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ON THE VECTOR PROCESS OBTAINED BY ITERATED INTEGRATION OF THE TELEGRAPH SIGNAL

ENZO ORSINGHER

ABSTRACT. We analyse the vector process $(X_0(t), X_1(t), \dots, X_n(t), t > 0)$ where $X_k(t) = \int_0^t X_{k-1}(s)ds$, $k = 1, \dots, n$, and $X_0(t)$ is the two-valued telegraph process.

In particular, the hyperbolic equations governing the joint distributions of the process are derived and analysed.

Special care is given to the case of the process $(X_0(t), X_1(t), X_2(t), t > 0)$ representing a randomly accelerated motion where some explicit results on the probability distribution are derived.

1. General Results Concerning the Integrated Telegraph Signal

Let us consider the two-valued telegraph process

$$X_0(t) = X(0)(-1)^{N(t)}, \tag{1.1}$$

where $X(0)$ is a random variable which is independent of $N(t)$ and takes values $\pm a$ with equal probability. By $N(t)$ we denote the number of events of a homogeneous Poisson process up to time t (with rate λ).

Let also

$$X_1(t) = \int_0^t X_0(s)\,ds \tag{1.2}$$

and, in general,

$$X_k(t) = \int_0^t X_{k-1}(s)\,ds, \quad k = 1, 2, \dots, n. \tag{1.3}$$

When $n = 2$, the vector process $(X_0(t), X_1(t), X_2(t), t \geq 0)$ has a straightforward physical interpretation. Indeed, $X_2(t)$ represents the position of

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a particle with acceleration X_0 and velocity X_1. The same interpretation is possible for the triple $(X_j(t), X_{j+1}(t), X_{j+2}(t),\ t \geq 0)$, where $j = 0, 1, \dots, n-2$.

Most of the current literature (see [1–4]) concerns the process $(X_0(t), X_1(t),\ t \geq 0)$ and our effort here is to examine the general situation, with special attention to the case $n = 2$.

Let us now introduce

$$\begin{cases} F(x_1, \dots, x_n, t) = \Pr\{X_1(t) \leq x_1, \dots, X_n(t) \leq x_n,\ X_0(t) = a\}, \\ B(x_1, \dots, x_n, t) = \Pr\{X_1(t) \leq x_1, \dots, X_n(t) \leq x_n,\ X_0(t) = -a\} \end{cases} \tag{1.4}$$

and denote by $f = f(x_1, \dots, x_n, t)$, $b = b(x_1, \dots, x_n, t)$ the corresponding densities. Our first result is the differential system governing f and b.

Theorem 1.1. *The densities f and b are solutions of*

$$\begin{cases} \dfrac{\partial f}{\partial t} = -a \dfrac{\partial f}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1} \dfrac{\partial f}{\partial x_j} + \lambda(b - f), \\ \dfrac{\partial b}{\partial t} = a \dfrac{\partial b}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1} \dfrac{\partial b}{\partial x_j} + \lambda(f - b). \end{cases} \tag{1.5}$$

Proof. For the sake of simplicity, we divide the time domain into intervals of length Δt and assume that changes of the process X_0 occur only at the endpoints of these intervals. The effect of this assumption disappears as $\Delta t \to 0$.

In order to be at $(x_1, \dots, x_n)$ at time $t + \Delta t$ one of the following cases need to occur:

(i) $\Delta N = 0$ and the starting point for each process must be

$$\overline{x}_k = x_k - x_{k-1}\Delta t + \dots + (-1)^j x_{k-j} \frac{(\Delta t)^j}{j!} + \dots + (-1)^k a \frac{(\Delta t)^k}{k!},$$

where $k = 1, 2, \dots, n$ and $x_0 = 0$.

(ii) $\Delta N = 1$ and the starting point for each process is

$$\overline{x}_k = x_k - x_{k-1}\Delta t + \dots + (-1)^j x_{k-j} \frac{(\Delta t)^j}{j!} + \dots + (-1)^{k-1} a \frac{(\Delta t)^k}{k!}.$$

(iii) $\Delta N > 1$.

Restricting ourselves only to the first-order terms we have

$$\begin{aligned} &f(x_1, \dots, x_n, t + \Delta t) = \\ &= (1 - \lambda\Delta t) f(x_1 - a\Delta t, x_2 - x_1\Delta t, \dots, x_n - x_{n-1}\Delta t, t) + \\ &+ \lambda\Delta t b(x_1 + a\Delta t, x_2 - x_1\Delta t, \dots, x_n - x_{n-1}\Delta t, t) + o(\Delta t). \end{aligned} \tag{1.6}$$

Expanding in Taylor series and passing to the limit as $\Delta t \to 0$, we obtain the first equation of system (1.5). An analogous treatment leads to the second equation of (1.5).

If $p = f + b$ and $w = f - b$, the above system can be rewritten as

$$\begin{cases} \dfrac{\partial p}{\partial t} = -a\, \dfrac{\partial w}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1}\, \dfrac{\partial p}{\partial x_j}, \\ \dfrac{\partial w}{\partial t} = -a\, \dfrac{\partial p}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1}\, \dfrac{\partial w}{\partial x_j} - 2\lambda w. \end{cases} \tag{1.7}$$

A rather surprising fact is that the equation governing p (to be obtained from (1.7)) is of third order for any $n \geq 2$. □

Theorem 1.2. *The probability density $p = p(x_1, \dots, x_n, t)$ is a solution of*

$$\frac{\partial}{\partial x_1}\Big[\frac{\partial^2 p}{\partial t^2} + \sum_{j=2}^{n} x_{j-1}\, \frac{\partial}{\partial x_j}\Big(\frac{\partial p}{\partial t} + \sum_{r=2}^{n} x_{r-1}\, \frac{\partial p}{\partial x_r}\Big) + 2\lambda\Big(\frac{\partial p}{\partial t} + \sum_{j=2}^{n} x_{j-1}\, \frac{\partial p}{\partial x_j}\Big) + \\ + \sum_{j=2}^{n} x_{j-1}\, \frac{\partial^2 p}{\partial t \partial x_j} - a^2\, \frac{\partial^2 p}{\partial x_1^2}\Big] = -\frac{\partial}{\partial x_2}\Big(\frac{\partial p}{\partial t} + \sum_{j=2}^{n} x_{j-1}\, \frac{\partial p}{\partial x_j}\Big). \tag{1.8}$$

Proof. Deriving the first equation of (1.7) with respect to time t and inserting the other one derived with respect to x_1, we obtain

$$\frac{\partial^2 p}{\partial t^2} = -a\, \frac{\partial^2 w}{\partial x_1 \partial t} - \sum_{j=2}^{n} x_{j-1}\, \frac{\partial^2 p}{\partial t \partial x_j} =$$

$$= -a\Big[-a\, \frac{\partial^2 p}{\partial x_1^2} - \frac{\partial w}{\partial x_2} - \sum_{j=2}^{n} x_{j-1}\, \frac{\partial^2 w}{\partial x_j \partial x_1} - 2\lambda\, \frac{\partial w}{\partial x_1}\Big] - \sum_{j=2}^{n} x_{j-1}\, \frac{\partial^2 p}{\partial t \partial x_j} =$$

$$= a^2\, \frac{\partial^2 p}{\partial x_1^2} + a\, \frac{\partial w}{\partial x_2} - \sum_{j=2}^{n} x_{j-1}\, \frac{\partial}{\partial x_j}\left(\frac{\partial p}{\partial t} + \sum_{r=2}^{n} x_{r-1}\, \frac{\partial p}{\partial x_r}\right) -$$

$$-2\lambda\left(\frac{\partial p}{\partial t} + \sum_{j=2}^{n} x_{j-1}\, \frac{\partial p}{\partial x_j}\right) - \sum_{j=2}^{n} x_{j-1}\, \frac{\partial^2 p}{\partial t \partial x_j}.$$

A further derivation with respect to x_1 then leads to equation (1.8). □

Dealing with the third-order equation (1.8) implies substantial difficulties. In order to circumvent them we present the second-order equations governing the densities f and b.

For convenience we write $f = e^{-\lambda t}\overline{f}$, $b = e^{-\lambda t}\overline{b}$ and obtain from (1.5)

$$\begin{cases} \dfrac{\partial \overline{f}}{\partial t} = -a\,\dfrac{\partial \overline{f}}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial \overline{f}}{\partial x_j} + \lambda \overline{b}, \\ \dfrac{\partial \overline{b}}{\partial t} = a\,\dfrac{\partial \overline{b}}{\partial x_1} - \displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial \overline{b}}{\partial x_j} + \lambda \overline{f}. \end{cases} \tag{1.9}$$

Some calculations now suffice to obtain from the above system the second-order equations

$$\begin{cases} \dfrac{\partial^2 \overline{f}}{\partial t^2} = -2\displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial^2 \overline{f}}{\partial x_j \partial t} + a^2\,\dfrac{\partial^2 \overline{f}}{\partial x_1^2} + a\,\dfrac{\partial \overline{f}}{\partial x_2} - \\ \qquad - \displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial}{\partial x_j}\left(\sum_{r=2}^{n} x_{r-1}\,\dfrac{\partial \overline{f}}{\partial x_r}\right) + \lambda^2 \overline{f}, \\ \dfrac{\partial^2 \overline{b}}{\partial t^2} = -2\displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial^2 \overline{b}}{\partial x_j \partial t} + a^2\,\dfrac{\partial^2 \overline{b}}{\partial x_1^2} - a\,\dfrac{\partial \overline{b}}{\partial x_2} - \\ \qquad - \displaystyle\sum_{j=2}^{n} x_{j-1}\,\dfrac{\partial}{\partial x_j}\left(\sum_{r=2}^{n} x_{r-1}\,\dfrac{\partial \overline{b}}{\partial x_r}\right) + \lambda^2 \overline{b}. \end{cases} \tag{1.10}$$

Remark 1.1. If $\lambda \to \infty$ and $a \to \infty$ in such a way that $\frac{a^2}{\lambda} \to 1$, we obtain from (1.8) the equation

$$\frac{\partial p}{\partial t} + \sum_{j=2}^{n} x_{j-1}\,\frac{\partial p}{\partial x_j} = \frac{1}{2}\,\frac{\partial^2 p}{\partial x_1^2}, \tag{1.11}$$

which is satisfied by the probability law of the vector process $(X_1(t), \dots, X_n(t),\ t \geq 0)$, where X_1 is a standard Brownian motion and

$$X_k(t) = \int_0^t X_{k-1}(s)\,ds, \quad k = 2, \dots, n.$$

2. The Special Case $n = 2$

A deeper analysis is possible in the case of the vector process $(X_0(t), X_1(t), X_2(t),\ t \geq 0)$ representing a uniformly accelerated random motion.

In that case system (1.6) reads as

$$\begin{cases} \dfrac{\partial f}{\partial t} = -a\,\dfrac{\partial f}{\partial x_1} - x_1\,\dfrac{\partial f}{\partial x_2} + \lambda(b - f), \\ \dfrac{\partial b}{\partial t} = a\,\dfrac{\partial b}{\partial x_1} - x_1\,\dfrac{\partial b}{\partial x_2} + \lambda(f - b) \end{cases} \tag{2.1}$$

and equations (1.10) are reduced to the form

$$\begin{cases} \dfrac{\partial^2 \overline{f}}{\partial t^2} = -2x_1 \dfrac{\partial^2 \overline{f}}{\partial t \partial x_2} - x_1^2 \dfrac{\partial^2 \overline{f}}{\partial x_2^2} + a^2 \dfrac{\partial^2 \overline{f}}{\partial x_1^2} + a \dfrac{\partial \overline{f}}{\partial x_2} + \lambda^2 \overline{f}, \\ \dfrac{\partial^2 \overline{b}}{\partial t^2} = -2x_1 \dfrac{\partial^2 \overline{b}}{\partial t \partial x_2} - x_1^2 \dfrac{\partial^2 \overline{b}}{\partial x_2^2} + a^2 \dfrac{\partial^2 \overline{b}}{\partial x_1^2} - a \dfrac{\partial \overline{b}}{\partial x_2} + \lambda^2 \overline{b}. \end{cases} \tag{2.2}$$

By combining equations (2.2) it is easy to obtain (1.8) once again provided that functions f and b are inserted.

In our view the most interesting result concerning equations (2.2) is given in the next theorem.

Theorem 2.1. *The function*

$$\overline{f}(x_1, x_2, t) = q\Big(x_2 - \frac{1}{2}\, x_1 t + \frac{a^2t^2 - x_1^2}{4a}\,, x_2 - \frac{1}{2}\, x_1 t - \frac{a^2t^2 - x_1^2}{4a}\Big) \tag{2.3}$$

is a solution of the first equation of (2.2), *provided that* $q = q(u, w)$ *is a solution of*

$$(w - u)\, \frac{\partial^2 q}{\partial u \partial w} = \frac{\partial q}{\partial w} + \frac{\lambda^2}{2a}\, q. \tag{2.4}$$

Analogously,

$$\overline{b}(x_1, x_2, t) = g\Big(x_2 - \frac{1}{2}\, x_1 t + \frac{a^2t^2 - x_1^2}{4a}\,, x_2 - \frac{1}{2}\, x_1 t - \frac{a^2t^2 - x_1^2}{4a}\Big) \tag{2.5}$$

is a solution of the second equation of (2.2) *provided that* g *is a solution of*

$$(w - u)\, \frac{\partial^2 g}{\partial u \partial w} = -\frac{\partial g}{\partial w} + \frac{\lambda^2}{2a}\, g. \tag{2.6}$$

Proof. Since only simple calculations are involved, we omit the details. □

Remark 2.1. It is interesting that equations (2.4) and (2.6) are reduced by the transformation $z = \sqrt{w - u}$ to the Bessel equation

$$\frac{\partial^2 q}{\partial z^2} + \frac{1}{z}\, \frac{\partial q}{\partial z} + \frac{2\lambda^2}{a}\, q = 0. \tag{2.7}$$

This result is due to the fact that $q = q(s)$ is a function depending only on x_1 through z. This is related to the well-known fact (see [1]) that the marginals

$$\int f(x_1, x_2, t)\, dx_2 \quad \text{and} \quad \int b(x_1, x_2, t)\, dx_2$$

are expressed in terms of Bessel functions of order zero with imaginary arguments and depending on $z = \sqrt{a^2t^2 - x_1^2}$.

To increase our insight into the vector process $(X_1(t), X_2(t),\ t > 0)$ we first note that possible values of X_1 at time t are within $[-at, at]$ and possible values of X_2 are located in the interval $[-\frac{1}{2}at^2, \frac{1}{2}at^2]$. However, not all couples of the set

$$R = \Big\{x_1, x_2 \ : \ -at \le x_1 \le at, \ \ -\frac{1}{2}\, at^2 \le x_2 \le \frac{1}{2}\, at^2\Big\}$$

can be occupied. For example, it is impossible for the process X_1 to take values close to $-at$ and for X_2 to occupy positions near $\frac{1}{2}at^2$ (the interpretation of X_1 as velocity and X_2 as the current position of a moving particle can help here).

We now present the following result.

Theorem 2.2. *At time t the support of $(X_1(t), X_2(t))$ is the set*

$$S = \Big\{x_1, x_2 \ : \ -at \le x_1 \le at, \ \frac{1}{2}\, x_1 t - \frac{a^2t^2 - x_1^2}{4a} \le x_2 \le \le \frac{1}{2}\, x_1 t + \frac{a^2t^2 - x_1^2}{4a}\Big\}, \tag{2.8}$$

which can be rewritten as

$$S = \Big\{x_1, x_2 \ : \ -\frac{1}{2}at^2 \le x_2 \le \frac{1}{2}at^2, \ at - \sqrt{2a^2t^2 - 4ax_2} \le x_1 \le \le -at + \sqrt{2a^2t^2 + 4ax_2}\Big\}. \tag{2.9}$$

Proof. Assume that at time t, $X_1(t) = x_1$. If

$$T_+ = \int_0^t I_{\{X_0(s)>0\}}\, ds = \text{meas}\{s < t : X_0(s) > 0\},$$
$$T_- = \int_0^t I_{\{X_0(s)<0\}}\, ds = \text{meas}\{s < t : X_0(s) < 0\},$$

it is clear that $a(T_+ - T_-) = x_1$. In that case the farthest position to the right of the origin is reached when the entire rightward motion occurs initially, during time T_+.

The final position is

$$\max X_2 = \frac{1}{2}\, aT_+^2 + aT_+T_- - \frac{1}{2}\, aT_-^2.$$

Since $T_+ + T_- = t$, we obtain

$$\max X_2 = \frac{1}{2}\, x_1 t + \frac{a^2t^2 - x_1^2}{4a}.$$

Conversely, min X_2 is reached when the leftward motion is performed initially.

In writing down (2.9), the sign must be chosen in such a way that for $x_2 = \pm\frac{1}{2}at^2$ we should have $x_1 = \pm at$. $\square$

All the information which can be read from the form of S coincides with the intuition. In particular, when $x_1 = \pm at$ we have $x_2 = \frac{1}{2}at^2$ and the closer x_1 is to the origin, the bigger the interval of possible values of X_2 becomes.

We note that, if $X_1(t) = x_1$, at time t the random variable $X_2(t)$ can take any value from the interval $[\frac{1}{2}x_1 t - \frac{a^2t^2-x_1^2}{4a}, \frac{1}{2}x_1 t + \frac{a^2t^2-x_1^2}{4a}]$.

To realize this we present

Theorem 2.3. *If $N(t) = m$, $X_0(0) = a$, T_i is the random time at which the i-th event of the driving Poisson process ($i \le m$) occurs, then*

$$\begin{cases} X_1(t) = 2a \sum_{i=1}^{m} (-1)^{i-1} T_i + a(-1)^m t, \\ X_2(t) = a \sum_{i=1}^{m} (-1)^i T_i^2 + 2at \sum_{i=1}^{m} (-1)^{i-1} T_i + (-1)^m \frac{1}{2}\, at^2. \end{cases} \tag{2.10}$$

Proof. At the instant the m-th Poisson event takes place, the processes $(X_1(t), X_2(t),\, t \ge 0)$ take values $X_1(T_m)$ and $X_2(T_m)$ and thus, after some substitutions and simplifications, at $t < T_m$ we have (by induction)

$$\begin{aligned} X_2(t) &= X_2(T_m) + X_1(T_m)(t - T_m) + \frac{1}{2}\, a(-1)^m (t - T_m)^2 = \\ &= a \sum_{i=1}^{m} (-1)^i T_i^2 + 2at \sum_{i=1}^{m} (-1)^{i-1} T_i + (-1)^m \frac{1}{2}\, at^2. \qquad \square \end{aligned}$$

Corollary 2.1. *It $N(t) = m$, $X_1(t) = x_1$, $X_0(0) = a$, possible positions which can be occupied at time t are given by*

$$\begin{aligned} X_2(t) = a \sum_{i=1}^{m-1} (-1)^i T_i^2 + \frac{(-1)^m}{4a} \Big(x_1 - a(-1)^m t + \\ + 2a \sum_{i=1}^{m-1} (-1)^{i-1} T_i \Big)^2 + t x_1 - \frac{1}{2}\, (-1)^m a t^2. \end{aligned} \tag{2.11}$$

Proof. From the first equation of (2.10) we get

$$T_m = \frac{x_1 - a(-1)^m t - 2a \sum_{i=1}^{m-1} (-1)^{i-1} T_i}{2a(-1)^m}$$

and rewriting the second equation as,

$$X_2(t) = a \sum_{i=1}^{m-1} (-1)^i T_i^2 + a(-1)^m T_m^2 + t\big(x_1 - a(-1)^m t\big) + \frac{1}{2}\, (-1)^m a t^2,$$

after a substitution the desired result emerges. $\square$

Remark 2.2. If $N(t)=m$, $X_0(0)=a$, and the random times $T_1,\dots,T_{m-1}$ are fixed, possible couples (x_1,x_2) of velocities and positions form a parabola.

Remark 2.3. If $X_2(t)=x_2$, $B_i=\frac{1}{2}x_1t-\frac{a^2t^2-x_1^2}{4a}$, $B_s=\frac{1}{2}x_1t+\frac{a^2t^2-x_1^2}{4a}$, then from (2.11) we derive the special cases.

$$\begin{aligned} m=2 \quad & x_2=T_1(at-x_1)+B_i,\\ m=3 \quad & x_2=B_s+(T_1-T_2)(x_1+at-2aT_1),\\ m=4 \quad & x_2=B_i+(T_2-T_1)\big(2aT_2-(at-x_1)\big)+\\ & \qquad +T_3\big(at-x_1+2a(T_1-T_2)\big). \end{aligned} \tag{2.12}$$

These formulas permit us to show that, if $N(t)\geq 2$, at time t for fixed values of $X_1(t)$, possible positions (namely the values of $X_2(t)$) cover the whole interval $[B_i,B_s]$.

It must be observed that various curves formed by the couples (x_1,x_2) must be analysed taking into account the constraints concerning times T_i.

For example, if $m=2$, we have $0\leq T_1\leq\frac{at+x_1}{2a}$. Thus if $T_1=0$, then $x_2=B_i$ and the couples (x_1,x_2) form one of the parabolas bounding the set S.

Analogously, if $T_1=\frac{at+x_1}{2a}$, then $x_2=B_s$ and the other curve bounding S is obtained.

If $T_1=\frac{at+x_1}{4a}$, we obtain the line of symmetry of S, whose equation is $x_2=\frac{1}{2}x_1t$.

Remark 2.4. The direct calculation of

$$\Pr\big\{X_2(t)\in dx_2\,\big|\,X_0(0)=a,\ X_1(t)=x_1,\ N(t)=m\big\} \tag{2.13}$$

presents considerable difficulties even when m is small.

We have been able to derive these results when $m=2,3$. From the formulas obtained it follows that general expressions of the distribution densities of $(X_1(t),X_2(t),t\geq 0)$ are of form (2.3) and (2.5) and are solutions of equations (2.4) and (2.6).

To evaluate (2.13) it is necessary to know the conditional distribution:

$$\Pr\big\{T_1\in dt_1,\dots,T_{m-1}\in dt_{m-1}\big|N(t)=m,\ X_1(t)=x_1,\ X_0(0)=a\big\}. \tag{2.14}$$

In particular, we have obtained

Theorem 2.4.

$$\Pr\big\{T_1\in dt_1\,\big|\,X_1(t)=x_1,\ N(t)=2,\ X_0(0)=a\big\}=2a/(at+x_1) \quad \textit{when} \quad 0<t_1<(at+x_1)/2a, \tag{2.15}$$

$$\Pr\big\{T_1\in dt_1,\ T_2\in dt_2\,\big|\,X_1(t)=x_1,\ N(t)=3,\ X_0(0)=a\big\}=\frac{4a^2}{a^2t^2-x_1^2}$$

when $0 < t_1 < (at + x_1)/2a$ *and* $t_1 < t_2 < t_1 + (at - x_1)/2a$. (2.16)

Proof. When $m = 3$, from (2.12) we immediately have

$$\Pr\left\{N(t) = 3,\ X_1(t) \le x_1,\ X_0(0) = a\right\} =$$

$$= \frac{3!}{t^3}\left\{ \int_0^{(at+x_1)/2a} dt_1 \int_{t_1}^{t_1+(at-x_1)/2a} dt_2 \int_{t_2}^{t_2-t_1+(at+x_1)/2a} dt_3 + \right.$$

$$\left. + \int_0^{(at+x_1)/2a} dt_1 \int_{t_1+(at-x_1)/2a}^{t} dt_2 \int_{t_2}^{t} dt_3 \right\} = \frac{(at+x_1)^2(2at-x_1)}{(2a)^2 t^3 a}.$$

Furthermore, when $0 < t_1 < \frac{at+x_1}{2a}$, we have

$$\Pr\left\{T_1 \in dt_1,\ T_2 \in dt_2 \,\middle|\, X_1(t) = x_1,\ N(t) = 3,\ X_0(0) = a\right\} =$$

$$= \begin{cases} \dfrac{3!}{t^3}\left(\dfrac{at+x_1}{2a} - t_1\right) dt_1 dt_2 & \text{if } t_1 < t_2 < t_1 + \frac{at-x_1}{2a}, \\ \dfrac{3!}{t^3}(t - t_2)\, dt_1\, dt_2 & \text{if } t > t_2 > t_1 + \frac{at-x_1}{2a}. \end{cases}$$

distribution (2.16) follows from the above results. □

On the basis of all the previous results we can present the following explicit formulas.

Theorem 2.5.

$$\Pr\left\{X_2(t) \in dx_2 \,\middle|\, X_1(t) = x_1,\ N(t) = 2\right\} = \frac{dx_2}{B_s - B_i}, \quad B_i < x_2 < B_s, \quad (2.17)$$

$$\Pr\left\{X_2(t) \in dx_2 \,\middle|\, X_1(t) = x_1,\ N(t) = 3\right\} =$$

$$= -\frac{dx_2}{B_s - B_i} \log\left(1 - \frac{x_2 - B_i)}{B_s - B_i}\right), \quad B_i < x_2 < B_s. \quad (2.18)$$

Proof. From the first formula of (2.12) we readily have

$$\Pr\left\{X_2(t) < x_2 \,\middle|\, X_1(t) = x_1,\ N(t) = 2\right\} =$$

$$= \Pr\left\{T_1 < \frac{x_2 - B_i}{at - x_1} \,\middle|\, X_1(t) = x_1,\ N(t) = 2\right\} = \frac{2a}{at + x_1} \int_0^{\frac{x_2-B_i}{at-x_1}} dt_1,$$

where the last step is justified by (2.15).

The derivation of (2.18) requires some additional details.

Taking into account the second formula of (2.12), we have

$$\Pr\left\{X_2(t) < x_2 \,\middle|\, X_1(t) = x_1,\ N(t) = 3\right\} =$$

$$= \Pr\left\{T_2 < T_1 + \frac{B_s - x_2}{x_1 + at - 2T_2} \;\Big|\; X_1(t) = x_1,\ N(t) = 3\right\} =$$

$$= \iint\limits_{\substack{0<t_1<\frac{x_2-B_i}{at-x_1} \\ t_1+\frac{B_s-x_2}{x_1+at-2at_1}<t_2<t_1+\frac{at-x_1}{2a}}} \frac{4a^2}{a^2t^2 - x_1^2}\, dt_1\, dt_2 =$$

$$= \frac{x_2 - B_i}{B_s - B_i} + \frac{B_s - x_2}{B_s - B_i} \log\left(1 - \frac{x_2 - B_i}{B_s - B_i}\right). \qquad \square$$

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References

1. E. Orsingher, Probability law flow function, maximum distribution of wave-governed random motions and their connections with Kirchoff's laws. *Stochastic Process. Appl.* **34**(1990), 49–66.

2. M. O. Hongler and L. Streit, Generalized master equations and the telegrapher's equation. *Physica* **A 165**(1990), 196–206.

3. A. D. Kolesnik, One-dimensional models of random Markovian evolutions. (Russian) *Preprint,* 1990.

4. E. Orsingher and B. Bassan, On a $2n$-valued telegraph signal and the related integrated process. *Stochastics Stochastics Rep.* **38**(1992), 159–173.

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Author's address:
Dipartimento di Statistica
Probabilità e Statistiche Applicate
Università degli Studi di Roma "La Sapienza"
Piazzale A. Moro 5, 00185 Roma
Italy