GEORGIAN MATHEMATICAL JOURNAL: Vol. 6, No. 6, 1999, 567-590

MULTIPLE SOLUTIONS OF GENERALIZED MULTIPOINT CONJUGATE BOUNDARY VALUE PROBLEMS

PATRICIA J. Y. WONG AND RAVI P. AGARWAL

Abstract. We consider the boundary value problem

$$y^{(n)}(t) = P(t, y), \quad t \in (0, 1)$$
$$y^{(j)}(t_i) = 0, \quad j = 0, \ldots, n_i - 1, \ \ i = 1, \ldots, r,$$

where $r \geq 2$, $n_i \geq 1$ for $i = 1, \ldots, r$, $\sum_{i=1}^{r} n_i = n$ and $0 = t_1 < t_2 < \cdots < t_r = 1$. Criteria are offered for the existence of double and triple 'positive' (in some sense) solutions of the boundary value problem. Further investigation on the upper and lower bounds for the norms of these solutions is carried out for special cases. We also include several examples to illustrate the importance of the results obtained.

1. Introduction

Let $0 = t_1 < t_2 < \cdots < t_r = 1$ be r (≥ 2) fixed points and let n_i (≥ 1), $i = 1, \ldots, r$, be integers with $\sum_{i=1}^{r} n_i = n$. In this paper we shall consider the multipoint conjugate boundary value problem

$$\begin{aligned} y^{(n)}(t) &= P(t, y), \quad t \in (0, 1), \\ y^{(j)}(t_i) &= 0, \quad j = 0, \ldots, n_i - 1, \ \ i = 1, \ldots, r, \end{aligned} \tag{M}$$

where P is continuous at least in the interior of the domain of definition.

We shall define a *positive solution* y of (M) as follows: $y \in C^{(n)}(0, 1)$ is a nontrivial function that fulfills (M) and for each $1 \leq i \leq r - 1$, $(-1)^{n_{i+1}+\cdots+n_r}\, y$ is nonnegative on $[t_i, t_{i+1}]$. Our first task is to develop criteria for the existence of double positive solutions of (M). Next, we shall consider two special cases of (M), namely,

$$y^{(3)}(t) = q(t)\left[|y(t)|^{\alpha} + |y(t)|^{\beta}\right], \ \ t \in (0, 1), \ \ y(0) = y(c) = y(1) = 0 \tag{$M1$}$$

1991 *Mathematics Subject Classification.* 34B15.

Key words and phrases. Generalised multipoint conjugate boundary value problem, multiple solution.

1072-947X/99/1100-0567$12.50/0

and

$$y^{(3)}(t) = q(t)e^{\gamma|y(t)|}, \ \ t \in (0,1), \ \ y(0) = y(c) = y(1) = 0, \tag{$M2$}$$

where $0 < c < 1$, $0 \leq \alpha < 1 < \beta$, $\gamma > 0$ and q is continuous at least in the interior of the domain of definition. It is noted that the importance of $(M1)$ is well illustrated in [1, 2] and that of $(M2)$ is related to the analysis of diffusion of heat generated by positive temperature-dependent sources [3]. In addition to providing conditions for the existence of double positive solutions of $(M1)$ and $(M2)$, we also derive upper and lower bounds for the norms of these solutions. Finally, we shall examine the existence of triple positive solutions of a 'separable' case of (M), namely,

$$\begin{gathered} y^{(n)}(t) = b(t)f(y), \quad t \in (0,1), \\ y^{(j)}(t_i) = 0, \quad j = 0, \ldots, n_i - 1, \ \ i = 1, \ldots, r, \end{gathered} \tag{$M3$}$$

where b and f are continuous at least in the interior of the domain of definition. The criterion established will also provide estimates on the norms of these positive solutions.

The present work is motivated by the fact that a multipoint boundary value problem of the type (M) models various dynamical systems with n degrees of freedom in which n states are observed at n times, see Meyer [4]. In particular, when $n = r = 2$ the boundary value problem (M) describes a vast spectrum of nonlinear phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, e.g., see [5–11]. It is important to note that in most of these models, only positive solutions are meaningful. Recently, special cases of (M), namely, second order problem ($n = r = 2$) and two-point problem ($r = 2$), have been tackled by several authors [12–15]. Further, related investigations on other boundary value problems such as Sturm–Liouville type, focal type, Lidstone type as well as (n, p) type are documented in the monographs [3, 16–18, 1]. Our results, besides complement and extend the literature to multipoint problem, also improve the work of Eloe and Henderson [19].

The paper is organized as follows. In Section 2, we shall state the fixed point theorems required, and provide some properties of certain Green's functions which are needed later. By defining an appropriate Banach space and cone, in Section 3 we shall establish existence criteria for double positive solutions of (M). The cases $(M1)$ and $(M2)$ are respectively treated in Sections 4 and 5. Finally, in Section 6 we shall discuss the existence of triple positive solutions of $(M3)$.

2. Preliminaries

Let B be a Banach space equipped with the norm $\|\cdot\|$.

Definition 2.1. Let $C(\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:
(a) If $y \in C$ and $\ell \geq 0$, then $\ell y \in C$;
(b) If $y \in C$ and $-y \in C$, then $y = 0$.

Definition 2.2. Let $C(\subset B)$ be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:
(a) $\psi : C \to [0, \infty)$ is continuous;
(b) $\psi(\ell x + (1-\ell)y) \geq \ell\psi(x) + (1-\ell)\psi(y)$ for all $x, y \in C$ and $0 \leq \ell \leq 1$.

With C and ψ defined as above, we shall introduce the following notations. For $k, \ell, m > 0$, we shall denote

$$C(k) = \{y \in C \mid \|y\| < k\} \text{ and } C(\psi, \ell, m) = \{y \in C \mid \psi(y) \geq \ell,\ \|y\| \leq m\}.$$

The following fixed point theorems are needed later.

Theorem 2.1 ([20]). *Let $C(\subset B)$ be a cone. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\bar{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that either
(a) *$\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_2$, or*
(b) *$\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, and $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$.*
Then S has a fixed point in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Theorem 2.2 ([2, 21]). *Let $C(\subset B)$ be a cone and let $\nu > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(y) \leq \|y\|$ for all $y \in \bar{C}(\nu)$, and let $S : \bar{C}(\nu) \to \bar{C}(\nu)$ be a completely continuous operator. Suppose that there exist numbers k, ℓ, m, where $0 < k < \ell < m \leq \nu$, such that*
(a) *$\{y \in C(\psi, \ell, m) \mid \psi(y) > \ell\} \neq \varnothing$, and $\psi(Sy) > \ell$ for all $y \in C(\psi, \ell, m)$;*
(b) *$\|Sy\| < k$ for all $y \in \bar{C}(k)$;*
(c) *$\psi(Sy) > \ell$ for all $y \in C(\psi, \ell, \nu)$ with $\|Sy\| > m$.*
Then S has (at least) three fixed points y_1, y_2 and y_3 in $\bar{C}(\nu)$. Further we have

$$\begin{aligned} &y_1 \in C(k), \quad y_2 \in \big\{y \in C(\psi, \ell, \nu) | \psi(y) > \ell\big\}, \\ &y_3 \in \bar{C}(\nu) \backslash \big(C(\psi, \ell, \nu) \cup \bar{C}(k)\big). \end{aligned} \tag{2.1}$$

To obtain a solution for (M), we require a mapping whose kernel $G(t, s)$ is the Green's function of the boundary value problem

$$y^{(n)}(t) = 0, \ \ t \in [0, 1] \ \ y^{(j)}(t_i) = 0, \ \ j = 0, \ldots, n_i - 1, \ \ i = 1, \ldots. \tag{2.2}$$

We shall use the following notation. For each $1 \le i \le r-1$, we shall denote $\alpha_i = \sum_{j=i+1}^{r} n_j$ and $I_i = \big[(3t_i + t_{i+1})/4, (t_i + 3t_{i+1})/4\big]$. Further, for each $s \in (0,1)$, we define

$$\|G(\cdot,s)\| = \sup_{t\in[0,1]} |G(t,s)|. \tag{2.3}$$

It is well known [3, 22–24] that

$$(-1)^{\alpha_i} G(t,s) > 0,\ (t,s) \in (t_i, t_{i+1}) \times (0,1),\ i = 1,\dots,r-1. \tag{2.4}$$

In view of (2.3) we readily obtain

$$(-1)^{\alpha_i} G(t,s) \le \|G(\cdot,s)\|,\ \ (t,s) \in [t_i, t_{i+1}] \times [0,1],\ i=1,\dots,r-1. \tag{2.5}$$

Moreover, we have the following lemma which improves the result of Eloe and Henderson [23].

Lemma 2.1 ([24]). *For $(t,s) \in I_i \times (0,1)$, $i = 1,\dots,r-1$, we have*

$$(-1)^{\alpha_i} G(t,s) \ge L_i \|G(\cdot,s)\|,$$

where $0 < L_i \le 1$ is a constant defined by

$$\begin{aligned} L_i = \min\bigg\{ &\min\Big\{ g\Big(\frac{3t_i + t_{i+1}}{4}\Big), g\Big(\frac{t_i + 3t_{i+1}}{4}\Big)\Big\} \Big/ \max_{t\in[0,1]} g(t),\\ &\min\Big\{ h\Big(\frac{3t_i + t_{i+1}}{4}\Big), h\Big(\frac{t_i + 3t_{i+1}}{4}\Big)\Big\} \Big/ \max_{t\in[0,1]} h(t) \bigg\}, \end{aligned} \tag{2.6}$$

where $g(t) = \prod_{j=1}^{r-1} |t-t_j|^{n_j} (1-t)^{n_r - 1}$ and $h(t) = t^{n_1 - 1} \prod_{j=2}^{r} |t - t_j|^{n_j}$.

Lemma 2.2. *Let $0 < c < 1$ be fixed. Consider the following special case of* (2.2)*:*

$$cy^{(3)}(t) = 0,\ \ t \in [0,1],\quad y(0) = y(c) = y(1) = 0. \tag{2.7}$$

Let G_1 denote the Green's function of (2.7)*. We have*

$$\|G_1(\cdot,s)\| \le \phi(s) \equiv \begin{cases} \frac{1}{2(1-c)}(1-s)^2 \max\{c, 1-c\}, & c \le s \le 1\\ \frac{1}{2c} s^2 \max\{c, 1-c\}, & 0 \le s \le c. \end{cases} \tag{2.8}$$

Further, the constants L_1 and L_2 in (2.6) *are respectively given as*

$$L_1 = \min\bigg\{ \frac{3c^2}{16} \Big/ \max\Big\{ \frac{c^2}{4}, 1-c\Big\}, \frac{c(4-3c)}{16} \Big/ \max\Big\{ \frac{(1-c)^2}{4}, c\Big\}\bigg\}, \tag{2.9}$$

$$L_2 = \min\bigg\{ \frac{(3c+1)(1-c)}{16} \Big/ \max\Big\{ \frac{c^2}{4}, 1-c\Big\}, \frac{3(1-c)^2}{16} \Big/ \max\Big\{ \frac{(1-c)^2}{4}, c\Big\}\bigg\}. \tag{2.10}$$

Proof. Here $n = r = 3$, $n_1 = n_2 = n_3 = 1$, $t_1 = 0$, $t_2 = c$ and $t_3 = 1$. The explicit expression of the Green's function G_1 is given by [25]

$$-G_1(t,s) = \begin{cases} \frac{1}{2(1-c)}(1-s)^2 t(t-c), & c \le s \le 1,\ 0 \le t \le s \\ \frac{1}{2(1-c)}(1-s)^2 t(t-c) - \frac{1}{2}(t-s)^2, & c \le s \le 1,\ s \le t \le 1 \\ \frac{1}{2c} s^2 (t-c)(1-t) + \frac{1}{2}(s-t)^2, & 0 \le s \le c,\ 0 \le t \le s \\ \frac{1}{2c} s^2 (t-c)(1-t), & 0 \le s \le c,\ s \le t \le 1 \end{cases} \tag{2.11}$$

from which we find

$$\|G_1(\cdot,s)\| \le \begin{cases} \frac{1}{2(1-c)}(1-s)^2 s \max\{s-c, c\}, & c \le s \le 1,\ 0 \le t \le s \\ \frac{1}{2(1-c)}(1-s)^2 (1-c), & c \le s \le 1,\ s \le t \le 1 \\ \frac{1}{2c} s^2 c, & 0 \le s \le c,\ 0 \le t \le s \\ \frac{1}{2c}\ s^2 \max\{c-s, 1-c\}(1-s), & 0 \le s \le c,\ s \le t \le 1 \end{cases} \le$$

$$\le \begin{cases} \frac{1}{2(1-c)}(1-s)^2 \max\{c, 1-c\}, & c \le s \le 1,\ 0 \le t \le 1 \\ \frac{1}{2c} s^2 \max\{c, 1-c\}, & 0 \le s \le c,\ 0 \le t \le 1 \end{cases} = \phi(s).$$

The constants L_1 and L_2 are obtained by direct computation. $\square$

For clarity, we shall list the conditions that are needed later. In these conditions it is assumed that the functions $f : \mathbb{R} \to [0,\infty)$ and $a, b : (0,1) \to \mathbb{R}$ are continuous.

(A1) If $|u_1| \ge |u_2|$, then $f(u_1) \ge f(u_2)$.

(A2) For $(t,u) \in (0,1) \times \mathbb{R}$, $a(t) \le P(t,u)/f(u) \le b(t)$.

(A3) The function a is nonnegative and not identically zero on any nondegenerate subinterval of $(0,1)$; and there exists a number $0 < \rho \le 1$ such that $a(t) \ge \rho b(t)$ for $t \in (0,1)$.

(A4) $\displaystyle\int_0^1 \|G(\cdot,s)\| b(s)ds < \infty$.

(A5) Functions $a, b : [0,1] \to \mathbb{R}$ are continuous; and (A2) and (A3) hold with $(0,1)$ replaced by $[0,1]$.

Further, we introduce the notation $f_0 = \lim_{|u|\to 0^+} f(u)/|u|$ and $f_\infty = \lim_{|u|\to\infty} f(u)/|u|$.

3. Double Positive Solutions of (M)

For $y \in C[0,1]$ let

$$\theta = \int_0^1 \|G(\cdot,s)\| b(s) f(y(s)) ds \tag{3.1}$$

and

$$\Gamma = \int_0^1 \|G(\cdot,s)\| a(s) f(y(s)) ds. \tag{3.2}$$

If (A2) and (A3) hold, then it is clear that

$$\theta \geq \Gamma \geq \rho\theta \geq 0. \tag{3.3}$$

Further, we define the constants

$$\sigma_i = \rho L_i, \quad i = 1, \ldots, r-1, \tag{3.4}$$

where L_i is given in (2.6) and ρ appears in (A3). It is noted that $0 < \sigma_i \leq 1$ for each $1 \leq i \leq r-1$.

Let the Banach space $B = \{y \mid y \in C[0,1]\}$ be equipped with the norm $\|y\| = \sup_{t\in[0,1]} |y(t)|$. Define

$$C = \Big\{ y \in B \;\Big|\; \text{for each } 1 \leq i \leq r-1,\ (-1)^{\alpha_i} y(t) \geq 0 \text{ for } t \in [t_i, t_{i+1}] \\ \text{and } \min_{t\in I_i} (-1)^{\alpha_i} y(t) \geq \sigma_i \|y\| \Big\}.$$

Clearly, C is a cone in B.

We define the operator $S : C \to B$ by

$$Sy(t) = \int_0^1 G(t,s)P(s, y(s))ds,\ t \in [0,1]. \tag{3.5}$$

To obtain a positive solution of (M), we shall seek a fixed point of the operator S in the cone C.

If (A2) and (A3) hold, then in view of (2.4) it is clear that for $t \in [t_i, t_{i+1}],\ 1 \leq i \leq r-1$,

$$(-1)^{\alpha_i} Uy(t) \leq (-1)^{\alpha_i} Sy(t) \leq (-1)^{\alpha_i} Vy(t), \tag{3.6}$$

where

$$Uy(t) = \int_0^1 G(t,s)a(s)f(y(s))ds \text{ and } Vy(t) = \int_0^1 G(t,s)b(s)f(y(s))ds.$$

Now we shall state two lemmas whose proof is available in [26].

Lemma 3.1. *Let (A1)–(A4) hold. Then the operator S is compact on the cone C.*

Remark 3.1. From the proof of Lemma 3.1 we observe that the conclusion of Lemma 3.1 still holds if the conditions (A1)–(A4) are replaced by (A5).

Lemma 3.2. *Let (A2) and (A3) hold. Then the operator S maps C into itself.*

Theorem 3.1. *Let (A5) or (A1)–(A4) hold and let $w > 0$ be given. Suppose that f satisfies*

$$f(u) \leq w \left[\int_0^1 \|G(\cdot, s)\| b(s) ds \right]^{-1}, \quad |u| \leq w. \tag{3.7}$$

(a) *If* $f_0 = \infty$, *then* (M) *has a positive solution* y_1 *such that*

$$0 < \|y_1\| \le w. \tag{3.8}$$

(b) *If* $f_\infty = \infty$, *then* (M) *has a positive solution* y_2 *such that*

$$\|y_2\| \ge w. \tag{3.9}$$

(c) *If* $f_0 = f_\infty = \infty$, *then* (M) *has two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| \le w \le \|y_2\|. \tag{3.10}$$

Proof. (a) Let

$$A = \left[\sigma_1 \int_{I_1} (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) ds\right]^{-1}. \tag{3.11}$$

Since $f_0 = \infty$, there exists $0 < r < w$ such that

$$f(u) \ge A|u|, \quad 0 < |u| \le r. \tag{3.12}$$

First, let $y \in C$ be such that $\|y\| = r$. Using (3.6), (3.12) and (3.11) successively, we get

$$\begin{aligned}(-1)^{\alpha_1} Sy\Big(t_2/2\Big) &\ge \int_0^1 (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) f(y(s)) ds \ge \\ &\ge \int_0^1 (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) A|y(s)| ds \ge \\ &\ge \int_{I_1} (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) A\sigma_1 \|y\| ds = \|y\|.\end{aligned}$$

This immediately implies

$$\|Sy\| \ge \|y\|. \tag{3.13}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < r\}$, then (3.13) holds for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Then, employing (3.6), (2.5) and (3.7), we find for $t \in [t_i, t_{i+1}],\ 1 \le i \le r-1$,

$$(-1)^{\alpha_i} Sy(t) \le \int_0^1 \|G(\cdot, s)\| b(s) f(y(s)) ds \le w = \|y\|.$$

Consequently,

$$\|Sy\| \le \|y\|. \tag{3.14}$$

If we set $\Omega_2 = \{y \in B \mid \|y\| < w\}$, then (3.14) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (3.13) and (3.14), it follows from Theorem 2.1 that S has a fixed point $y_1 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $r \le \|y_1\| \le w$. Clearly, this y_1 is a positive solution of (M) fulfilling (3.8).

(b) As seen in the proof of Case (a), condition (3.7) leads to (3.14). Hence, if we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (3.14) holds for $y \in C \cap \partial\Omega_1$.

Next, noting that $f_\infty = \infty$, we may choose $T > w$ such that

$$f(u) \ge A|u|, \quad |u| \ge T, \tag{3.15}$$

where A is defined in (3.11). Let $T_1 = \max\left\{2w,\ T/\min_{1\le j\le r-1}\sigma_j\right\}$ and let $y \in C$ be such that $\|y\| = T_1$. Then we have for $s \in I_1$,

$$|y(s)| = (-1)^{\alpha_1} y(s) \ge \sigma_1 \|y\| \ge \sigma_1 \cdot \frac{T}{\min_{1\le j\le r-1}\sigma_j} \ge T.$$

So in view of (3.15) it follows that

$$f(y(s)) \ge A|y(s)|, \quad s \in I_1. \tag{3.16}$$

Applying (3.6), (3.16), and (3.11), we again find $(-1)^{\alpha_1} Sy\big(t_2/2\big) \ge \|y\|$. Therefore (3.13) holds. By setting $\Omega_2 = \{y \in B \mid \|y\| < T_1\}$ we have (3.13) for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (3.14) and (3.13), it follows from Theorem 2.1 that S has a fixed point $y_2 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \le \|y_2\| \le T_1$. It is clear that this y_2 is a positive solution of (M) satisfying (3.9).

(c) This follows from Cases (a) and (b). $\square$

Theorem 3.2. *Let (A5) or (A1)–(A4) hold and let $w > 0$ be given. Suppose that f satisfies*

$$f(u) \ge w\left[\sum_{i=1}^{r-1}\int_{I_i}(-1)^{\alpha_1} G\Big(t_2/2, s\Big)a(s)ds\right]^{-1}, \ \min_{1\le j\le r-1}\sigma_j w \le |u| \le w. \tag{3.17}$$

(a) *If $f_0 = 0$, then (M) has a positive solution y_1 such that (3.8) holds.*

(b) *If $f_\infty = 0$, then (M) has a positive solution y_2 such that (3.9) holds.*

(c) *If $f_0 = f_\infty = 0$, then (M) has two positive solutions y_1 and y_2 such that (3.10) holds.*

Proof. (a) Define

$$Q = \left[\int_0^1 \|G(\cdot, s)\| b(s)ds\right]^{-1}. \tag{3.18}$$

Since $f_0 = 0$, there exists $0 < r < w$ such that

$$f(u) \le Q|u|, \quad 0 < |u| \le r. \tag{3.19}$$

First, let $y \in C$ be such that $\|y\| = r$. Then the application of (3.6), (2.5), (3.19) and (3.18) yields for $t \in [t_i, t_{i+1}],\ 1 \le i \le r-1$,

$$(-1)^{\alpha_i} Sy(t) \le \int_0^1 \|G(\cdot,s)\| b(s) f(y(s)) ds \le \int_0^1 \|G(\cdot,s)\| b(s) Q\|y\| ds = \|y\|.$$

Hence (3.14) follows immediately. Set $\Omega_1 = \{y \in B \mid \|y\| < r\}$; then (3.14) holds for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Noting that for $s \in I_i$, $1 \le i \le r-1$,

$$\min_{1 \le j \le r-1} \sigma_j w \le \sigma_i \|y\| \le |y(s)| \le w,$$

it follows from (3.6) and (3.17) that

$$\begin{aligned}(-1)^{\alpha_1} Sy\Big(t_2/2\Big) &\ge \int_0^1 (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) f(y(s)) ds \ge \\ &\ge \sum_{i=1}^{r-1} \int_{I_i} (-1)^{\alpha_1} G\Big(t_2/2, s\Big) a(s) f(y(s)) ds \ge w = \|y\|.\end{aligned}$$

Thus we get (3.13). By setting $\Omega_2 = \{y \in B \mid \|y\| < w\}$ we see that (3.13) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (3.14) and (3.13), it follows from Theorem 2.1 that S has a fixed point $y_1 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $r \le \|y_1\| \le w$. Clearly, this y_1 is a positive solution of (M) satisfying (3.8).

(b) It is seen in the proof of Case (a) that condition (3.17) gives rise to (3.13). So if we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (3.13) holds for $y \in C \cap \partial\Omega_1$.

Next, let Q be defined as in (3.18). Since $f_\infty = 0$, we may choose $T > w$ such that

$$f(u) \le Q|u|, \quad |u| \ge T. \tag{3.20}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

Case 1. Suppose that f is bounded. Then, there exists some $J > 0$ such that

$$f(u) \le J, \quad u \in \mathbb{R}. \tag{3.21}$$

We define

$$T_1 = \max\Big\{2w,\ J\int_0^1 \|G(\cdot,s)\| b(s) ds\Big\}.$$

Let $y \in C$ be such that $\|y\| = T_1$. Using (3.6), (2.5), and (3.21), we find for $t \in [t_i, t_{i+1}]$, $1 \le i \le r-1$,

$$(-1)^{\alpha_i} Sy(t) \le \int_0^1 \|G(\cdot,s)\| b(s) f(y(s)) ds \le \int_0^1 \|G(\cdot,s)\| b(s) J ds \le T_1 = \|y\|.$$

Hence (3.14) follows immediately.

Case 2. Suppose that f is unbounded. Then there exists $T_1 > \max\{2w, T\}$ such that

$$f(u) \le \max\{f(T_1), f(-T_1)\}, \quad |u| \le T_1. \tag{3.22}$$

Let $y \in C$ be such that $\|y\| = T_1$. Then applying (3.6), (2.5), (3.22), (3.20) and (3.18) successively gives for $t \in [t_i, t_{i+1}]$, $1 \le i \le r-1$,

$$\begin{aligned}(-1)^{\alpha_i} Sy(t) &\le \int_0^1 \|G(\cdot, s)\| b(s) f(y(s)) ds \le \\ &\le \int_0^1 \|G(\cdot, s)\| b(s) \max\{f(T_1), f(-T_1)\} ds \le \\ &\le \int_0^1 \|G(\cdot, s)\| b(s) Q T_1 ds = T_1 = \|y\|.\end{aligned}$$

Therefore we have (3.14) immediately.

In Cases 1 and 2, if we set $\Omega_2 = \{y \in B \mid \|y\| < T_1\}$, then (3.14) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (3.13) and (3.14), it follows from Theorem 2.1 that S has a fixed point $y_2 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \le \|y_2\| \le T_1$. Obviously, this y_2 is a positive solution of (M) such that (3.9) holds.

(c) This is immediate by Cases (a) and (b). $\square$

Example 3.1. Consider the boundary value problem

$$y^{(3)} = t(2y^2+1) + t^2(y^2+1), \quad t \in (0,1), \quad y(0) = y(0.5) = y(1) = 0.$$

Here, $n = r = 3$. Take $f(y) = y^2 + 1$. Then $f_0 = f_\infty = \infty$ and

$$\frac{P(t,y)}{f(y)} = t\,\frac{2y^2+1}{y^2+1} + t^2.$$

Thus we may take $a(t) = t + t^2$ and $b(t) = 2t + t^2$. The condition (A5) is fulfilled with $\rho = 1/2$. Since $f(u) \le w^2 + 1$ for $|u| \le w$, by Lemma 2.2 $(c = 0.5)$ we have

$$\int_0^1 \phi(s) b(s) ds \ge \int_0^1 \|G_1(\cdot, s)\| b(s) ds.$$

For condition (3.7) to be satisfied, we impose

$$\begin{aligned}f(u) \le w^2 + 1 \le w\left[\int_0^1 \phi(s) b(s) ds\right]^{-1} &= \\ = 18.82w \le w\left[\int_0^1 \|G_1(\cdot, s)\| b(s) ds\right]^{-1}&, \quad |u| \le w.\end{aligned}$$

The above inequality holds if

$$0.05329 \le w \le 18.76. \tag{3.23}$$

Hence (3.7) is fulfilled for any $w \in [0.05329, 18.76]$. By Theorem 3.1(c), the boundary value problem has two positive solutions y_1 and y_2 such that $0 < \|y_1\| \le w \le \|y_2\|$. In view of (3.23) we further conclude that

$$0 < \|y_1\| \le 0.05329 \qquad \text{and} \qquad \|y_2\| \ge 18.76. \tag{3.24}$$

Example 3.2. Consider the boundary value problem

$$y^{(3)} = h(t)y^2 e^{-|y|}, \;\; t \in (0,1), \quad y(0) = y(0.5) = y(1) = 0,$$

where $h \in C[0,1]$ is nonnegative.

Let $f(y) = y^2 e^{-|y|}$. Then we have $f_0 = f_\infty = 0$ and we may take $a(t) = b(t) = h(t)$ so that (A5) is satisfied with $\rho = 1$. By Lemma 2.2 ($c = 0.5$) we compute that $\sigma_i = \rho L_i = L_i = 3/32,\; i = 1,2$. Our aim is to find some $w > 0$ such that condition (3.17) be fulfilled.

Case 1. Let $h(t) = (t+0.1)^{-10}$. Suppose that $w \le 2$. Then, since f is nondecreasing in $|u|$ for $|u| \in [0,2]$, we have $f(u) \ge \Big(3w/32\Big)^2 e^{-3w/32}$, $3w/32 \le |u| \le w$. Therefore (3.17) is satisfied if we set

$$f(u) \ge \Big(\frac{3w}{32}\Big)^2 e^{-3w/32} \ge w\left[\sum_{i=1}^{r-1}\int_{I_i}(-1)^{\alpha_1}G\Big(t_2/2, s\Big)a(s)ds\right]^{-1} =$$

$$= w\left[\sum_{i=1}^{2}\int_{I_i}G_1(0.25, s)h(s)ds\right]^{-1} = 0.002939w, \;\; 3w/32 \le |u| \le w.$$

The explicit expression of the Green's function (see (2.11)) is used in the computation of the above integrals. It can be checked that the above inequality holds if

$$0.3454 \le w \le 2. \tag{3.25}$$

Thus by Theorem 3.2(c) the boundary value problem has two positive solutions y_1 and y_2 such that $0 < \|y_1\| \le w \le \|y_2\|$. Moreover, it follows from (3.25) that

$$0 < \|y_1\| \le 0.3454 \;\; \text{and} \quad \|y_2\| \ge 2. \tag{3.26}$$

Case 2. Let $h(t) = (t^6 + 0.1)^{-16}$. Suppose that $3w/32 \ge 2$. Then, since f is nonincreasing in $|u|$ for $|u| \ge 2$, we have $f(u) \ge w^2 e^{-w}$, $3w/32 \le |u| \le w$. Thus, for (3.17) to be satisfied we impose

$$f(u) \ge w^2 e^{-w} \ge w\left[\sum_{i=1}^{r-1}\int_{I_i}(-1)^{\alpha_1}G\Big(t_2/2, s\Big)a(s)ds\right]^{-1} =$$

$$= 4.001 \times 10^{-14} w, \quad 3w/32 \le |u| \le w.$$

The above inequality holds if

$$21.34 \le w \le 34.38. \tag{3.27}$$

It follows from Theorem 3.2(c) that the boundary value problem has two positive solutions y_1 and y_2 such that $0 < \|y_1\| \le w \le \|y_2\|$. Moreover, in view of (3.27) we have

$$0 < \|y_1\| \le 21.34 \qquad \text{and} \qquad \|y_2\| \ge 34.38. \tag{3.28}$$

4. Double Positive Solutions of $(M1)$

We assume that the function $q : (0,1) \to [0,\infty)$ is continuous and satisfies (B1) and (B2), or (B3), where

(B1) q is not identically zero on any nondegenerate subinterval of $(0,1)$;

(B2) $\int_0^1 \|G_1(\cdot,s)\| q(s)ds < \infty$, where G_1 is given in (2.11);

(B3) $q : [0,1] \to [0,\infty)$ is continuous and (B1) holds with $(0,1)$ replaced by $[0,1]$.

Theorem 4.1. *Let $w > 0$ be given. Suppose that*

$$\int_0^1 \phi(s)q(s)ds \le \frac{w}{w^\alpha + w^\beta}, \tag{4.1}$$

where ϕ is defined in (2.8)*. Then the boundary value problem* $(M1)$ *has two positive solutions y_1 and y_2 such that* $0 < \|y_1\| \le w \le \|y_2\|$.

Proof. Let $f(u) = |u|^\alpha + |u|^\beta$. Then $f_0 = f_\infty = \infty$. Further, we may take $a(t) = b(t) = q(t)$. Clearly, $f(u) \le w^\alpha + w^\beta$ for $|u| \le w$. So to ensure that (3.7) is satisfied we apply Lemma 2.2 and impose

$$w^\alpha + w^\beta \le w\left[\int_0^1 \phi(s)b(s)ds\right]^{-1} \le w\left[\int_0^1 \|G_1(\cdot,s)\|b(s)ds\right]^{-1}$$

which is exactly condition (4.1). The conclusion now follows from Theorem 3.1(c). $\square$

Example 4.1. Consider the boundary value problem $(M1)$ with $c = 0.2$. Let $w = 1$. Then condition (4.1) reduces to

$$\int_0^1 \phi(s)q(s)ds \le 1/2. \tag{4.2}$$

By Theorem 4.1 the boundary value problem has double positive solutions y_1 and y_2 such that $0 < \|y_1\| \le 1 \le \|y_2\|$ if $q(t)$ fulfills (4.2). Examples of such $q(t)$ include $q(t) = 1,\ t+1,\ \sin^2(t+1)$.

The next result offers upper and lower bounds for the norms of two positive solutions of $(M1)$.

Theorem 4.2. *We define*

$$q_i = \inf_{t\in I_i} q(t), \quad i = 1, 2, \tag{4.3}$$

$$Q_1(u) = L_1^u q_1 \frac{c^3(92 - 39c)}{3072}, \quad Q_2(u) = L_2^u q_2 \frac{13(1 - c)^3(3c + 1)}{3072},$$

$$w_1 = [\max\{Q_1(\alpha), Q_2(\alpha)\}]^{\frac{1}{1-\alpha}} \quad \text{and} \quad w_2 = [\max\{Q_1(\beta), Q_2(\beta)\}]^{\frac{1}{1-\beta}}.$$

Let $w > 0$ be given. Suppose that (4.1) *holds. Then the boundary value problem* $(M1)$ *has double positive solutions y_1 and y_2 such that*

(a) *if* $w < \min\{w_1, w_2\}$, *then* $0 < \|y_1\| \le w \le \|y_2\| \le \min\{w_1, w_2\}$;

(b) *if* $\min\{w_1, w_2\} < w < \max\{w_1, w_2\}$, *then* $\min\{w_1, w_2\} \le \|y_1\| \le w \le \|y_2\| \le \max\{w_1, w_2\}$;

(c) *if* $w > \max\{w_1, w_2\}$, *then* $\max\{w_1, w_2\} \le \|y_1\| \le w \le \|y_2\|$.

Proof. Since (4.1) is satisfied, it follows from Theorem 4.1 that (M1) has double positive solutions y_3 and y_4 such that

$$0 < \|y_3\| \le w \le \|y_4\|. \tag{4.4}$$

Let C_1 be a cone in B defined by

$$C_1 = \left\{ y \in B \,\middle|\, (-1)^{\alpha_i} y(t) \ge 0,\ t \in [t_i, t_{i+1}],\ \min_{t\in I_i} (-1)^{\alpha_i} y(t) \ge L_i \|y\|,\ i = 1, 2 \right\}, \tag{4.5}$$

where L_i, $i = 1, 2$, are given in (2.9) and (2.10). Define the operator $T : C_1 \to B$ by

$$Ty(t) = \int_0^1 G_1(t, s) q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds, \quad t \in [0, 1].$$

To obtain a positive solution of $(M1)$ we shall seek a fixed point of T in the cone C_1.

First, we shall show that $T(C_1) \subseteq C_1$. For this, let $y \in C_1$. Clearly, for $i = 1, 2$, $(-1)^{\alpha_i} Ty(t)$ is nonnegative on $[t_i, t_{i+1}]$. Further, for $t \in [t_i, t_{i+1}]$, $i = 1, 2$, we have

$$(-1)^{\alpha_i} Ty(t) \le \int_0^1 \|G_1(\cdot, s)\| q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds$$

which implies

$$\|Ty\| \le \int_0^1 \|G_1(\cdot, s)\| q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds. \tag{4.6}$$

Now, applying Lemma 2.2 and (4.6), we find for $t \in I_i,\ i = 1, 2,$

$$(-1)^{\alpha_i} Ty(t) \geq \int_0^1 L_i \|G_1(\cdot, s)\| q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds \geq L_i \|Ty\|.$$

Thus $\min_{t \in I_i} (-1)^{\alpha_i} Ty(t) \geq L_i \|Ty\|$, $i = 1, 2$, and so $Ty \in C_1$. Also, the standard arguments yield that T is completely continuous.

Let $y \in C_1$ be such that $\|y\| = w$. Then, using Lemma 2.2 and (4.1), we find for $t \in [t_i, t_{i+1}]$, $i = 1, 2$,

$$\begin{aligned}(-1)^{\alpha_i} Ty(t) \leq \int_0^1 \|G_1(\cdot, s)\| q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds \leq \\ \leq \int_0^1 \phi(s) q(s)(w^\alpha + w^\beta) ds \leq w = \|y\|.\end{aligned}$$

Therefore

$$\|Ty\| \leq \|y\|. \tag{4.7}$$

By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we have (4.7) for $y \in C_1 \cap \partial\Omega$.

Now, let $y \in C_1$. It follows that

$$\begin{aligned}\|Ty\| &= \sup_{\substack{t \in [t_i, t_{i+1}],\\ i=1,2}} \int_0^1 |G_1(t, s)| q(s)[|y(s)|^\alpha + |y(s)|^\beta] ds \geq \\ &\geq \sup_{\substack{t \in [t_i, t_{i+1}],\\ i=1,2}} \int_{I_i} |G_1(t, s)| q(s)[(L_i\|y\|)^\alpha + (L_i\|y\|)^\beta] ds \geq \\ &\geq \max\Big\{ \int_{I_1} \Big|G_1\big(c/4, s\big)\Big| q_1[(L_1\|y\|)^\alpha + (L_1\|y\|)^\beta] ds, \\ &\qquad \int_{I_2} \Big|G_1\Big((3c+1)/4, s\Big)\Big| q_2[(L_2\|y\|)^\alpha + (L_2\|y\|)^\beta] ds \Big\}.\end{aligned}$$

From (2.11) we have

$$\begin{aligned}\big|G_1(c/4, s)\big| &= \frac{3}{8}\Big(1 - \frac{c}{4}\Big)s^2 - \frac{1}{2}\Big(s - \frac{c}{4}\Big)^2, \quad s \in I_1, \\ \big|G_1((3c+1)/4, s)\big| &= \frac{3c+1}{32}(1-s)^2, \quad s \in I_2,\end{aligned} \tag{4.8}$$

which upon substituting into the above inequality yields

$$\|Ty\| \geq \max\Big\{ Q_1(\alpha)\|y\|^\alpha + Q_1(\beta)\|y\|^\beta,\ Q_2(\alpha)\|y\|^\alpha + Q_2(\beta)\|y\|^\beta \Big\}. \tag{4.9}$$

Let $y \in C_1$ be such that $\|y\| = w_1$. It follows from (4.9) that

$$\|Ty\| \geq \max\Big\{ Q_1(\alpha)\|y\|^\alpha, Q_2(\alpha)\|y\|^\alpha \Big\} =$$

$$= \max\Big\{Q_1(\alpha), Q_2(\alpha)\Big\}\|y\|^{\alpha-1}\|y\| = \|y\|. \tag{4.10}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$, then (4.10) holds for $y \in C_1 \cap \partial\Omega_1$. Now that we have obtained (4.7) and (4.10), it follows from Theorem 2.1 that T has a fixed point y_5 such that

$$\min\{w_1, w\} \le \|y_5\| \le \max\{w_1, w\}. \tag{4.11}$$

On the other hand, if we let $y \in C_1$ be such that $\|y\| = w_2$, then from (4.9) we get

$$\|Ty\| \ge \max\Big\{Q_1(\beta)\|y\|^\beta, Q_2(\beta)\|y\|^\beta\Big\} = \|y\|. \tag{4.12}$$

Take $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$, then (4.12) holds for $y \in C_1 \cap \partial\Omega_2$. Having obtained (4.7) and (4.12), by Theorem 2.1 we conclude that T has a fixed point y_6 such that

$$\min\{w_2, w\} \le \|y_6\| \le \max\{w_2, w\}. \tag{4.13}$$

Now, a combination of (4.4), (4.11) and (4.13) yields our result. More precisely, in Case (a) we may take $y_1 = y_3$ and $y_2 = \begin{cases} y_5, & w_1 \le w_2 \\ y_6, & w_1 \ge w_2 \end{cases}$. In Case (b) it is clear that $y_1 = \begin{cases} y_5, & w_1 \le w_2 \\ y_6, & w_1 \ge w_2 \end{cases}$ and $y_2 = \begin{cases} y_6, & w_1 \le w_2 \\ y_5, & w_1 \ge w_2 \end{cases}$. Finally, in Case (c) we shall take $y_1 = \begin{cases} y_6, & w_1 \le w_2 \\ y_5, & w_1 \ge w_2 \end{cases}$ and $y_2 = y_4$. $\square$

Example 4.2. Consider the boundary value problem

$$y^{(3)} = (t+1)\Big(|y|^{0.1} + |y|^3\Big),\ \ t \in (0,1),\quad y(0) = y(0.3) = y(1) = 0.$$

Here $\alpha = 0.1,\ \beta = 3,\ c = 0.3$ and $q(t) = t + 1$. Condition (4.1) is the same as

$$\frac{w}{w^{0.1} + w^3} \ge \int_0^1 \phi(s)q(s)ds = 0.09718$$

which is satisfied for any $w \in [0.07505, 3.151]$.

Further, with $q_1 = 1.075,\ q_2 = 1.475,\ L_1 = 0.02411$ and $L_2 = 0.1188$, we compute

$$w_1 = [\max\{Q_1(\alpha), Q_2(\alpha)\}]^{1/(1-\alpha)} = [Q_2(\alpha)]^{1/(1-\alpha)} = 2.836 \times 10^{-4}$$

and

$$w_2 = [\max\{Q_1(\beta), Q_2(\beta)\}]^{1/(1-\beta)} = [Q_2(\beta)]^{1/(1-\beta)} = 382.9.$$

Since $w \in (w_1, w_2)$, by Theorem 4.2(b) the boundary value problem has two positive solutions y_1 and y_2 such that

$$2.836 \times 10^{-4} \le \|y_1\| \le w \le \|y_2\| \le 382.9. \tag{4.14}$$

Taking into account the range of w, (4.14) leads to

$$2.836 \times 10^{-4} \le \|y_1\| \le 0.07505 \quad \text{and} \quad 3.151 \le \|y_2\| \le 382.9. \tag{4.15}$$

5. Double Positive Solutions of $(M2)$

As in Section 4, it is assumed that the function $q : (0,1) \to [0,\infty)$ is continuous and satisfies (B1) and (B2), or (B3).

Theorem 5.1. *Let $w > 0$ be given. Suppose that*

$$\int_0^1 \phi(s)q(s)ds \le we^{-\gamma w}. \tag{5.1}$$

Then the boundary value problem $(M2)$ has two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| \le w \le \|y_2\|. \tag{5.2}$$

Proof. Let $f(u) = e^{\gamma|u|}$ and $a(t) = b(t) = q(t)$. Noting that $f(u) \le e^{\gamma w}$ for $|u| \le w$, an argument as in the proof of Theorem 4.1 yields the conclusion. $\square$

Example 5.1. Consider the boundary value problem

$$y^{(3)} = q(t)e^{|y|}, \ \ t \in (0,1), \quad y(0) = y(0.7) = y(1) = 0.$$

Let $w = 1/3$ be given. Then condition (5.1) reduces to

$$\int_0^1 \phi(s)q(s)ds \le \frac{1}{3}\ e^{-1/3}. \tag{5.3}$$

By Theorem 5.1, for those $q(t)$ which fulfill (5.3), the boundary value problem has two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| \le \frac{1}{3} \le \|y_2\|.$$

Some examples of such $q(t)$ are $q(t) = (2t+12)^{-1}$, $(t^2+3)/2$, $\left[\cos^2(t+2)\right]/2$.

Once again we shall establish upper and lower bounds for the norms of two positive solutions of $(M2)$.

Theorem 5.2. *Let k, ℓ $(k \ne \ell)$ be given integers in the set $\{0, 2, 3, \dots\}$. We define*

$$R_1(u) = q_1 \frac{(\gamma L_1)^u}{u!} \frac{c^3(92-39c)}{3072}, \quad R_2(u) = q_2 \frac{(\gamma L_2)^u}{u!} \frac{13(1-c)^3(3c+1)}{3072},$$

$$w_1 = [\max\{R_1(k), R_2(k)\}]^{\frac{1}{1-k}} \quad \textit{and} \quad w_2 = [\max\{R_1(\ell), R_2(\ell)\}]^{\frac{1}{1-\ell}},$$

where $q_i,\ i = 1, 2$, *are given in* (4.3). *Let* $w > 0$ *be given. Suppose that* (5.1) *holds. Then, the boundary value problem* $(M2)$ *has double positive solutions* y_1 *and* y_2 *such that conclusions* (a)–(c) *of Theorem* 4.2 *hold.*

Proof. Since (5.1) is fulfilled, by Theorem 5.1 the boundary value problem $(M2)$ has double positive solutions y_3 and y_4 such that (4.4) holds.

To proceed, let C_1 be a cone in B defined by (4.5) and let the operator $\bar{T} : C_1 \to B$ be defined by

$$\bar{T}y(t) = \int_0^1 G_1(t, s)q(s)e^{\gamma|y(s)|}ds, \quad t \in [0, 1].$$

To obtain a positive solution of $(M2)$, we shall seek a fixed point of $\bar{T}$ in the cone C_1. Using an argument as in the proof of Theorem 4.2, it can be verified that $\bar{T}(C_1) \subseteq C_1$ and $\bar{T}$ is completely continuous.

Let $y \in C_1$ be such that $\|y\| = w$. Applying Lemma 2.2 and (5.1), we get for $t \in [t_i, t_{i+1}],\ i = 1, 2,$

$$(-1)^{\alpha_i}\bar{T}y(t) \le \int_0^1 \phi(s)q(s)e^{\gamma}|y(s)|ds \le \int_0^1 \phi(s)q(s)e^{\gamma w}ds \le w = \|y\|.$$

Hence

$$\|\bar{T}y\| \le \|y\|. \tag{5.4}$$

If we set $\Omega = \{y \in B \mid \|y\| < w\}$, then (5.4) holds for $y \in C_1 \cap \partial\Omega$.

Next, let $y \in C_1$. We find that

$$\|\bar{T}y\| \ge \sup_{\substack{t\in[t_i,t_{i+1}]\\ i=1,2}} nt_{I_i}|G_1(t, s)|q_i e^{\gamma L_i\|y\|}ds.$$

Using the relation

$$e^u \ge \frac{u^k}{k!} + \frac{u^\ell}{\ell!},\ u > 0,$$

in the above inequality, we find

$$\begin{aligned}\|\bar{T}y\| \ge \sup_{\substack{t\in[t_i,t_{i+1}]\\ i=1,2}} \int_{I_i} |G_1(t, s)|q_i\Big[\frac{(\gamma L_i)^k}{k!}\ \|y\|^k + \frac{(\gamma L_i)^\ell}{\ell!}\ \|y\|^\ell\Big] \ge \\ \ge \max\bigg\{\int_{I_1}\Big|G_1\Big(\frac{c}{4}, s\Big)\Big|q_1\Big[\frac{(\gamma L_1)^k}{k!}\ \|y\|^k + \frac{(\gamma L_1)^\ell}{\ell!}\ \|y\|^\ell\Big], \\ \int_{I_2}\Big|G_1\Big(\frac{3c+1}{4}, s\Big)\Big|q_2\Big[\frac{(\gamma L_2)^k}{k!}\ \|y\|^k + \frac{(\gamma L_2)^\ell}{\ell!}\ \|y\|^\ell\Big]\bigg\}.\end{aligned}$$

On substituting (4.8), it follows that

$$|\bar{T}y\| \ge \max\big\{R_1(k)\|y\|^k + R_1(\ell)\|y\|^\ell, R_2(k)\|y\|^k + R_2(\ell)\|y\|^\ell\big\}. \tag{5.5}$$

Employing a technique as in the proof of Theorem 4.2, from (5.5) we obtain

$$|\bar{T}y\| \geq \|y\| \tag{5.6}$$

for $y \in C_1 \cap \partial\Omega_1$ as well as for $y \in C_1 \cap \partial\Omega_2$, where

$$\Omega_1 = \{y \in B \mid \|y\| < w_1\} \qquad \text{and} \qquad \Omega_2 = \{y \in B \mid \|y\| < w_2\}.$$

Now that we have obtained (5.4) and (5.6), by Theorem 2.1 $\bar{T}$ has a fixed point y_5 satisfying

$$\min\{w_1, w\} \leq \|y_5\| \leq \max\{w_1, w\}, \tag{5.7}$$

and also a fixed point y_6 such that

$$\min\{w_2, w\} \leq \|y_6\| \leq \max\{w_2, w\}. \tag{5.8}$$

As in the proof of Theorem 4.2, the combination of (4.4), (5.7) and (5.8) readily gives rise to conclusions (a)–(c). □

Example 5.2. Consider the boundary value problem

$$y^{(3)} = \frac{6}{\exp(0.5t(1-t)|t-0.3|)} e^{|y|/2}, \ t \in (0,1), \ \ y(0) = y(0.3) = y(1) = 0.$$

With $c = 0.3$ and $q(t) = 6[\exp(0.5t(1-t)|t-0.3|)]^{-1}$, we get $L_1 = 0.02411$, $L_2 = 0.1188$, $q_1 = q\Big((13-\sqrt{79})/30\Big) = 5.942$ and $q_2 = q((13+\sqrt{79})/30) = 5.751$. It can be checked that condition (5.1) is satisfied provided

$$0.5166 \leq w \leq 5.093. \tag{5.9}$$

Let $k = 0$ and $\ell = 9$. We find $w_1 = [R_2(k)]^{1/(1-k)} = 0.01586$ and $w_2 = [R_1(\ell)]^{1/(1-\ell)} = 1415$. Since $w \in (w_1, w_2)$, it follows from Theorem 5.2(b) that the boundary value problem has two positive solutions y_1 and y_2 such that $0.01586 \leq \|y_1\| \leq w \leq \|y_2\| \leq 1415$. In view of (5.9) we further conclude that

$$0.01586 \leq \|y_1\| \leq 0.5166 \qquad \text{and} \qquad 5.093 \leq \|y_2\| \leq 1415. \tag{5.10}$$

Indeed, the boundary value problem has a positive solution given by $y(t) = t(t-0.3)(t-1)$ and we note that $\|y\| = y\Big((13+\sqrt{79})/30\Big) = 0.08475$ is within the range given in (5.10).

6. Triple Positive Solutions of ($M3$)

It is assumed that the functions $f : \mathbb{R} \to [0, \infty)$ and $b : (0, 1) \to \mathbb{R}$ are continuous and satisfy (A1), (A3)$'$ and (A4), or (A5)$'$, where

(A3)$'$ b is nonnegative and not identically zero on any nondegenerate subinterval of $(0, 1)$;

(A5)$'$ $b : [0, 1] \to \mathbb{R}$ is continuous and (A3)$'$ holds with $(0, 1)$ replaced by $[0, 1]$.

With the same Banach space B $(= C[0, 1])$, let

$$C_2 = \{y \in B \mid (-1)^{\alpha_i} y(t) \geq 0,\ t \in [t_i, t_{i+1}],\ 1 \leq i \leq r - 1\}.$$

We note that C_2 is a cone in B.

Let the operator $V : C_2 \to B$ be defined by

$$Vy(t) = \int_0^1 G(t, s)b(s)f(y(s))ds, \quad t \in [0, 1]. \tag{6.1}$$

To obtain a positive solution of ($M3$), we shall seek a fixed point of the operator V in the cone C_2.

By using an argument as in Section 3, we see that the operator V is compact on the cone C_2. Next, it is clear from (6.1) and (2.4) that if $y \in C_2$, then $(-1)^{\alpha_i} Vy(t) \geq 0$ for $t \in [t_i, t_{i+1}],\ 1 \leq i \leq r - 1$. Hence $Vy \in C_2$ and we have shown that V maps C_2 into itself. Also, the standard arguments yield that V is completely continuous.

It is clear that

$$\|Vy\| = \int_0^1 \|G(\cdot, s)\| b(s)f(y(s))ds. \tag{6.2}$$

We define the constants

$$z = \int_0^1 \|G(\cdot, s)\| b(s)ds \text{ and } \mu = \min_{1 \leq i \leq r-1} \min_{t \in I_i} \int_0^1 (-1)^{\alpha_i} G(t, s)b(s)ds. \tag{6.3}$$

Lemma 6.1. *Suppose that there exists $\nu > 0$ such that $f(u) < \frac{\nu}{z}$ for $|u| \in [0, \nu]$. Then*

$$V(\bar{C}_2(\nu)) \subseteq C_2(\nu) \subset \bar{C}_2(\nu).$$

Proof. Let $y \in \bar{C}_2(\nu)$. Then we have for $t \in [t_i, t_{i+1}],\ 1 \leq i \leq r - 1$,

$$(-1)^{\alpha_i} Vy(t) \leq \int_0^1 \|G(\cdot, s)\| b(s)f(y(s))ds < \int_0^1 \|G(\cdot, s)\| b(s)\ \frac{\nu}{z} ds = \nu.$$

Consequently, $\|Vy\| < \nu$ and so $Vy \in C_2(\nu)$. This immediately implies that $V(\bar{C}_2(\nu)) \subseteq C_2(\nu) \subset \bar{C}_2(\nu)$. $\square$

Theorem 6.1. *Suppose that there exist numbers k, ℓ, m, where $0 < k < \ell < m$, $m \geq 1$ and $\ell \leq \big[\min_{1\le i\le r-1}(t_{i+1}-t_i)/4\big]^n$, such that the following conditions are satisfied:*

(H1) *one of the following holds:*

(i) $\limsup_{|u|\to\infty} f(u)/|u| < 1/z$;

(ii) *there exists a number η, where $\eta > m$, such that if $|u| \in [0,\eta]$, then $f(u) < \eta/z$;*

(H2) *if $|u| \in [\ell, m]$, then $f(u) > \ell/\mu$;*

(H3) *if $|u| \in [0,k]$, then $f(u) < k/z$.*

Then the boundary value problem $(M3)$ has (at least) three positive solutions y_1, y_2 and y_3 such that

$$\begin{aligned} &\|y_1\| < k; \qquad (-1)^{\alpha_i} y_2(t) > \ell,\ t\in I_i,\ 1\le i\le r-1;\\ &\|y_3\| > k \qquad \text{and} \qquad \min_{1\le i\le r-1}\min_{t\in I_i}(-1)^{\alpha_i} y_3(t) < \ell. \end{aligned} \tag{6.4}$$

Proof. We shall show that the conditions of Theorem 2.2 are fulfilled. First we shall prove that condition (H1) leads to the existence of a number ν, where $\nu > m$, such that

$$V(\bar{C}_2(\nu)) \subset \bar{C}_2(\nu). \tag{6.5}$$

For this, it is clear that if (ii) holds, then by Lemma 6.1 we immediately have (6.5) where $\nu = \eta$. Suppose now that (i) is satisfied. Then there exist $T > 0$ and $\epsilon < \frac{1}{z}$ such that

$$f(u)/|u| < \epsilon, \quad |u| > T. \tag{6.6}$$

Define $M = \max_{|u|\in[0,T]} f(u)$. In view of (6.6) it is obvious that

$$f(u) \le M + \epsilon|u|, \quad |u| \ge 0. \tag{6.7}$$

Now let ν be such that

$$\nu > \max\big\{m,\ M(1/z-\epsilon)^{-1}\big\}. \tag{6.8}$$

For $y \in \bar{C}_2(\nu)$ and $t \in [t_i, t_{i+1}]$, $1 \le i \le r-1$, we use (2.5), (6.7), (6.3) and (6.8) to get

$$\begin{aligned} (-1)^{\alpha_i} Vy(t) &\le \int_0^1 \|G(\cdot,s)\| b(s) f(y(s))ds \le \\ &\le \int_0^1 \|G(\cdot,s)\| b(s)(M+\epsilon|y(s)|)ds \le \\ &\le \int_0^1 \|G(\cdot,s)\| b(s)(M+\epsilon\nu)ds = \\ &= z(M+\epsilon\nu) < z\big[\nu(1/z-\epsilon)+\epsilon\nu\big] = \nu. \end{aligned}$$

Hence $\|Vy\| < \nu$ and so (6.5) follows immediately.

Let $\psi : C_2 \to [0, \infty)$ be defined by

$$\psi(y) = \min_{1\le i\le r-1} \min_{t\in I_i} (-1)^{\alpha_i} y(t).$$

Clearly, ψ is a nonnegative continuous concave functional on C_2 such that $\psi(y) \le \|y\|$ for all $y \in C_2$.

We shall now show that condition (a) of Theorem 2.2 is satisfied. For this, note that $y_1(t) \equiv \prod_{i=1}^{r} (t - t_i)^{n_i}$ has the properties

$$(-1)^{\alpha_i} y_1(t) \ge 0,\ t \in [t_i, t_{i+1}],\ 1 \le i \le r-1,\ \|y_1\| < 1 \le m$$

and

$$\begin{aligned}\psi(y_1) &= \min_{1\le i\le r-1} \min_{t\in I_i} (-1)^{\alpha_i} y_1(t) = \\ &= \min_{1\le i\le r-1} \left\{ \Big| y_1\Big(\frac{3t_i + t_{i+1}}{4}\Big)\Big|, \Big| y_1\Big(\frac{t_i + 3t_{i+1}}{4}\Big)\Big| \right\} > \\ &> \big[\min_{1\le i\le r-1} (t_{i+1} - t_i)/4 \big]^n \ \ge\ \ell.\end{aligned}$$

Hence

$$y_1 \in \{ y \in C_2(\psi, \ell, m) \mid \psi(y) > \ell \} \ne \varnothing. \tag{6.9}$$

Next, let $y \in C_2(\psi, \ell, m)$. Then $\ell \le \|y\| \le m$ and so $|y(s)| \in [\ell, m]$ for all $s \in [0, 1]$. Using this together with (H2) and (6.3), we get

$$\begin{aligned}\psi(Vy) &= \min_{1\le i\le r-1} \min_{t\in I_i} (-1)^{\alpha_i} Vy(t) = \\ &= \min_{1\le i\le r-1} \min_{t\in I_i} \int_0^1 (-1)^{\alpha_i} G(t, s) b(s) f(y(s)) ds > \\ &> \min_{1\le i\le r-1} \min_{t\in I_i} \int_0^1 (-1)^{\alpha_i} G(t, s) b(s)\ (\ell/\mu)\ ds\ =\ \ell.\end{aligned}$$

Therefore $\psi(Vy) > \ell$ for all $y \in C_2(\psi, \ell, m)$.

Moreover, it follows from Lemma 6.1 and condition (H3) that $V(\bar{C}_2(k)) \subseteq C_2(k)$. Hence condition (b) of Theorem 2.2 is satisfied.

It remains to verify that condition (c) of Theorem 2.2 holds. Let $m > \ell(\min_{1\le i\le r-1} L_i)^{-1}$ $(\ge \ell)$ and let $y \in C_2(\psi, \ell, \nu)$ with $\|Vy\| > m$. Applying Lemma 2.1 and (6.2), we find

$$\begin{aligned}\psi(Vy) &= \min_{1\le i\le r-1} \min_{t\in I_i} \int_0^1 (-1)^{\alpha_i} G(t, s) b(s) f(y(s)) ds \ge \\ &\ge \min_{1\le i\le r-1} \int_0^1 L_i \|G(\cdot, s)\| b(s) f(y(s)) ds =\end{aligned}$$

$$= \min_{1\le i\le r-1} L_i\|Vy\| > m \min_{1\le i\le r-1} L_i \;>\; \ell.$$

This shows that $\psi(Vy) > \ell$ for all $y \in C_2(\psi,\ell,\nu)$ with $\|Vy\| > m$.

Consequently, it follows from Theorem 2.2 that the boundary value problem $(M3)$ has (at least) three *positive* solutions $y_1, y_2, y_3 \in \bar{C}_2(\nu)$. Further, we have (2.1) which reduces to (6.4). $\square$

Example 6.1. Consider the boundary value problem

$$y^{(3)} = (t^2+1)f(y), \;\; t \in (0,1), \quad y(0) = y(0.5) = y(1) = 0,$$

where

$$f(y) = \begin{cases} |\sin y|, & |y| \le 0.001 \\ \sin 0.001 + (|y| - 0.001)^{1/8}, & 0.001 \le |y| \le 1 \\ \sin 0.001 + 0.999^{1/8}, & |y| \ge 1 \end{cases}.$$

Here $n = r = 3$ and $b(t) = t^2 + 1$. It is clear that $(A5)'$ is satisfied. Using (2.11), we find

$$z = \int_0^1 \|G_1(\cdot,s)\| b(s)ds =$$
$$= \max\left\{ \max_{t\in[0,0.5]} \int_0^1 |G_1(t,s)|b(s)ds, \;\; \max_{t\in[0.5,1]} \int_0^1 |G_1(t,s)|b(s)ds \right\} = 0.0109$$

and

$$\mu = \min\left\{ \min_{t\in[\frac{1}{8},\frac{3}{8}]} \int_0^1 |G_1(t,s)|b(s)ds, \;\; \min_{t\in[\frac{5}{8},\frac{7}{8}]} \int_0^1 |G_1(t,s)|b(s)ds \right\} = 0.006081.$$

Obviously, condition (H1) holds as

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = \lim_{|u|\to\infty} \frac{\sin 0.001 + 0.999^{1/8}}{|u|} = 0.$$

Next, take $k = 0.001$. Then (H3) is fulfilled, since for $|u| \in [0,k]$,

$$f(u) \le \sin 0.001 < k/z.$$

Finally, let $m = 1$ and take ℓ $\left(\le \left[\min_{1\le i\le r-1}(t_{i+1} - t_i)/4\right]^n = 0.001953\right)$ such that condition (H2) is satisfied. Clearly, (H2) is fulfilled provided that for $|u| \in [\ell, m]$,

$$f(u) \ge \sin 0.001 + (\ell - 0.001)^{1/8} > \ell/\mu.$$

The above inequality is satisfied if

$$0.001001 \le \ell \le 0.001953. \tag{6.10}$$

By Theorem 6.1 the boundary value problem has (at least) three positive solutions $y_1,\ y_2$ and y_3. Further, in view of (6.10), it follows from (6.4) that

$$\begin{aligned} &\|y_1\| < 0.001; \ \ |y_2(t)| > 0.001953, \ t \in \left[\frac{1}{8}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{7}{8}\right]; \\ &\|y_3\| > 0.001 \ \text{ and } \min_{t\in\left[\frac{1}{8},\frac{3}{8}\right]\cup\left[\frac{5}{8},\frac{7}{8}\right]} |y_3(t)| < 0.001001. \end{aligned} \tag{6.11}$$

References

1. R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, Positive solutions of differential, difference and integral equations. *Kluwer Academic Publishers, Dordrecht*, 1999.

2. D. Guo and V. Lakshmikantham, Nonlinear problems in abstract cones. *Academic Press, San Diego*, 1988.

3. R. P. Agarwal, Boundary value problems for higher order differential equations. *World Scientific, Singapore*, 1986.

4. G. H. Meyer, Initial value methods for boundary value problems. *Academic Press, New York*, 1973.

5. D. Aronson, M. G. Crandall, and L. A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlinear Anal.* **6**(1982), 1001–1022.

6. Y. S. Choi and G. S. Ludford, An unexpected stability result of the near-extinction diffusion flame for non-unity Lewis numbers. *Quart. J. Mech. Appl. Math.* **42**, part **1**(1989), 143–158.

7. D. S. Cohen, Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory. *SIAM J. Appl. Math.* **20** (1971), 1–13.

8. E. N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large. *J. Differential Equations* **37**(1980), 404-437.

9. H. Fujita, On the nonlinear equations $\Delta u + e^u = 0$ and $\frac{\partial v}{\partial t} = \Delta v + e^v$. *Bull. Amer. Math. Soc.* **75**(1969), 132–135.

10. I. M. Gel'fand, Some problems in the theory of quasilinear equations. *Uspehi Mat. Nauk* **14**(1959), 87–158; *English translation: Trans. Amer. Math. Soc.* **29**(1963), 295–381.

11. S. Parter, Solutions of differential equations arising in chemical reactor processes. *SIAM J. Appl. Math.* **26**(1974), 687–716.

12. R. P. Agarwal, M. Bohner, and P. J. Y. Wong, Positive solutions and eigenvalues of conjugate boundary value problems. *Proc. Edinburgh Math. Soc.* (*to appear*).

13. R. P. Agarwal, M. Bohner, and P. J. Y. Wong, On eigenvalues and eigenfunctions of discrete conjugate boundary value problems. *Comput.*

Math. Appl. (*to appear*).

14. P. W. Eloe and J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problems. *Differential Equations Dynam. Systems* (*to appear*).

15. L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations. *Proc. Amer. Math. Soc.* **120**(1994), 743–748.

16. R. P. Agarwal, Difference equations and inequalities. *Marcel Dekker, New York*, 1992.

17. R. P. Agarwal, Focal boundary value problems for differential and difference equations. *Kluwer Academic Publishers, Dordrecht*, 1998.

18. R. P. Agarwal and P. J. Y. Wong, Advanced topics in difference equations. *Kluwer Academic Publishers, Dordrecht*, 1997.

19. P. W. Eloe and J. Henderson, Twin solutions for nonlinear multipoint conjugate boundary value problems. *Dynam. Contin. Discrete Impuls. Systems* (*to appear*).

20. M. A. Krasnosel'skii, Positive solutions of operator equations. *Noordhoff, Groningen*, 1964.

21. R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **28**(1979), 673–688.

22. W. Coppel, Disconjugacy. *Lecture Notes in Math.* 220, *Springer-Verlag, New York*, 1971.

23. P. W. Eloe and J. Henderson, Inequalities for solutions of multipoint boundary value problems. *Rocky Mountain J. Math.* (*to appear*).

24. P. J. Y. Wong, Sharp inequalities for solutions of multipoint boundary value problems. *Math. Ineq. Appl.* (*to appear*).

25. R. P. Agarwal, Boundary value problems for differential equations with deviating arguments. *J. Math. Phys. Sci.* **6**(1972), 425–438.

26. P. J. Y. Wong and R. P. Agarwal, Generalized multipoint conjugate eigenvalue problems. (*Submitted*).

(Received 14.12.1998)

Authors' addresses:

P. J. Y. Wong
Division of Mathematics
Nanyang Technological University,
Singapore

R. P. Agarwal
Department of Mathematics
National University of Singapore
Singapore