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# ON SOME RINGS OF ARITHMETICAL FUNCTIONS

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ABSTRACT. In this paper we consider several constructions which from a given $B$-product $*_B$ lead to another one $\widetilde{*}_B$. We shall be interested in finding what algebraic properties of the ring $R_B = \langle C^{\mathbb{N}}, +, *_B\rangle$ are shared also by the ring $R_{\widetilde{B}} = \langle C^{\mathbb{N}}, +, *_B\rangle$. In particular, for some constructions the rings $R_B$ and $R_{\widetilde{B}}$ will be isomorphic and therefore have the same algebraic properties.

## § 1. Introduction

In [1] the author shows a new kind of convolution product called the $B$-product defined as follows. For every natural number $n$ let $B_n$ be the set of some pairs $(r, s)$ of divisors of $n$.

For arithmetical functions $f$ and $g$ we define their $B$-product $f*_B$ as

$$(f *_B g)(n) = \sum_{(r,s)\in B_n} f(r)g(s) \quad \text{for} \quad n = 1, 2, 3, \dots . \tag{1}$$

This $B$-product generalizes simultaneously the $A$ product of W. Narkiewicz [2] and the l.c.m. product and has a nonempty intersection with the $\psi$-product of D. H. Lehmer [3]. The $\tau$-product of H. Scheid [4] is also a particular case of the $B$-product.

In [5] the author consideres a special kind of the $B$-product called the "multiplicative $B$-product". A $B$-product is multiplicative iff the following condition holds:

For every pair $(m, n)$ of relatively prime natural numbers we have

$$(r, s) \in B_{mn} \quad \text{iff} \quad (r^{(m)}, s^{(m)}) \in B_m \quad \text{and} \quad (r^{(n)}, s^{(n)}) \in B_n, \tag{2}$$

where $k^{(n)}$ denotes the g.c.d. of $k$ and $n$.

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In this paper we consider several constructions which, given a $B$-product $*_B$, lead to another one $\widetilde{*}_B$. We shall be interested in finding what algebraic properties of the ring $R_B = \langle C^{\mathbb{N}}, +, *_B \rangle$ are shared by the ring $R_{\widetilde{B}} = \langle C^{\mathbb{N}}, +, *_B \rangle$, where $C^{\mathbb{N}}$ denotes the set of all arithmetical functions. In particular, for some constructions the rings $R_B$ and $R_{\widetilde{B}}$ will be isomorphic and therefore have the same algebraic properties.

## § 2. Twisted Products

Let $R_B$ be a commutative and associative ring with a unit $e$. For a fixed invertible element $h \in R_B$, let $f^{(h)} = f *_B h$, where $f \in R_B$.

Evidently, $f \mapsto f^{(h)}$ is the one-to-one mapping of $R_B$ onto itself preserving the set of invertible elements. It is also an isomorphism of the additive group of $R_B$.

We define the twisted product $*_B^h$ as follows:

$$f *_B^h g = f *_B g *_B h \quad \text{for} \quad f,\, g \in R_B.$$

In other words,

$$f^{(h)} *_B g^{(h)} = (f *_B^h g)(h).$$

This means that the rings $R_B^h = \langle C^{\mathbb{N}}, +, *_B \rangle$ are isomorphic. The isomorphism $R_B \to R_B^h$ is given by the twisting $f \mapsto f^{(h^{-1})}$, where $h^{-1}$ is the inverse of $h$ in $R_B$.

Therefore the ring $R_B^h$ is also commutative and associative and $e^{(h^{-1})} = e *_B h^{-1} = h^{-1}$ is its unit element.

Let us remark that if the product $*_B$ is multiplicative and if the function $h$ is multiplicative, then the product $*_B^h$ is multiplicative. In fact, if functions $f$ and $g$ are multiplicative, then the function $f *_B^h g = f *_B g *_B h$ is multiplicative, since $*_B$ preserves the multiplicativity.

In general, the twisted multiplication $*_B^h$ is not a $B$-product. We shall give below some conditions on $*_B$ and on $h$ for $*_B^h$ to be a $B$-product.

**Theorem 2.1.** *The twisted product $*_B^h$ is a $B$-product iff for every $r$, $s$, and $n$*

$$\sum_{\substack{d_1,\, d_2 \\ (r,s)\in B_{d_1} \\ (d_1,d_2)\in B_n}} h(d_2) = 0 \ \text{ or } \ 1. \tag{3}$$

*Moreover if we denote by $B_n^h$ the set corresponding to the $B$-product $*_B^h$, then $(r,s) \in B_n^h$ iff sum (3) is equal to 1.*

*Proof.* $\Longrightarrow$ We have

$$(e_r *_{B} e_s)(n) = \begin{cases} 1 & \text{if} \quad (r,s) \in B_n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since by the assumption $*_{B}^{h}$ is a $B$-product, we have

$$(e_r *_{B}^{h} e_s)(n) = \begin{cases} 1 & \text{if} \quad (r,s) \in B_n^n, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$(e_r *_{B}^{h} e_s)(n) = (e_r *_{B} e_s *_{B} h)(n) = \sum_{\substack{d_1,\, d_2 \\ (d_1,d_2)\in B_n}} (e_r *_{B} e_S)(d_1)h(d_2) =$$

$$= \sum_{\substack{d_1,\, d_2 \\ (d_1,d_2)\in B_n \\ (r,s)\in B_{d_1}}} h(d_2). \tag{4}$$

Hence the result follows.

$\Longleftarrow$ Define $B_n^h$ as the set of pairs $(r,s)$ such that sum (3) is equal to 1. In view of (4), for any functions $f$ and $g$, we have

$$(f *_{B}^{h} g)(n) = \Big[\Big(\sum_{r=1}^{n} f(r)e^r\Big) *_{B}^{h} *_{B}^{h} \Big(\sum_{s=1}^{n} g(s)e_s\Big)\Big](n) =$$

$$= \sum_{r,s=1}^{n} f(r)g(s)(e_r *_{B}^{h} e_s)(n) = \sum_{(r,s)\in B_n^h} f(r)g(s) = (f *_{B^h} g)(n).$$

If we start from the Dirichlet convolution $*$, then condition (3) of Theorem 2.1 takes the form

$$\sum_{\substack{d_1,\, d_2 \\ r_s=d_1 \\ d_1 d_2=n}} h(d_2) = h\Big(\frac{n}{r_s}\Big) = 0 \ \text{ or } \ 1$$

for every $r$, $s$, and $n$.

Thus $h(m) = 0$ or 1 for every $m$. Moreover, since $h$ is invertible, we conclude that $h(1) \neq 0$, i.e., $h(1) = 1$. $\square$

**Corollary 2.2.** *Let $h(1) = 1$ and $h(n) = 0$ or 1 for every $n$. Then the multiplication $*^h$ defined by*

$$f *^h g = f * g * h,$$

*where $*$ is the Dirichlet convolution, is a $B$-product and*

$$B_n^h = \Big\{(r,s): \ \ rs|n \ \ \text{and} \ \ h\Big(\frac{n}{rs}\Big) = 1\Big\}.$$

*If, moreover, the function $h$ is multiplicative, then the twisted product $*^h$ preserves the multiplicativity.*

Rings $R = \langle C^{\mathbb{N}}, +, *\rangle$ and $R^h = \langle C^{\mathbb{N}}, +, *^h\rangle$ are isomorphic. The isomorphism is given by $f \mapsto f * h^{-1}$, where $h^{-1}$ is the Dirichlet inverse of $h$. The function $h^{-1}$ is the unit of the ring $R^h$.

Consequently $R^h$ is a local ring without zero divisors.

A function $f$ is invertible in $R$ iff it is invertible in $R^h$ iff $f(1) \neq 0$, and the inverse $f^{(-1)}$ of $f$ in $R^h$ is given by

$$f^{(-1)} = f^{-1} * h^{-2}.$$

## § 3. Strong Associativity

We introduce the important notion of a strong associativity. We say that a $B$-product is strongly associative iff for fixed $d_1$, $d_2$, $d_3$, $n$, the fulfilment of

$$(r, d_1) \in B_n \quad \text{and} \quad (d_2, d_3) \in B_r \tag{5}$$

for some $r$ implies that $w = \frac{d_1 r}{d_2}$ (which is evidently a natural number) satisfies the condition

$$(d_2, w) \in B_n \quad \text{and} \quad (d_3, d_1) \in B_w \tag{6}$$

and, conversely, the fulfilment of (6) for $w$ implies that $r = \frac{d_2 w}{d_1}$ (which is a natural number) satisfies (5).

From this definition and Theorem 2.1 of [1] it follows that every strongly associative $B$-product is associative. The converse does not hold in generaL. Nevertheless the following theorem is true.

**Theorem 3.1.** *An associative $\tau$-product is strongly associative iff*

$$\begin{gathered} d_2\tau(d_3, d_1) = \tau(d_2, d_3)d_1 \quad \text{for all} \quad d_1,\ d_2,\ d_3 \quad \text{satisfying} \\ \tau\big(\tau(d_2, d_3), d_1\big) \neq 0. \end{gathered} \tag{7}$$

*Proof.* $\Longleftarrow$ Let $r = \tau(d_2, d_3)$ and $n = \tau(r, d_1) = \tau(\tau(d_2, d_3), d_1) \neq 0$. Then (5) holds and by the strong associativity we have (6), i.e., $w = \tau(d_3, d_1)$, where $w = \frac{d_1 r}{d_2}$. Therefore we get (7).

$\Longrightarrow$ Suppose that (5) holds for some $d_1$, $d_2$, $d_3$, $n$. Then $r = \tau(d_2, d_3)$ and $n = \tau(r, d_1) = \tau(\tau(d_2, d_3), d_1)$. Since $n \neq 0$, we have (7) by the assumption. Hence

$$w = \frac{d_1 r}{d_2} = \tau(d_3, d_1), \quad \text{i.e.,} \quad (d_3, d_1) \in B_w.$$

Therefore

$$\tau(d_2, w) = \tau(d_2, \tau(d_3, d_1)) = \tau(\tau(d_2, d_3), d_1) = n,$$

i.e., $(d_2, w) \in B_n$. Thus (6) holds.

Similarly, we can prove that (6) implies (5). $\square$

## § 4. Unitary $B$-products

For a given $B$-product $*_B$ we define the corresponding unitary $B$-product denoted by $\circ_B$ as follows. Let

$$B_n^0 = \Big\{(r,s): \ \ (r,s) \in B_n, \ \ (r,s) = 1, \ \ \Big(rs, \frac{n}{rs}\Big) = 1\Big\}.$$

Then $(f \circ_B g)(n) = \sum\limits_{(r,s)\in B_n^0} f(r)g(s)$.

We shall investigate relations between the corresponding properties of $B$-products $*_B$ and $\circ_B$.

**Theorem 4.1.**
(i) *If $*_B$ is commutative, then $\circ_B$ is commutative.*
(ii) *If $*_B$ is strongly associative, then $\circ_B$ is associative.*

*Proof.* (i) is clear. To prove (ii) we have to show that for every $d_1$, $d_2$, $d_3$, and $n$

$$\sum_{\substack{r \\ (r,d_1)\in B_n^0 \\ (d_2,d_3)\in B_r^0}}{}' = \sum_{\substack{w \\ (d_2,w)\in B_n^0 \\ (d_3,d_1)\in B_w^0}}{}' \ . \tag{8}$$

Suppose that $r$ satisfies $(r, d_1) \in B_n^0$, $(d_1, d_2) \in B_r^0$, i.e., $(r, d_1) \in B_n$, $(d_2, d_3) \in B_r$ and, moreover,

$$(r,d_1) = 1, \quad \Big(rd_1, \frac{n}{rd_1}\Big) = 1, \quad (d_2,d_3) = 1, \quad \Big(d_2d_3, \frac{r}{d_2d_3}\Big) = 1. \tag{$*$}$$

By the strong associativity of $*_B$ we find for $w = \frac{rd_1}{d_2}$ that $(d_2, w) \in B_n$, $(d_3, d_1) \in B_w$.

To prove that $w$ satisfies

$$(d_2, w) \in B_n^0, \quad (d_3, d_1) \in B_w^0,$$

it is sufficient to show that

$$\begin{aligned} &(d_2,w) = 1, \quad \Big(d_2w, \frac{n}{d_2w}\Big) = 1, \\ &(d_3,d_1) = 1, \quad \Big(d_3d_1, \frac{w}{d_3d_1}\Big) = 1 \end{aligned} \tag{$**$}$$

in view of (6). The formulas $(**)$ follow from $(*)$. We have proved that to every summand of $L$, $H$, $S$ of (8) there corresponds a summand in $R$, $H$, $S$. Similarly, one can give the inverse correspondence. □

**Theorem 4.2.** *If $e_1$ is the unit in the ring $R_B$, then $e_1$ is also the unit for the corresponding unitary product $\circ_B$.*

*Proof.* In view of Corollary 2.6 of [1] $e_1$ is the unit in the ring $R_B$ iff $(1,n)$, $(n,1) \in B_n$ for $n \geq 1$ and $(k,1) \notin B_n$, $(1,k) \notin B_n$ for $k \neq n$, $n > 1$.

These conditions clearly imply that $(1,n)$, $(n,1) \in B_n^0$ and $(k,1) \notin B_n^0$ and $(1,k) \notin B_n^0$ for $k \neq n$, $n > 1$.

Therefore, using once more Corollary 2.6 of [1], we deduce that $e_1$ is the unit with respect to $\circ_B$. □

**Theorem 4.3.** *If the $B$-product $*_B$ is multiplicative, then the corresponding unitary product $*_B$ is multiplicative.*

*Proof.* Let $m$ and $n$ be coprime natural numbers and suppose that

$$(r,s) \in B_{mn}^0. \tag{A}$$

Then $(r,s) \in B_{mn}^0$ and hence by the multiplicativity of $*_B$ we get $(r^{(m)}, s^{(m)}) \in B_m$ and, similarly, $(r^{(n)}, s^{(n)}) \in B_n$. Moreover, from $(r,s)=1$ and $(rs, \frac{mn}{rs})=1$ we conclude that

$$(r^{(m)}, s^{(m)}) = 1, \quad (r^{(n)}, s^{(n)}) = 1 \quad \text{and futher}$$
$$\left(r^{(m)}s^{(m)}, \frac{m}{r^{(m)}s^{(m)}}\right) = 1, \quad \left(r^{(n)}s^{(n)}, \frac{n}{r^{(n)}s^{(n)}}\right) = 1.$$

Thus

$$(r^{(m)}, s^{(m)}) \in B_m^0 \quad \text{and} \quad (r^{(n)}, s^{(n)}) \in B_n^0. \tag{B}$$

Similarly, one can prove that (B) implies (A). □

## § 5. The Narkiewicz Product

For a given $B$-product $*_B$ we define a new $B$-product $\Delta_B$ as follows: Let

$$B_n^{\Delta} = \{(r,s) : \ (r,s) \in B_n, \ rs = n\}.$$

Then

$$(f \Delta_B g)(n) = \sum_{(r,s) \in B_n^{\Delta}} f(r)f(s) = \sum_{(r, \frac{n}{r}) \in B_n} f(r) g\Big(\frac{n}{r}\Big).$$

Evidently, $\Delta_B$ is the Narkiewicz product. It will be called the Narkiewicz product corresponding to the $B$-product $*_B$. We shall investigate some important properties of the product $\Delta_B$.

**Theorem 5.1.** *If the product $*_B$ is commutative, then $\Delta_B$ is also commutative.*

*Proof* is clear. □

**Theorem 5.2.** *If the B-product $*_B$ is strongly associative, then the corresponding Narkiewicz product $\Delta_B$ is associative.*

*Proof.* For fixed $d_1$, $d_2$, $d_3$, $n$ we have

$$\sum_{\substack{r \\ (r,d_1)\in B_n^{\Delta} \\ (d_2,d_3)\in B_r^{\Delta}}}' 1 = \sum_{\substack{r \\ (r,d_1)\in B_n,\ rd_1=n \\ (d_2,d_3)\in B_r,\ d_2d_3=r}}' 1.$$

If $n \neq d_1d_2d_3$, then this sum is equal to 0. We assume that $n = d_1d_2d_3$. Then the sum is equal to

$$\sum_{\substack{(\frac{n}{d_1},d_1)\in B_n \\ (d_2,d_3)\in B_{\frac{n}{d_1}}}} 1$$

for $r = \frac{n}{d_1}$.

By the strong associativity of $*_B$ for $w = \frac{rd_1}{D_2}$ the above sum is equal to

$$\sum_{\substack{w \\ (d_2,w)\in B_n \\ (d_3,d_1)\in B_w}} 1 = \sum_{\substack{(d_2,\frac{n}{d_2})\in B_n \\ (d_3,d_1)\in B_{\frac{n}{d_2}}}} 1 = \sum_{\substack{(d_2\overline{w})\in B_n^{\Delta} \\ (d_3,d_1)\in B_{\overline{w}}^{\Delta}}} 1.$$

Therefore the product $\Delta_B$ is associative. □

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