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THE CONTACT PROBLEM FOR AN ELASTIC ORTHOTROPIC PLATE SUPPORTED BY PERIODICALLY LOCATED BARS OF EQUAL RESISTANCE

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Abstract. The contact problem of the plane theory of elasticity is studied for an elastic orthotropic half-plane supported by periodically located (infinitely many) stringers of equal resistance. Using the methods of the theory of a complex variable, the problem is reduced to the Keldysh–Sedov type problem for a circle. The solution of the problem is constructed.

Let an elastic orthotropic plate occupying a lower half-plane of a complex plane $z = x + iy$ be supported by periodically located elastic absolutely flexible (infinitely many) bars of equal resistance. Longitudinal forces p and q are applied to the bar ends. The bars are to be free assumed from other external loads. The problem consists in finding the cross-sectional areas $S(x)$ of the bars and the contact the tangential stresses $\tau_{xy}(x,0)$ provided that the longitudinal stresses $\sigma_x^{(0)}(x)$ in the bars are constant and equal to a.

Similar problems for isotropic elastic domains have been investigated in [1–4]. In the case of an anisotropic half-plane this problem has been studied by the author in [5]. Periodic problems dealing with springers of constant rigidity can be found in [6–7].

Without restriction of generality, the length of the stringer bases is assumed to be equal to unity. Denote the distance between the stringers by $2l$. The stringers are located symmetrically with respect to the ordinate axis. In such a case the stringers will be located as follows: $[(2k+1)l + k; (2k+1)l + k + 1]$, $k = 0, \pm 1, \pm 2, \dots$.

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From the equilibrium condition of stringer elements on the reinforced sections we obtain

$$S(x)\sigma_x^{(0)}(x) - h\int_{(2k+1)l+k}^{x} \tau_{xy}(s)ds - q = 0, \quad x \in L_k, \tag{1}$$

where h is the bar thickness and L_k denotes a segment $[(2k+1)l+k; (2k+1)l+k+1]$.

Taking into account the fact that the bars are absolutely flexible and their resistance under bending is a negligibly small value, we may assume that $\sigma_y = \sigma_y^{(0)} = 0$ for $-\infty < x < \infty$. As far as the stringers are located periodically, we may consider the problem on a half-strip $-\infty < y < 0$, $0 < x < 2l+1$.

On the boundary we have the following conditions:

$$S(x)\sigma_x^{(0)}(x) - h\int_{l}^{x} \tau_{xy}(s)ds - q = 0, \quad x \in (l; l+1); \tag{2}$$

$$\begin{aligned} &\sigma_y(x) = 0, \quad x \in (0; 2l+1), \\ &\tau_{xy} = 0, \qquad x \in (0; l) \cup (l+1; 2l+1), \\ &\sigma_x^{(0)}(x) = a, \quad x \in (l; l+1); \end{aligned} \tag{3}$$

$$\begin{aligned} &\tau_{xy}(0; y) - \tau_{xy}(2l+1; y) = \sigma_y(0, y) - \sigma_y(2l+1; y) = \sigma_x(0; y) - \\ &-\sigma_x(2l+1; y) = u(0; y) - u(2l+1; y) = v(0; y) - v(2l+1; y) = 0. \end{aligned} \tag{4}$$

According to Hooke's law, we have respectively for the bar and for the plate:

$$\frac{du_0(x)}{dx} = \frac{\sigma_x^{(0)}(x)}{E_0}; \quad \frac{du(x,0)}{dx} = \frac{\sigma_x(x,0)}{E_1},$$

where E_0 is the modulus of elasticity of the bar; $a_{11} = 1/E_1$ is the elastic constant of the plate.

The conditions of full contact between the elastic bar and the plate

$$\frac{du_0(x)}{dx} = \frac{du(x,0)}{dx}, \quad \tau_{xy}^{(0)}(x) = \tau_{xy}(x)$$

result in the equality $\sigma_x^{(0)}(x) = \frac{E_0}{E_1}\,\sigma_x(x,0)$. Now the boundary conditions (2) and (3) can be written as

$$\begin{aligned} &\frac{E_0}{E_1}\sigma_x(x,0) = a, \quad x \in (l; l+1), \\ &\sigma_y = 0, \quad x \in (0; 2l+1), \quad \tau_{xy} = 0, \quad x \in (0; l) \cup (l+1; 2l+1), \\ &aS(x) - h\int_l^x \tau_{xy}(s)ds = q, \quad x \in (l; l+1). \end{aligned} \tag{5}$$

As is known, the stress components are calculated by the formulas [8]

$$\begin{aligned}\sigma_x &= 2\,\mathrm{Re}\left[\mu_1^2\Phi_1(z_1)+\mu_2^2\Phi_2(z_2)\right],\\ \sigma_y &= 2\,\mathrm{Re}\left[\Phi_1(z_1)+\Phi_2(z_2)\right],\\ \tau_{xy} &= -2\,\mathrm{Re}\left[\mu_1\Phi_1(z_1)+\mu_2\Phi_2(z_2)\right],\end{aligned} \tag{6}$$

where $z_k = x_k + \mu_k y$, $k = 1, 2$, and μ_k are the roots of the characteristic equation corresponding to the generalized biharmonic equation.

Due to the periodicity of the boundary conditions, the functions $\Phi_1(z)$ and $\Phi_2(z)$ are also periodic in the half-plane $y < 0$ with period $2l + 1$, that is,

$$\Phi_1(i\beta_1 y) = \Phi_1(2l+1+i\beta_1 y), \quad \Phi_2(i\beta_2 y) = \Phi_2(2l+1+i\beta_2 y). \tag{7}$$

Since the body is orthotropic and the axes of elastic symmetry are parallel to the coordinate axes, $\mu_1 = i\beta_1$, $\mu_2 = i\beta_2$ (we assume $\beta_1 > \beta_2 > 0$), using formulas (6) the boundary conditions take the form

$$\mathrm{Re}\left[\beta_1^2\Phi_1(x)+\beta_2^2\Phi_2(x)\right] = -\frac{E_1 a}{2E_0}, \quad x \in (l; l+1), \tag{8}$$

$$\mathrm{Re}\left[\Phi_1(x)+\Phi_2(x)\right] = 0, \quad x \in (0; 2l+1), \tag{9}$$

$$\mathrm{Im}\left[\beta_1\Phi_1(x)+\beta_2\Phi_2(x)\right] = 0, \quad x \in (0;l)\cup(l+1;2l+1), \tag{10}$$

$$aS(x) - h\int_l^x \tau_{xy}(s)ds = q, \quad x \in (l; l+1).$$

Let us prove the validity of the following proposition.

Theorem. *If the boundary conditions* (8), (9), *and* (10) *are fulfilled, then the stress components are expressed in terms of one analytic function.*

Proof. The function $\Phi_1(x) + \Phi_2(x)$ is a boundary value of the function $\mathrm{Im}\, z < 0$ which is holomorphic in the half-plane $\Phi_1(z) + \Phi_2(z)$ and periodic with period $2l+1$, that is, $\Phi_1(x+2l+1)+\Phi_2(x+2l+1) = \Phi_1(x)+\Phi_2(x)$, bounded in the half-strip $0 \le x \le 2l+1$, $y < 0$, continuously extendible to the boundary $0 \le x \le 2l+1$, with the exclusion maybe of the points $x = l$, $x = l + 1$. In the vicinity of these points the function under consideration satisfies the condition

$$|\Phi_1(z)+\Phi_2(z)| < \frac{c}{|z-(l+k)|^\delta}, \quad k = 0; 1, \quad 0 \le \delta < 1. \tag{11}$$

Since the function $\Phi_1(z)+\Phi_2(z)$ takes imaginary values on the real axis, on the basis of the Riemann-Schwarz symmetry principle it is analytically extendible on the whole strip $0 < x < 2l+1$, $-\infty < y < \infty$, with the exclusion maybe of the above-mentioned points in whose vicinity the estimate (11) holds.

It follows from the above that these points are removable. Since the function $\Phi_1(z)+\Phi_2(z)$ is periodic, it is bounded on the whole plane.

According to Liouville's theorem, we can conclude that the function $\Phi_1(z)+\Phi_2(z)$ is constant. If we use the conditions (9) and (10), then we can say that the function $\Phi_1(z)+\Phi_2(z)$ equals zero on the whole plane.

$$\Phi_1(z) = -\Phi_2(z) \quad \text{for} \quad \operatorname{Im} z \leq 0. \quad \square \tag{12}$$

Applying the above-obtained equality, the boundary conditions (8) and (10) can be written as

$$\begin{gathered}(\beta_1^2-\beta_2^2)\operatorname{Re}\Phi_1(x) = -\frac{aE_1}{2E_0}, \quad x\in(l;l+1), \\ \operatorname{Im}\Phi_1(x)=0, \quad x\in(0;l)\cup(l+1;2l+1).\end{gathered} \tag{13}$$

Thus the problem under consideration is reduced to the problem of finding an analytic in the half-strip $0<x<2l+1$, $y<0$ function $\Phi(z)$ with the boundary conditions (13).

The function

$$z=(2l+1)\Big(1-\frac{1}{2\pi i}\ln\zeta\Big) \tag{14}$$

maps the half-strip $0<\operatorname{Re} z<2l+1$, $\operatorname{Im} z<0$ onto a circle $|\zeta|<1$ cut along the segment $(0;1)$; besides, the point $x=2l+1$ transfers to the point $\zeta=1$, the segment $(0;2l+1)$ maps onto the circumference $|\zeta|=1$, the half-line $x=0$, $y<0$ transfers to the lower end of the cut, and the half-line $x=2l+1$, $y<0$ to the upper end of the cut.

We introduce the notation

$$\Psi(\zeta)=\Phi_1\Big[(2l+1)\Big(1-\frac{1}{2\pi i}\ln\zeta\Big)\Big]. \tag{15}$$

The function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta|<1$ cut along the segment $0<\zeta<1$. From the periodicity of the function $\Phi_1(z)$ we find the equality $\Psi^+(\zeta)=\Psi^-(\zeta)$, $0<\zeta<1$, where $\Psi^+(\zeta)$ and $\Psi^-(\zeta)$ denote the boundary values of the function $\Psi(\zeta)$ on the upper and lower ends, respectively.

From the above we conclude that the function $\Psi(\zeta)$ is holomorphic in the circle $|\zeta|<1$. In this case the boundary conditions (13) take the form

$$\operatorname{Re}[\Psi(\zeta)] = -\frac{aE_1}{2E_0(\beta_1^2-\beta_2^2)}, \quad \zeta\in\gamma_1, \tag{16}$$

$$\operatorname{Im}[\Psi(\zeta)]=0, \quad \zeta\in\gamma_2, \tag{17}$$

where γ_1 denotes an arc of the circumference of unit radius which is the mapping of the segment $(l;l+1)$, and γ_2 denotes the remaining part of the circumference onto which the segments $(0;l)\cup(l+1;2l+1)$ are mapped.

Moreover, to the points $x = 0$ and $x = 2l + 1$ there corresponds the point $\zeta = 1$, i.e., γ_2 is a continuous arc.

If we introduce the notation

$$\psi(\zeta) = \Psi(\zeta) + \frac{E_1 a}{2E_0(\beta_1^2 - \beta_2^2)}, \tag{18}$$

then we obtain

$$\operatorname{Re}\psi(\zeta) = 0 \quad \text{for} \quad \zeta \in \gamma_1, \quad \operatorname{Im}\psi(\zeta) = 0 \quad \text{for} \quad \zeta \in \gamma_2, \tag{19}$$

or

$$\psi(\zeta) + \overline{\psi(\zeta)} = 0 \quad \text{for} \quad \zeta \in \gamma_1, \quad \psi(\zeta) - \overline{\psi(\zeta)} = 0 \quad \text{for} \quad \zeta \in \gamma_2. \tag{20}$$

Introducing a piecewise holomorphic function

$$W(\zeta) = \begin{cases} \psi(\zeta) & \text{for} \quad |\zeta| < 1, \\ \overline{\psi(\bar{\zeta})} & \text{for} \quad |\zeta| > 1, \end{cases} \tag{21}$$

we obtain the problem

$$\begin{cases} W^+(\sigma) + W^-(\sigma) = 0 & \text{for} \quad \sigma \in \gamma_1, \\ W^-(\sigma) - W^-(\sigma) = 0 & \text{for} \quad \sigma \in \gamma_2. \end{cases} \tag{22}$$

A general solution of problem (22) belonging to the class h_0 and bounded at infinity is given by the formula [9]

$$W(\zeta) = \frac{c_0\zeta + \bar{c}_0}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}}, \tag{23}$$

where σ_1 and σ_2 are the ends of the arc γ_1, $\sigma_1 = e^{\frac{2\pi l i}{2l+1}}$, $\sigma_2 = e^{\frac{2\pi(l+1)i}{2l+1}}$.

By $\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}$ we mean a function which is holomorphic on the plane cut along γ_1 and satisfies the condition

$$\frac{\zeta}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}} \to 1 \quad \text{as} \quad \zeta \to \infty.$$

Taking into account the equalities (21) and (23), from the equality (18) we obtain

$$\Psi(\zeta) = \frac{c_0\zeta + \bar{c}_0}{\sqrt{(\zeta - \sigma_1)(\zeta - \sigma_2)}} - M, \quad M = \frac{E_1 a}{2E_0(\beta_1^2 - \beta_2^2)}. \tag{24}$$

Getting back to the variable z which is connected with the variable ζ by the relation (14), i.e., $\zeta = \exp 2\pi i\bigl(1 - \frac{z}{2l+1}\bigr)$, and introducing the variables

$$\rho = e^{\frac{2\pi y}{2l+1}}, \quad \theta = 2\pi\Bigl(1 - \frac{x}{2l+1}\Bigr), \tag{25}$$

from formula (24) we find that

$$\Psi[\rho e^{i\theta}] = \Phi_1(z) = \frac{c_0\rho e^{i\theta} + \overline{c_0}}{\sqrt{(\rho e^{i\theta} - e^{i\theta_1})(\rho e^{i\theta} - e^{i\theta_2})}} - M, \tag{26}$$

where $\theta_1 = \frac{2\pi l}{2l+1}$, $\theta_2 = \frac{2\pi(l+1)}{2l+1}$.

If now instead of y we substitute in the formula (26) the values $\beta_1 y$ and $\beta_2 y$, then applying equalities (25) we get

$$\Phi_1(z) = \frac{c_0\rho_1 e^{i\theta} + \overline{c_0}}{\sqrt{(\rho_1 e^{i\theta} - e^{i\theta_1})(\rho_1 e^{i\theta} - e^{i\theta_2})}} - M, \ \ \theta_1 < \theta < \theta_2, \tag{27}$$

where $\rho_1 = e^{\frac{2\pi\beta_1 y}{2l+1}}$.

With regard for the condition $\Phi_2(z) = -\Phi_1(z)$ for the function $\Phi_2(z)$ we obtain the formula

$$\Phi_2(z) = -\frac{c_0\rho_2 e^{i\theta} + \overline{c_0}}{\sqrt{(\rho_2 e^{i\theta} - e^{i\theta_1})(\rho_2 e^{i\theta} - e^{i\theta_2})}} + M, \ \ \theta_1 < \theta < \theta_2, \tag{28}$$

where $\rho_2 = e^{\frac{2\pi\beta_2 y}{2l+1}}$.

To find a complex constant c_0, we take advantage of the conditions $\sigma_x(x-i\infty) = \sigma_y(x-i\infty) = 0$.

The external forces acting on the stringer are, in general, not in equilibrium ($p - q \neq 0$), that is, the principal vector of tangential stresses does not equal to zero. Therefore, the tangential stresses do not tend to zero as $y \to -\infty$, while σ_y and σ_x vanish as $y \to -\infty$.

Passing to the limit in the equalities (27) and (28), we obtain the following relations:

$$\Phi_1(x - i\infty) = -\Phi_2(x - i\infty) = \bar{c}_0 - M. \tag{29}$$

Using now formulas (6), we arrive at

$$\begin{gathered} \operatorname{Re}\big[\beta_1^2\Phi_1(x-i\infty) + \beta_2^2\Phi_2(x-i\infty)\big] = 0, \\ \operatorname{Re}\big[\Phi_1(x-i\infty) + \Phi_2(x-i\infty)\big] = 0. \end{gathered}$$

The second condition, due to the equalities (29), is satisfied for any $\bar{c}_0$, and from the first condition it follows that

$$\operatorname{Re} c_0 = M. \tag{30}$$

We use the following equilibrium condition of the stringer:

$$\int_l^{l+1} \tau_{xy}(s)ds = \frac{p-q}{h}. \tag{31}$$

Applying the third equality from formulas (6), we obtain the following expression for contact tangential stresses:

$$\tau_{xy}(x) = -\frac{2(\beta_1-\beta_2)[\operatorname{Re} c_0 \cos\frac{\theta}{2} - \operatorname{Im} c_0 \sin\frac{\theta}{2}]}{\sqrt{\sin\frac{\theta-\theta_1}{2}\sin\frac{\theta_2-\theta_1}{2}}}. \tag{32}$$

Since $\theta_2-\theta_1 = \frac{2\pi}{2l+1} < 2\pi$, the value under the radical sign is positive.

In order to determine a form of the stringer, we have to calculate the integral $\int_l^x \tau_{xy}(s)ds$.

With the use of the condition $\theta_1+\theta_2 = 2\pi$, after elementary transformations we find from the equality (32) that

$$\tau_{xy}(x) = \frac{2(\beta_1-\beta_2)\operatorname{Re} c_0 \cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\theta}{2}-\sin^2\frac{\theta_1}{2}}} + \frac{2(\beta_1-\beta_2)\operatorname{Im} c_0 \cos\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_1}{2}-\cos^2\frac{\theta}{2}}}.$$

Integrating the last equality, we obtain

$$\int_l^x \tau_{xy}(s)ds = 2(\beta_1-\beta_2)\operatorname{Re} c_0 \int_l^x \frac{\cos\frac{\pi s}{2l+1}ds}{\sqrt{\sin^2\frac{\pi s}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}} +$$

$$+2(\beta_1-\beta_2)\operatorname{Im} c_0 \int_l^x \frac{\sin^2\frac{\pi s}{2l+1}ds}{\sqrt{\cos^2\frac{(l+1)\pi}{2l+1}-\cos^2\frac{\pi s}{2l+1}}} =$$

$$= 2(\beta_1-\beta_2)\frac{2l+1}{\pi}\Big[\operatorname{Re} c_0 \ln\frac{\sin\frac{\pi x}{2l+1}+\sqrt{\sin^2\frac{\pi x}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}}{\sin\frac{\pi l}{2l+1}} +$$

$$+\operatorname{Im} c_0\Big(\arcsin\frac{\cos\frac{\pi x}{2l+1}}{\cos\frac{\pi l}{2l+1}} - \frac{\pi}{2}\Big)\Big] \tag{33}$$

whence we have the formula $\int_l^{l+1} \tau_{xy}(s)ds = 2(\beta_2-\beta_1)(2l+1)\operatorname{Im} c_0$.

If the use is made of the equilibrium equation (31), then we get

$$\operatorname{Im} c_0 = \frac{p-q}{2h(\beta_2-\beta_1)(2l+1)}. \tag{34}$$

Substituting the integral value defined by the equality (33) into the last of the boundary conditions (formulas (5)) for an elastic half-plane, we obtain for an unknown profile of the stinger the following relation:

$$aS(x) = q + 2h(\beta_1-\beta_2)\frac{2l+1}{\pi}\Big[\operatorname{Re} c_0 \ln\frac{\sin\frac{\pi x}{2l+1}+\sqrt{\sin^2\frac{\pi x}{2l+1}-\sin^2\frac{(l+1)\pi}{2l+1}}}{\sin\frac{\pi l}{2l+1}} +$$

$$+\operatorname{Im} c_0\Big(\arcsin\frac{\cos\frac{\pi x}{2l+1}}{\cos\frac{\pi l}{2l+1}} - \frac{\pi}{2}\Big)\Big]. \quad \square$$

References

1. B.M. Nuller, Contact problem for an elastic wedge reinforced by a bar of equal resistance. (Russian) *Dokl. Akad. Nauk SSSR*, No. 3, **225**(1975) 532–534.

2. B.M. Nuller, Optimal choice of rigidity of thin-shelled supporting elements in contact problems of the theory of elasticity. (Russian) *Izv. Vsesojuzn. Nauchn.-Issled. Inst. Gidrotekhniki* **108**(1975), 115–123.

3. A.N. Zlatin and B.M. Nuller, Optimal rigidity of circular plates lying on an elastic halfplane. (Russian) *Izv. Vsesojuzn. Nauchn.-Issled. Inst. Gidrotekhniki* **110**(1976) 91–94.

4. E.L. Nakhmein and B.M. Nuller, Periodical contact problems for an elastic half-plane reinforced by stringers or beams of equal resistance. (Russian) *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela* 1978, No. 5, 81–88.

5. L.A. Gogolauri, Contact problem for an elastic anisotropic plate reinforced by a stringer of equal resistance. (Russian) *Soobshch. Bull. Acad. Sci. Georgia* **141**(1991), No. 1, 57–60.

6. N.Kh. Arutunyan and S.M. Mkhitaryan, Periodical contact problem for a half-plane with elastic straps. (Russian) *Prikl. Mat. Mekh.* **33**(1969), No. 5, 813–843.

7. G.A. Morar' and G.Ya. Popov, To the periodic contact problem for a half-plane with elastic straps. (Russian) *Prikl. Mat. Mekh.* **35**(1971), No. 1, 172–178.

8. S.G. Lekhnitskii, Anisotropic plates. (Russian) *Gostekhizdat, Moscow–Laningrad*, 1947.

9. N.I. Muskhelishvili, Singular integral equations. (Translation from the Russian) *P. Noordhoff, Groningen*, 1953.

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