for which the generalized theta-function introduced by T. V. Vepkhvadze [4] is used.

1. Let $f = f(x) = f(x_1, x_2, \dots, x_s) = \frac{1}{2}X'AX = \frac{1}{2}\sum_{j,k=1} a_{jk}x_jx_k$ be an integral positive quadratic form. Here and in what follows X is a column-vector, and X' is a row-vector with components $x_1, x_2, \dots, x_s$. Let further $r(n;f)$ denote the number of representations of a natural number n by the form f.

For our discussion we shall need the following results.

As is well known, for each quadratic form f we have the corresponding series

$$\vartheta(\tau, f) = 1 + \sum_{n=1}^{\infty} r(n, f)Q^n, \tag{1}$$

$$\theta(\tau, f) = 1 + \sum_{n=1}^{\infty} \rho(n, f)Q^n, \tag{2}$$

where $Q = e^{2\pi i\tau}(\mathcal{I}_m\tau > 0)$ and $\rho(n, f)$ is a singular series. In the cases considered here the sum of the singular series can be calculated by means of the following two lemmas.

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Lemma 1 (see [1]). *Let* $2 \nmid s$, $\Delta = 2^s\Delta_0$, $n\Delta_0 = 2^{\alpha+\gamma}v_1v_2 = r^2\omega$, $2^\alpha \| n$, $2^\gamma \| \Delta_0$, $p^l \| \Delta_0$, $p^\omega \| n$, $v_1 = \prod\limits_{\substack{p|n \\ p\nmid 2\Delta_0}} p^\omega = r_1^2\omega_1$, $v_1 = \prod\limits_{\substack{p|\Delta_0 n \\ p|\Delta_0, p>2}} p^{\omega+l} = r_2^2\omega_2$, ($\omega$, ω_1 *and* ω_2 *are square-free integers*).

Then

$$\rho(n,f) = \frac{2^{2-\frac{s}{2}}\pi^{1-\frac{s}{2}}(s-1)!}{\Gamma(\frac{s}{2})\Delta_0^{\frac{1}{2}}B_{\frac{s-1}{2}}} n^{\frac{s}{2}-1} r_1^{2-s}\chi(2)\Pi_{\substack{p|\Delta_0 \\ p>2}}\chi(p) \times$$

$$\times \Pi_{p|2\Delta_0}(1-p^{1-s})^{-1} L\Big(\frac{s-1}{2}, (-1)^{\frac{s-1}{2}}\omega\Big)\Pi_{\substack{p|r_2 \\ r_2>2}}\bigg(1-\Big(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\Big)p^{\frac{1-s}{2}}\bigg)\times$$

$$\times \sum_{d|r_1} d^{s-2}\Pi_{p|d}\bigg(1-\Big(\frac{(-1)^{\frac{s-1}{2}}\omega}{p}\Big)p^{\frac{1-s}{2}}\bigg), \tag{3}$$

where $B_{\frac{s-1}{2}}$ *are Bernoulli's numbers,* $(\frac{\cdot}{p})$ *is Jacobi's symbol, and the values of* $\chi(2)$ *are given in* [2] (*p.* 66, *formulas* (28)–(33)).

For the case $s = 5$ the values of $L(\cdot,\cdot)$ are given in

Lemma 2 (see, e. g., [3]).

$$L(2;1) = \frac{\pi^2}{8}, \quad L(2;2) = \frac{2^{\frac{1}{2}}\pi^2}{16},$$

$$L(2;\omega) = -\frac{\pi}{\omega^{\frac{3}{2}}}\sum_{1\le h\le\frac{\omega}{2}} h\Big(\frac{h}{\omega}\Big), \quad \text{if} \quad \omega \equiv 1 \pmod 4, \ \ \omega > 1;$$

$$L(2;\omega) = \frac{\pi^2}{2\omega^{\frac{3}{2}}}\bigg\{2\sum_{1\le h\le\frac{\omega}{4}} h\Big(\frac{h}{\omega}\Big) + \sum_{\frac{\omega}{4}<h\le\frac{\omega}{2}}(\omega-2h)\Big(\frac{h}{\omega}\Big)\bigg\}, \quad \text{if} \quad \omega \equiv 3 \pmod 4;$$

$$L(2;\omega) = \frac{\pi^2}{4\omega^{\frac{3}{2}}}\bigg\{\omega\sum_{1\le h\le\frac{\omega}{16}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) + \sum_{\frac{\omega}{16}<h\le\frac{3\omega}{16}}(\omega-16h)\Big(\frac{h}{\frac{1}{2}\omega}\Big) - 2\omega\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}}\Big(\frac{h}{\frac{1}{2}\omega}\Big)\bigg\}, \quad \text{if} \quad \omega \equiv 2 \pmod 8, \ \ \omega > 2;$$

$$L(2;\omega) = \frac{\pi^2}{4\omega^{\frac{3}{2}}}\bigg\{16\sum_{1\le h\le\frac{\omega}{16}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) + \omega\sum_{\frac{\omega}{16}<h\le\frac{3\omega}{16}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) + 4\omega\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}}\Big(\frac{h}{\frac{1}{2}\omega}\Big) - 16\omega\sum_{\frac{3\omega}{16}<h\le\frac{\omega}{4}} h\Big(\frac{h}{\frac{1}{2}\omega}\Big)\bigg\}, \quad \text{if} \quad \omega \equiv 6 \pmod 8.$$

In [4] Vepkhvadze constructed generalized theta-functions with characteristic and spherical functions

$$\vartheta_{gh}(\tau; P_\nu, f) = \sum_{X \equiv g \pmod{N}} (-1)^{\frac{h' A(X-g)}{N^2}} P_\nu(X) e^{\frac{\pi i \tau X' A X}{N^2}}. \qquad (4)$$

Here g and h are special vectors with respect to the matrix A of form f, i.e.,

$$Ag \equiv 0 \pmod{N}, \quad Ah \equiv 0 \pmod{N},$$

where N is a level of the form f, i.e., the smallest integer for which NA^{-1} is a symmetric integral matrix with even diagonal elements; $P_\nu = P_\nu(x) = P_\nu(x_1, \dots, x_s)$ is a spherical function of ν-th order with respect to f.

The properties of functions (4) are investigated in [4], where these functions are used to derive a formula for the number of representations of a quadratic form with seven variables.

In this paper we use the method of [4] to obtain formulas for the number of representations of natural numbers by all positive diagonal quadratic forms with five variables of level 16.

Lemma 3 (see, e.g., [4], Lemma 4). *Let k be an arbitrary integral vector, and l a special vector with respect to the matrix A of the form f. Then the equalities*

$$\vartheta_{g+Nk,h}(\tau; P_\nu, f) = (-1)^{\frac{h' A k}{N}} \vartheta_{gh}(\tau; P_\nu, f), \quad \vartheta_{g,h+2l}(\tau; P_\nu, f) = \vartheta_{gh}(\tau; P_\nu, f)$$

are valid.

For $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ denote

$$v(M) = \left(i^{\frac{1}{2}\eta(\gamma)(\operatorname{sgn}\delta - 1)}\right)^{s+2\nu} (\operatorname{sgn}\delta)^\nu \left(i^{(\frac{|\delta|-1}{2})^2}\right)^{s+2\nu} \Big(\frac{2\Delta(\operatorname{sgn}\delta)\beta}{|\delta|}\Big)\Big(\frac{-1}{|\delta|}\Big), (5)$$

$\eta(\gamma) = 1$ for $\gamma \geq 0$, $\eta(\gamma) = -1$ for $\gamma < 0$. By $v_0(M)$ we denote $v(M)$ in the case $\nu = 0$.

Lemma 4 (see, e.g., [4], Theorem 2). *Let $f = f(x)$ be an integral positive quadratic form with an odd number of variables s, Δ the determinant of the matrix A of the form f, and N the level of the form f. Then function* (1) *is an integral modular form of type $\big(-\frac{s}{2}, N, v_0(M)\big)$.*

Lemma 5 (see, e.g., [4], Theorem 2). *Let $f_k = f_k(x)$ $(k = 1, \dots, j)$ be integral positive quadratic forms with the number of variables s, $P_\nu^{(k)} = P_\nu^{(k)}(x)$ $(k = 1, 2, \dots, j)$ the corresponding spherical functions, A_k a matrix of the form $f_k(x)$, Δ_k the determinant of the matrix A_k, and N_k the level* *of the form f_k. Let further $g^{(k)}$ and $h^{(k)}$ be vectors with even components, and B_k arbitrary complex numbers. Then the function*

$$\Phi(\tau)=\sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k)$$

is an integral modular form of the type $\big(-(\frac{s}{2}+\nu), N, v(M)\big)$, where $v(M)$ are determined by formula (5)*, if and only if the conditions*

$$N_k|N, \quad N_k^2|f_k(g^{(k)}), \quad 4N_k\Big|\frac{N}{N_k}f_k(h^{(k)}) \tag{6}$$

are fulfilled, and for all α and δ satisfying the condition $\alpha\delta\equiv 1 \pmod N$ we have

$$\sum_{k=1}^{j} B_k \vartheta_{\alpha g^{(k)},-h^{(k)}}(\tau; P_\nu^{(k)}, f_k)(\operatorname{sgn}\delta)^\nu\Big(\frac{(-1)^{\frac{s-1}{2}}\Delta_k}{|\delta|}\Big)=$$
$$=\Big(\frac{(-1)^{\frac{s-1}{2}+\nu}\Delta}{|\delta|}\Big)\sum_{k=1}^{j} B_k \vartheta_{g^{(k)}h^{(k)}}(\tau; P_\nu^{(k)}, f_k). \tag{7}$$

Lemma 6 (see, e.g., [5], Theorem 4). *If all the conditions of Lemma 5 are fulfilled and $\nu>0$, then the function $\Phi(\tau)$ is a cusp form of the type $\big(-(\frac{s}{2}+\nu), N, v(M)\big)$.*

Lemma 7 (see, e.g., [4], Theorem 1). *Let F be an integral modular form of the type $(-\Gamma, N, v(M))$, where $v(M)$ are determined by formula* (5)*. Then the function F is identically zero if in its expansion into powers $Q=e^{2\pi i\tau}$ the coefficients of Q^n are zero for all*

$$n\le \frac{r}{12}N\prod_{p|N}\Big(1+\frac{1}{p}\Big).$$

2. Positive diagonal quadratic forms with five variables of level 16 are written as

$$f_{s_1,s_2}=\sum_{j=1}^{s_1}x_j^2+2\sum_{j=s_1+1}^{s_2}x_j^2+4\sum_{j=s_2+1}^{5}x_j^2,$$

where $1\le s_1\le s_2\le 4$.

Theorem 1. *Let $f_1=4x_1^2+4x_2^2+2x_3^2$, $P_1=x_3$, $g'=(4,4,8)$, $h'=(2,2,4)$. Then the identity*

$$\vartheta(\tau; f_{s_1,s_2})=\theta(\tau; f_{s_1,s_2})+\Phi(\tau; f_{s_1,s_2}), \tag{8}$$

holds, where

$$\Phi(\tau; f_{1,2}) = \frac{1}{16}\vartheta_{gh}(\tau; P_1, f_1),$$
$$\Phi(\tau; f_{2,3}) = \Phi(\tau; f_{3,4}) = \frac{1}{4}\vartheta_{gh}(\tau; P_1, f_1),$$
$$\Phi(\tau; f_{s_1,s_2}) = 0 \quad \textit{in other cases.}$$

Proof. By Lemma 4 the function $\vartheta(\tau; f_{s_1,s_2})$ belongs to the space of integral modular forms of the type $\big(-\frac{5}{2}, 16, v_0(M)\big)$, where the system of multiplicators $v_0(M)$ is calculated by formula (5). Therefore by Siegel's theorem the function $\theta(\tau; f_{s_1,s_2})$ also belongs to this space.

It is easy to verify that the function $\Phi(\tau; f_{s_1,s_2})$ satisfies conditions (6) of Lemma 5.

If $\alpha\delta \equiv 1 \pmod{16}$, then $\alpha\delta \equiv 1 \pmod 4$, i.e., either $\alpha \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 4$ or $\alpha \equiv -1 \pmod 4$ and $\delta \equiv -1 \pmod 4$.

In our case condition (7) of Lemma 5 is written as

$$\vartheta_{\alpha g,-h}(\tau; P_1, f_1)(\operatorname{sgn}\delta)\Big(\frac{-2^8}{|\delta|}\Big) = \Big(\frac{2^{10}}{|\delta|}\Big)\vartheta_{gh}(\tau; P_1, f_1) \tag{9}$$

and we must check it.

1. Let $\alpha \equiv 1 \pmod 4$ and $\delta \equiv 1 \pmod 4$. It is easy to verify that

$$(\operatorname{sgn}\delta)\Big(\frac{-2^8}{|\delta|}\Big) = \Big(\frac{2^{10}}{|\delta|}\Big)$$

and since $\alpha g = Nk_1 + g$ with as an integral vector k_1, together with Lemma 3 this implies the validity of (9).

2. We now set $\alpha \equiv -1 \pmod 4$ and $\delta \equiv -1 \pmod 4$. Since

$$(\operatorname{sgn}\delta)\Big(\frac{-2^8}{|\delta|}\Big) = -\Big(\frac{2^{10}}{|\delta|}\Big)$$

and $\alpha g = Nk_2 - g$, where k_2 is an integral vector, and, as is easy to verify, $\vartheta_{-g,h}(\tau; P_1, f_1) = -\vartheta_{g,h}(\tau; P_1, f_1)$, Lemma 3 implies (9). From (9) it follows that the function $\vartheta_{gh}(\tau; P_1, f_1)$ satisfies conditions (7) of Lemma 5 as well. Hence, by Lemmas 5 and 6, the function $\vartheta_{gh}(\tau; P_1, f_1)$ is a cusp form of the type $\big(-\frac{5}{2}, 16, v_0(M)\big)$.

Therefore due to Lemma 7 the function

$$\psi(\tau; f_{s_1,s_2}) = \vartheta(\tau; f_{s_1,s_2}) - \theta(\tau; f_{s_1,s_2}) - \Phi(\tau; f_{s_1,s_2}) \tag{10}$$

will be identically zero if in its expansion into powers of $Q = e^{2\pi i\tau}$ all coefficients of Q^n for $n \leq 5$ are zero.

Let $n = 2^{\alpha} m$ $(2 \nmid m,\ \alpha \geq 0)$, $2^{10-s_1-s_2} n = r^2\omega$, $m = r_1^2\omega_1$, ω and ω_1 be square-free integers. Then by formulas (2) and (3) we have

$$\theta(\tau; f_{s_1,s_2}) = 1 + \sum_{n=1}^{\infty} \rho(n; f_{s_1,s_2}) Q^n,$$

where

$$\rho(n; f_{s_1,s_2}) = \frac{2^{\frac{3\alpha+s_1+s_2}{2}+2}\omega_1^{\frac{3}{2}}}{\pi^2} \sum_{d|r_1} d^3 \prod_{p|d} \Big(1 - \big(\frac{\omega}{p}\big)p^{-2}\Big) L(2;\omega)\chi(2). \quad (11)$$

The values of $L(2,\omega)$ are given by Lemma 2. Introduce the notation $\chi_{s_1,s_2}(2)$ for the values of $\chi(2)$ corresponding to the quadratic form f_{s_1,s_2}. Using formulas (28)–(33) from [3], we obtain

$$\chi_{2,3}(2) = \begin{cases} 1, & \text{for } \alpha = 0 \text{ or } \alpha = 2; \\ \frac{2^{-\frac{3\alpha}{2}-\frac{1}{2}}}{7}\Big(13 \cdot 2^{\frac{3\alpha}{2}-\frac{1}{2}} + 2 - 7\big(\frac{2}{m}\big)\Big), & \text{for } 2 \nmid \alpha,\ m \equiv 1 \pmod 4; \\ \frac{2^{-\frac{3\alpha}{2}+\frac{1}{2}}}{7}\big(13 \cdot 2^{\frac{3\alpha}{2}-\frac{3}{2}} + 15\big), & \text{for } 2 \nmid \alpha,\ m \equiv 3 \pmod 4; \\ \frac{2^{-\frac{3\alpha}{2}+2}}{7}\big(13 \cdot 2^{\frac{3\alpha}{2}-3} + 15\big), & \text{for } 2 \mid \alpha,\ \alpha > 2. \end{cases} \quad (12)$$

After calculating the values of $\rho(n; f_{2,3})$ for all $n \leq 5$, by (2), (11) and (12) we have

$$\theta(\tau; f_{2,3}) = 1 + 2Q + 6Q^2 + 12Q^3 + 16Q^4 + 28Q^5 + \dots.$$

Formula (1) implies

$$\vartheta(\tau; f_{2,3}) = 1 + 4Q + 6Q^2 + 8Q^3 + 16Q^4 + 24Q^5 + \dots.$$

By (4) we obtain

$$\frac{1}{8}\vartheta_{gh}(\tau; P_1, f_1) = \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{4n = x_1^2 + x_2^2 + 2x_3^2 \\ x_1 \equiv 1 \pmod 4 \\ x_2 \equiv 1 \pmod 4 \\ 2 \nmid x_3}} (-1)^{\frac{x_1-1}{4} + \frac{x_2-1}{4} + \frac{x_3-1}{2}} x_3 Q^n \Bigg) =$$

$$= 2Q - 4Q^3 - 4Q^5 + \dots. \quad (13)$$

Now it is not difficult to verify that all coefficients of Q^n in the expansion into powers of Q of the function $\psi(\tau; f_{2,3})$ determined by (10) are zero for all $n \leq 5$. Thus identity (8) is proved for the case, where $s_1 = 2$ and $s_2 = 3$.

For other values of s_1 and s_2, the theorem is proved similarly. We give here a list of suitable values of $\chi(2)$ calculated by means of formulas (28)–(33) from [2]:

$$\chi_{1,1}(2)=\begin{cases}0, & \text{for } \alpha=1 \text{ or } \alpha=0,\\ & \quad m\equiv 3 \pmod 4;\\ 2, & \text{for } \alpha=0,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+1}}{7}\Big(5\cdot 2^{\frac{3\alpha}{2}}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2|\alpha,\ \alpha>1,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+2}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-1}+15\big), & \text{for } 2|\alpha,\ \alpha>1,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{7}{2}}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-\frac{5}{2}}+15\big), & \text{for } 2\nmid\alpha,\ \alpha>1;\end{cases}$$

$$\chi_{1,2}(2)=\begin{cases}1, & \text{for } \alpha=0 \text{ or } \alpha=1,\\ & \quad m\equiv 3 \pmod 4 \text{ or } \alpha=2;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{1}{2}}}{7}\Big(3\cdot 2^{\frac{3\alpha}{2}+\frac{1}{2}}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\nmid\alpha,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{3}{2}}}{7}\big(3\cdot 2^{\frac{3\alpha}{2}-\frac{1}{2}}+15\big), & \text{for } 2\nmid\alpha,\ \alpha>1,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+3}}{7}\big(3\cdot 2^{\frac{3\alpha}{2}-2}+15\big), & \text{for } 2\mid\alpha,\ \alpha>2;\end{cases}$$

$$\chi_{1,3}(2)=\begin{cases}1, & \text{for } \alpha=0 \text{ or } \alpha=1;\\ \dfrac{2^{-\frac{3\alpha}{2}}}{7}\Big(5\cdot 2^{\frac{3\alpha}{2}}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\mid\alpha,\ \alpha>1,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+1}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-1}+15\big), & \text{for } 2\mid\alpha,\ \alpha>1,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{5}{2}}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-\frac{5}{2}}+15\big), & \text{for } 2\nmid\alpha,\ \alpha>1;\end{cases}$$

$\chi_{1,4}(2)=\chi_{2,3}(2)$ (see (12));

$$\chi_{2,2}(2)=\begin{cases}1, & \text{for } 2\mid\alpha,\ \alpha\geq 0,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}}}{7}\Big(5\cdot 2^{\frac{3\alpha}{2}}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\mid\alpha,\ \alpha\geq 0,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+1}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-1}+15\big), & \text{for } 2\mid\alpha,\ \alpha\geq 0,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{5}{2}}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-\frac{5}{2}}+15\big), & \text{for } 2\nmid\alpha,\ \alpha>1;\end{cases}$$

$$\chi_{2,4}(2)=\begin{cases}1, & \text{for } \alpha=0 \text{ or } \alpha=1;\\ \dfrac{2^{-\frac{3\alpha}{2}-1}}{7}\Big(3\cdot 2^{\frac{3\alpha}{2}+2}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\mid\alpha,\ \alpha>1,\\ & \quad m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}}}{7}\big(3\cdot 2^{\frac{3\alpha}{2}+1}+15\big), & \text{for } 2\mid\alpha,\ \alpha>1,\\ & \quad m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{3}{2}}}{7}\big(5\cdot 2^{\frac{3\alpha}{2}-\frac{1}{2}}+15\big), & \text{for } 2\nmid\alpha,\ \alpha>1;\end{cases}$$

$$\chi_{3,3}(2)=\begin{cases}\dfrac{3}{2}, & \text{for } \alpha=1 \text{ or } \alpha=0,\ m\equiv 1 \pmod 4;\\ \dfrac{1}{2}, & \text{for } \alpha=0,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}-1}}{7}, & \text{for } 2\mid\alpha,\ \alpha>1,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}}}{7}, & \text{for } 2\mid\alpha,\ \alpha>1,\ m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+\frac{3}{2}}}{7}, & \text{for } 2\nmid\alpha,\ \alpha>1;\end{cases}$$

$$\chi_{3,4}(2)=\begin{cases}1, & \text{for } \alpha=0 \text{ or } \alpha=1,\\ & \quad m\equiv 3 \pmod 4 \text{ or } \alpha=2;\\ \dfrac{2^{-\frac{3\alpha}{2}-\frac{3}{2}}}{7}\Big(27\cdot 2^{\frac{3\alpha}{2}-\frac{1}{2}}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\nmid\alpha,,\ m\equiv 1 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}-\frac{3}{2}}}{7}\big(27\cdot 2^{\frac{3\alpha}{2}-\frac{1}{2}}+30\big), & \text{for } 2\nmid\alpha,\ \alpha>1,\\ & \quad m\equiv 3 \pmod 4;\\ \dfrac{2^{-\frac{3\alpha}{2}+1}}{7}\big(9\cdot 2^{\frac{3\alpha}{2}-3}+5\big), & \text{for } 2\mid\alpha,\ \alpha>2;\end{cases}$$

$$\chi_{4,4}(2)=\begin{cases}1, & \text{for } \alpha=0;\\ \dfrac{3\cdot 2^{-\frac{3\alpha}{2}+\frac{1}{2}}}{7}\big(2^{\frac{3\alpha}{2}-\frac{1}{2}}+5\big), & \text{for } 2\nmid\alpha;\\ \dfrac{2^{-\frac{3\alpha}{2}-2}}{7}\Big(3\cdot 2^{\frac{3\alpha}{2}+2}+2-7\Big(\dfrac{2}{m}\Big)\Big), & \text{for } 2\mid\alpha,\ m\equiv 1 \pmod 4;\\ \dfrac{3\cdot 2^{-\frac{3\alpha}{2}-1}}{7}\big(2^{\frac{3\alpha}{2}+1}+5\big), & \text{for } 2\mid\alpha,\ m\equiv 3 \pmod 4. \quad \square\end{cases}$$

Theorem 2. *Let* $n=2^{\alpha}m$ $(\alpha\geq 0,\ 2\nmid m)$, $m=r_1^2\omega_1$, $1\leq s_1\leq s_2\leq 4$,

$2^{10-s_1-s_2}n = r^2\omega$ (*ω and ω_1 are square-free integers*). *Then*

$$r(n;f_{s_1,s_2}) = \frac{2^{\frac{3\alpha+s_1+s_2}{2}+1}\omega_1^{\frac{3}{2}}}{\pi^2}\sum_{d|r_1} d^3 \prod_{p|d}\Big(1-(\frac{\omega}{p})p^{-2}\Big)L(2;\omega)\chi(2)+ \\ +\nu_{s_1,s_2}(n), \qquad (14)$$

where

$$2\nu_{1,2}(n) = \nu_{2,3}(n) = \nu_{3,4}(n) = 2\sum_{\substack{4n=x_1^2+x_2^2+2x_3^2\\ 2\nmid x_1,\ 2\nmid x_2,\ 2\nmid x_3\\ x_1>0,\ x_2>0,\ x_3>0}}\Big(\frac{2}{x_1x_2}\Big)\Big(\frac{-1}{x_3}\Big)x_3,$$

$$\nu_{s_1,s_2}(n) = 0 \quad \textit{in other cases.}$$

Proof. By equating the coefficients of equal powers of Q in both parts of identity (8) we obtain

$$r(n;f_{s_1,s_2}) = \rho(n;f_{s_1,s_2}) + \nu_{s_2,s_2}(n), \qquad (15)$$

where $\nu_{s_1,s_2}(n)$ denotes the coefficients of Q^n in the expansion of the function $\Phi(\tau;f_{s_1,s_2})$ into powers of Q.

When $s_1 = 2$ and $s_2 = 3$, by (13) we have

$$\nu_{2,3}(n) = \sum_{\substack{4n=x_1^2+x_2^2+2x_3^2\\ x_1\equiv 1 \pmod 4\\ x_2\equiv 1 \pmod 4\\ 2\nmid x_3}} (-1)^{\frac{x_1-1}{4}+\frac{x_2-1}{4}+\frac{x_3-1}{2}}x_3$$

i.e.,

$$\nu_{2,3}(n) = 2\sum_{\substack{4n=x_1^2+x_2^2+x_3^2\\ 2\nmid x_1, 2\nmid x_2, 2\nmid x_3\\ x_1>0, x_2>0, x_3>0}}\Big(\frac{2}{x_1x_2}\Big)\Big(\frac{-1}{x_3}\Big)x_3. \qquad (16)$$

From formulas (11), (15) and (16) it follows that the theorem is valid when $s_1 = 2$ and $s_2 = 3$. The validity of equality (14) for other values of s_1 and s_2 is proved in a similar manner. □

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